Homework 1.

(1) (a) Let \( X = \xi_1 + \cdots + \xi_n \), where \( \xi_i \) are independent indicator random variables which equal 1 with probability \( 1/2 \). What is \( \mathbb{P}(X = k) \). Compare \( \mathbb{P}(X = k) \) and \( \mathbb{P}(X = m) \) for all pairs of numbers \((k, m)\). Which value of \( k \) maximizes \( \mathbb{P}(X = k) \)? Compute the value of \( \mathbb{P}(|X - n| \leq 2\sqrt{n}) \) approximately (assuming that \( n \) is large).

(b) What are the answers if we replace \( 1/2 \) by a constant \( 0 < p < 1 \)? (again assuming that \( n \) is very large and does not depend on \( p \)).

Hint: Recall the central limit theorem.

(2) Prove that the function \( \binom{n}{k} \) is convex. Read the proof in page 7, Chapter 1.

(3) Let \( S \) be a set of \( n \) elements (\( n \) even). Choose a random subset \( A \) of \( S \) by selecting each element with probability \( 1/2 \). Repeat the process to choose another random subset \( B \). What is the expectation and variance of \( |A \cap B| \)? What happens if each element of \( B \) is chosen with probability \( 1/3 \)? What are the expectation and variance of \( |A \cup B| \) in each case?

(4) Problem 1, page 10.

(5) Problem 2, page 10.

(6) Problem 4, page 11.

Homework 2.

(1) Let \( \sigma \) be a random permutation from \( S_n \). Let \( X \) be the number of fixed points of \( \sigma \).

(a) Compute the expectation and variance of \( X \).

(b) Prove that probability that \( X = 0 \) tends to \( 1/e \) as \( n \) tends to infinity.

(2) Compute the probability that \( X \) contains a cycle of length more than \( n/2 \) (assuming that \( n \) is even).

(3) Problem 7, page 21.


(5) Problem 9, page 21.

Homework 3.
(1) Prove, without using the prime number theorem, that \( \pi(n) = \Theta(n/\log n) \), where \( \pi(n) \) is the number of primes between 1 and \( n \).

(2) Let \( X \) be the number of \( K_4 \) in \( G(n, p) \). Compute the variance of \( X \).

(3) Problem (2), page 58.

(4) Problem (4), page 59.

(5) Problem (5), page 59.

**Homework 4. Due Wednesday the week after the break**

(1) Prove the central limit theorem for the number of \( H \) in \( G(n, p) \), where \( H \) is a fixed balanced graph, and \( p \) is a constant. Find out if the proof works for \( p = n^{-c} \) for some constant \( c > 0 \). What is the largest \( c \) can you deduce?

(2) Find the smallest unbalanced graph. Call this \( H \). What is the threshold function for the appearance of \( H \) in \( G(n, p) \)?

(3) Section 5.8, exercise 3.

(4) Section 5.8, exercise 4.

(5) Read the proof of decomposable covering in 5.4. Prove the following statement: A set of \( n \) balls in \( \mathbb{R}^3 \) partition \( \mathbb{R}^3 \) in at most \( O(n^3) \) connected components. What happens in \( \mathbb{R}^d \) for a general \( d \) (we think of \( d \) fixed and \( n \) large). What happens if we replace ball by planes in \( \mathbb{R}^3 \) (and hyperplanes in \( \mathbb{R}^d \))?