LIGHT REVIEW OF WZW LAGRANGIAN

- WILL BASICALLY DISCUSS WITTEN'S 1983 PAPER
  - CLARIFIES AND REDERIVES WZ LAGRANGIAN WITH TOPOLOGICAL ARGUMENTS
  - SHOWS THAT IT NEEDS TO BE QUANTIZED
- NEED TO KNOW WHAT IS WZ LAGRANGIAN
  - J. WESS & B. ZUMINO IN 1971
  - THEY SHOW ANOMALLES OF WARD IDENTITIES SATISFY CONSISTENCY RELATIONS
  - THEY CALCULATE AN EFFECTIVE LAGRANGIAN WHICH DESCRIBE LOW ENERGY CONSEQUENCES OF ANOMALOUS WARD IDENTITIES
  - THEY SHOW EVEN THOUGH THE AXIAL VECTOR ANOMALY DEPENDS ON THE PARTICULAR SPINOR FIELDS IN THE THEORY, THIS EFFECTIVE ACTION IS DETERMINED BY THE STRUCTURE OF GAUGE GROUP ONLY: IT IS MODEL INDEPENDENT!

– NEED TO REVIEW ANOMALIES

Anomaly: quantum corrections destroying the symmetries of classical EOM

- Weyl anomaly (classical conformal symmetry is broken)
- Gravitational anomaly (general covariance of GR + some field is broken)
- Axial vector anomaly
- etc.

Axial vector anomaly: Theories with chiral symmetry and massless fermions classically have $\boxed{\partial_\mu J^\mu_A = 0}$

Quantum corrections ensures this is incompatible with gauge invariance!

– A nice demonstration in 1+1 Q.E.D. in Peskin, chapter 19.1

Different perspectives for axial vector anomaly:

1-) Regularization leaves traces: There are finite extra terms as regulator masses go to infinite (This is the source of anomaly stated in WZ paper as well (1971))

G. 't Hooft and M.J.G. Veltman introduces dimensional regularization next year (1972). Gauge invariance and $\partial_\mu J^\mu_A = 0$ can NOT be satisfied together.

For example, in 1+1 QED, one gets $\boxed{q_\mu \langle J^\mu \rangle = \frac{e}{\pi} q^\nu A_\nu}$

if one insists $\boxed{q_\mu \langle J^\mu_A \rangle = 0}$

depends on unphysical gauge degree of freedom

2-) Products of local operators are singular so they should be defined as limiting cases:

Instead of $\boxed{J_A^\mu(x) = \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)}$ we should use

$$\boxed{J_A^\mu(x) = \lim_{\varepsilon\to 0} \bar{\psi}(x+\tfrac{\varepsilon}{2})\gamma^\mu\gamma^5 \exp\left[-ie\int_{x-\varepsilon/2}^{x+\varepsilon/2} A(z)\,dz\right]\psi(x-\tfrac{\varepsilon}{2})}$$

Gauge invariance requires Wilson line.

If one now calculates, they find $\boxed{\partial_\mu J_\mu^A(x) = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}}$ in 1+1 QED

SIDENOT: One still gets $\partial_\mu J^\mu(x) = 0$. The difference in calculation $\mathrm{Tr}(\gamma^\alpha\gamma^\mu\gamma^5) = 2\epsilon^{\alpha\mu}$ for J_A, $\mathrm{Tr}(\gamma^\alpha\gamma^\mu) \propto g^{\alpha\mu}$ for J_V, and contraction symmetry

3-) Relevant symmetries are NOT that of action but that of Z:

Under $\delta\psi(x) = i\delta\alpha(x)\gamma^5\psi(x)$, $\mathcal{L} \to \mathcal{L} + \delta\alpha(x)\partial_\mu J_A^\mu$

$\delta\bar{\psi}(x) = i\delta\alpha(x)\bar{\psi}(x)\gamma^5$ so $\partial_\mu J_A^\mu = 0$

Fujikawa shows in 1979 that under this transformation

$$Z = \int [\mathcal{D}\psi][\mathcal{D}\bar{\psi}]\exp\left(i\int d^4x\,\mathcal{L}\right) \to Z = \int [\mathcal{D}\psi][\mathcal{D}\bar{\psi}]\exp\left(i\int d^4x\,\mathcal{L}'\right)$$

where $\boxed{\mathcal{L}' = \mathcal{L} + \delta\alpha(x)\left[\partial_\mu J_A^\mu + \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right]}$ (Adler-Bell-Jackiw Anomaly)

for extra contribution coming from $[\mathcal{D}\psi][\mathcal{D}\bar{\psi}] \to [\mathcal{D}\psi][\mathcal{D}\bar{\psi}]\,J^{-2}$

- Last perspective allows us to derive J_A^M in generic $d=2n$:

$$\partial_M J_A^M = (-1)^{n+1} \frac{2e^n}{n!\,(4\pi)^n} \epsilon^{M_1 \cdots M_{2n}} F_{M_1 M_2} \cdots F_{M_{2n-1} M_{2n}}$$

- IN THEIR PAPER, WESS AND ZUMINO ANALYZES EFFECTS OF THESE AXIAL VECTOR FIELD IN A MODEL WITH $SU(3) \times SU(3)$ SYMMETRY
- THEY ARGUE NO LOCAL FUNCTIONAL DEPENDING ONLY ON V^M AND V_A^M CAN SATISFY ANOMALOUS WARD IDENTITIES. HENCE THEY INTRODUCE AN OCTET OF PSEUDOSCALAR Π_i:

Π_i transforms linearly under $SU(3)_{\text{diagonal}}$
" " non-linearly " $SU(3) \times SU(3)$

- WE CAN UNDERSTAND THESE FIELDS AS GOLDSTONE BOSONS OF CHIRAL SSB
- TO UNDERSTAND THESE BETTER, LET US REVIEW CHIRAL SPONTANEOUS SYMMETRY BREAKING

- $\mathcal{L} = \sum_q \bar{q}(i\not{\partial} - m_q)q$, $q \in \{u,d\}$ or $q \in \{u,d,s\}$

- If we ignore m, we have $SU_L(2) \times SU_R(2) \times U_B(1) \times U_A(1)$

$$Q_{L,R} = \left(\frac{1 \mp \gamma^5}{2}\right)\begin{pmatrix} u \\ d \end{pmatrix}$$

$$J^M_{L,R} = \bar{Q}_{L,R}\gamma^M Q_{L,R} \qquad J^{M,a}_{L,R} = \bar{Q}_{L,R}\gamma^M \tau^a Q_{L,R}$$

$$J^M = J^M_R + J^M_L \qquad J^{M,a} = J^{M,a}_R + J^{M,a}_L$$

$$J^M_A = J^M_R - J^M_L \qquad J^{M,a}_A = J^{M,a}_R - J^{M,a}_L$$

- Classically all are conserved: $J^{M,a} \leftrightarrow$ isospin, $J^M \leftrightarrow$ baryon number conservation
- Since no conservation related to J_A observed in nature, Nambu and Lasinio hypothesized that these are spontaneously broken (1961).
- Since we have an attractive channel and quarks are massless (or very light), quantum fluctuations can easily create $\bar{Q}Q$ pairs in vacuum. We get quark pair condensate:

$$\langle 0|\bar{Q}Q|0\rangle \neq 0$$

- As $\bar{Q}Q$ has definite chirality, symmetry spontaneously break:

$$U(2) \times U(2) \to U(2)_{\text{diagonal}}$$
$$U(3) \times U(3) \to U(3)_{\text{diagonal}}$$

- We would expect 4 (9) Goldstone bosons in this naive picture. However Adler-Bell-Jackiw anomaly requires $U_A(1)$ not to be a symmetry to begin with, hence:

$SU(2)_L \times SU(2)_R \times U_B(1) \to SU(2) \times U_B(1)$ (3 goldstone bosons: $\pi^0, \pi^\pm$)

$SU(3)_L \times SU(3)_R \times U_B(1) \to SU(3) \times U_B(1)$ (meson octet: Π_i)

- These pseudoscalars are precisely those considered in WZ Lagrangian.

- LET US NOW SEE WITTEN'S DERIVATION OF THIS LAGRANGIAN

- FRAMEWORK

  - Low energy dynamics can be described by introducing a field $U(x^a)$ that transform in non-linear realization of $SU(3)_L \times SU(3)_R$

    $$U \to A U B^{-1} \quad \text{Under } SU(3)_L \times SU(3)_R$$

    U is an element of $SU(3)$

  - $\mathcal{L}_{eff}$ must have $SU(3)_L \times SU(3)_R$ symmetry.
  - To describe low energy limit correctly, it should have smallest number of derivatives.
  - Unique $\mathcal{L}_{eff}$:

$$\boxed{\mathcal{L} = \frac{1}{16\pi} F_\pi^2 \, \mathrm{Tr}\left(\partial_\mu U \, \partial^\mu U^{-1}\right)}$$

(Experiment: $F_\pi \approx 190$ MeV)

$$\boxed{U = 1 + \frac{2i}{F_\pi} \sum_{a=1}^{8} \lambda^a \pi^a + \text{H.O.T.}}$$

π^a: Goldstone boson fields

λ^a: $SU(3)$ generators

Explicit form: $\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \pi^a)(\partial^\mu \pi^a) + \text{H.O.T.}}$

π^a: Meson octet: $\boxed{\pi^0, \pi^\pm, K^0, \bar{K}^0, K^\pm, \eta}$

— SYMMETRIES OF $\mathcal{L}_{EFF}$

— $U \leftrightarrow U^T$: Charge conjugation

It is easier to see in SU(2) case:

$$(U \leftrightarrow U^T) = \begin{pmatrix} \sigma^1 \leftrightarrow \sigma^1 \\ \sigma^2 \leftrightarrow -\sigma^2 \\ \sigma^3 \leftrightarrow \sigma^3 \end{pmatrix} = \begin{pmatrix} \sigma^{\pm} = \frac{1}{\sqrt{2}}(\sigma^1 \pm i\sigma^2) & \leftrightarrow & \sigma^{\mp} \\ \sigma^3 & \leftrightarrow & \sigma^3 \end{pmatrix} = \begin{pmatrix} \pi^{\pm} \leftrightarrow \pi^{\mp} \\ \pi^0 \leftrightarrow \pi^0 \end{pmatrix}$$

Situation is similar with Gell-Mann matrices

— $x \leftrightarrow -x$: Naive parity $= P_0$
$t \leftrightarrow t$
$U \leftrightarrow V$

— $U \leftrightarrow U^{-1}$: $\pi^a \leftrightarrow -\pi^a$: $(-1)^{N_B}$: Counts modulo two the number of bosons

— PROBLEM : WE HAVE TOO MUCH SYMMETRY

QCD is parity invariant only if goldstone bosons are treated as pseudoscalars: Parity of QCD $= P = P_0 (-1)^{N_B}$

Example: $K^+K^- \to \pi^+\pi^0\pi^-$ is possible in QCD but violates $(-1)^{N_B}$

— SOLUTION: Can we add a term so that $\mathcal{L}_{eff}$ is invariant only under P, not P_0 and $(-1)^{N_B}$ separately?

EOM: $\partial_M \left(\frac{1}{8} F_\pi^2 U^{-1} \partial^M U \right) = 0$

- Since we want P_0 violation, Lorentz invariance forces that new term should have $\epsilon_{\mu\nu\alpha\beta}$.
- Unique term with minimum number of derivatives is:

$$\partial_\mu\left(\frac{1}{8}F_\pi^2 U^{-1}\partial^\mu U\right) + \lambda\,\epsilon^{\mu\nu\alpha\beta}(\partial_\mu U^{-1})(\partial_\nu U)(\partial_\alpha U^{-1})(\partial_\beta U) = 0$$

What is the Lagrangian for extra term?

- PROBLEM: NAIVE 4d PSEDOSCALAR TO BE ADDED TO LAGRANGIAN IS ZERO:

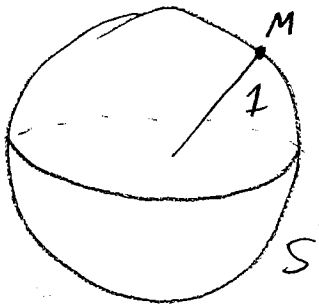
$$\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\left((\partial_\mu U^{-1})(\partial_\nu U)(\partial_\alpha U^{-1})(\partial_\beta U)\right) = 0$$

antisym × cyclic symmetric = 0

- LET US ANALYZE AN ANALOGOUS PROBLEM IN A SIMPLER SETTING

A PARTICLE CONSTRAINED TO MOVE ON A 2-D SPHERE

SIMPLE PROBLEM



constrained

- EOM:

$$\mathcal{L} = \frac{1}{2} m \dot{x}_i^2 + \frac{1}{2}\lambda(x_i^2 - 1)$$

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}$ $q \in \{x_i, \lambda\}$: $x_i^2 - 1 = 0 \Rightarrow x_i \dot{x}_i = 0 \Rightarrow -\ddot{x}_i x_i = \dot{x}_i^2$

$m\ddot{x}_i = \lambda x_i \Rightarrow m\ddot{x}_i x_i = \lambda x_i x_i$

$\Rightarrow \lambda = -m\frac{\dot{x}_i^2}{x_i^2}$

hence $\boxed{m\ddot{x}_i + m x_i(\dot{x}_j^2) = 0}$ as $x_i^2 = 1$

- SYMMETRIES : $\boxed{x_i \to -x_i \,,\; t \to -t}$

- Assume we only want $\boxed{\{x_i, t\} \to \{-x_i, -t\}}$

- a simple solution: $\boxed{m\ddot{x}_i + m x_i(\dot{x}_k^2) = \alpha\, \epsilon_{ilm} \dot{x}_l x_m}$

$\boxed{\alpha = q_e q_\beta}$

- how do we generate RHS via a Lagrangian?

- Lorentz force: $\boxed{F_i = q_e E_i + q_e \epsilon_{ilm} \dot{x}_l B_m}$

Magnetic monopole: $B_i = \frac{q_\beta x_i}{(x^2)^{3/2}} = q_\beta x_i$ on S^2

$\Rightarrow \boxed{F_i = q_e q_\beta \epsilon_{ilm} \dot{x}_l x_m}$

FORCE FOR AN e-charge interacting with magnetic monopole

- CLAIM: $\boxed{\mathcal{L} = \frac{1}{2} m \dot{x}_i^2 + q_e A_i \dot{x}_i + \frac{1}{2}\lambda(x_i^2 - 1)}$

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_k}\right) = m\ddot{x}_k + q_e \frac{dA_k}{dt} = m\ddot{x}_k + q_e \dot{x}_l \partial_l A_k$

$\frac{\partial \mathcal{L}}{\partial x_k} = q_e \dot{x}_l \partial_k A_l + \lambda x_l$

$\Rightarrow m\ddot{x}_k - \lambda x_l = q_e \dot{x}_l (\partial_k A_l - \partial_l A_k)$

$= q_e \dot{x}_l \epsilon_{klm} B_m$

$= q_e q_\beta \epsilon_{klm} \dot{x}_l x_m$

– One can show that λ is same!

—PATH INTEGRAL FOR THE ACTION:

$$Z = \int dx_i(t)\, e^{i\int \mathcal{L}(x(t),A)\,dt} \supset \exp\left(iq_e \int A_i \dot{x}_i\, dt\right) = e^{iq_e \oint_\gamma A_i\, dx^i}$$

γ: closed orbit for simplest case

$D' := S^2 - D$

Stoke's theorem: $\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$

$$iq_e \int_\gamma \underbrace{A_i dx^i}_{\text{1-form}} = iq_e \int_\gamma A \quad \nearrow \; iq_e \int_{\partial D} A = iq_e \int_D dA$$

$$\searrow \; iq_e \int_{\partial D'} -A = -iq_e \int_{D'} dA$$

Consistency forces $\exp\left(iq_e \int_D dA\right) = \exp\left(-iq_e \int_{D'} dA\right)$

—Dirac quantization: $iq_e \int_{D+D'} dA = 2\pi n \to q_e \int_{\partial(S^2)} F = 2\pi n$

$$iq_e \int_{S^2} F = 2\pi n \to iq_e (4\pi q_B) = 2\pi n \Rightarrow \boxed{q_e q_B = \frac{n}{2},\ n \in \mathbb{Z}}$$

NOTE: Using $\int_{\partial(S^2)} F = \int_{S^2} dF$ is just a mathematical trick. Physically

$dF = 0$, $d{*F} = {*J}$, and F is not defined on origin!

— BACK TO ORIGINAL PROBLEM: WITTEN ARGUES THAT THE SITUATION WITH WZ MODEL IS SIMILAR

U: Mapping from E^4 to SU(3)

$\exists$ S^4 in SU(3) defined by U

This S^4 can be seen as boundary of 5d disc Q

$\therefore$ just like $\int_\gamma A_i dx^i = \int_D F_{ij} d\Sigma^{ij}$, we can find ω

$$\Gamma = \int_Q \omega_{ijklm} \, d\Sigma^{ijklm}$$

— JUST LIKE DIRAC QUANTIZATION, WE HAVE TWO CHOICES, Q and Q', WHICH ADD UPTO FIVE-SPHERE

$$\int_S \omega_{ijklm} \, d\Sigma^{ijklm} = 2\pi \cdot \text{INTEGER}$$

S: ANY FIVE SPHERE IN SU(3) MANIFOLD

— IF WE NORMALIZE ω OVER BASIC FIVE SPHERE S_0

$$\int_{S_0} \omega_{ijklm} \, d\Sigma^{ijklm} = 2\pi$$

THEN WE CAN WORK WITH THE ACTION

$$I = \frac{1}{16} F_\pi^2 \int d^4x \; \mathrm{Tr}\left(\partial_\mu U \partial^\mu U^{-1}\right) + n\Gamma \qquad n \in \mathbb{Z}$$

$$\Gamma = \int_Q \omega_{ijklm} \, d\Sigma^{ijklm} \; : \text{Wess-Zumino Lagrangian}$$

- CONCLUSION: WITTEN REDERIVES WZ LAGRANGIAN AND SHOWS THAT IT HAS TO BE QUANTIZED

- DETAILS REGARDING THE RESULT:

$$\omega_{ijklm} = -\frac{i}{240\pi^2} \mathrm{Tr}\left[(\partial_i U^{-1})(\partial_j U)(\partial_k U^{-1})(\partial_l U)(\partial_m U^{-1})U\right]$$

$$\Gamma = \frac{2}{15\pi^2 F_\pi^5} \int d^4x \; \varepsilon^{\mu\nu\alpha\beta} \mathrm{Tr}\left(A\partial_\mu A \partial_\nu A \partial_\alpha A \partial_\beta A\right) + \text{H.O.T.}$$

$$A = \sum \lambda^a \pi^a$$

In a hypothetical world of massless kaons and pions, this $\mathcal{L}_{eff}$ can describe low energy limits of $K^+K^- \to \pi^+\pi^0\pi^-$ which breaks $(-1)^{N_B}$ but is invariant under P.

By coupling this $\mathcal{L}_{eff}$ to EM, Witten shows that n in $n\Gamma$ should be number of colors: $n = N_c$

ADDITIONAL DETAILS IF TIME PERMITS

— THAT WE CAN FIND A 5d DISC Q AND WRITE $\Gamma = \int_Q \omega_{ijklm} d\Sigma^{ijklm}$ IS NONTRIVIAL

☆ This follows from the fact that fourth homotopy group of SU(3) is trivial, that is $\boxed{\pi_4(SU(3)) = 0}$

☆ The situation was similar for Dirac quantization, there we used $\boxed{\pi_1(S^2) = 0}$

— K'TH HOMOTOPY GROUP OF MANIFOLD M: $\boxed{\pi_k(M)}$

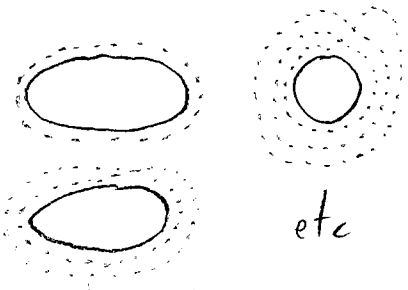
Definition is <u>roughly</u> as follows:

- consider all different mappings from S^k to M
- construct equivalence classes of mapping with respect to their topologies (whether they contract to a point, how they wrap, etc.
- construct the group of these equivalence classes
- Two mappings are <u>homotopic</u> if continuously deformed to one another

— <u>EXAMPLES:</u>

: All circles contract to zero, only 1 equivalence class

$\boxed{\pi_1(S^2) = 0}$



etc

: any integer number of winding

$\pi_1(S^1) = \mathbb{Z}$ infinite cyclic group

- No idea how $\pi_4(SU(3)) = 0$. However

$\pi_i(S^j) = 0$ for $j > i$

$SU(3) \sim S^3$ bundle over S^5

- COMPARISON OF DIRAC QUANTIZATION AND WESS-ZUMINO-WITTEN QUANTIZATION

Dirac's case: Constraint manifold $= S^2$
path on " " $= \gamma = S_1$
} $\pi_1(S^2) = 0 \Rightarrow \exists$ Discs D & D'

D & D' add upto $S^2 \Rightarrow \pi_2(S^2) = \mathbb{Z} \Rightarrow 2 q_e q_m$ should be integer

WZW: U: non linear sigma model field mapping $\mathbb{E}^4$ to $SU(3)$

Constraint manifold $= SU(3)$
path on " " $= S^4$
} $\pi_4(SU(3)) = 0 \Rightarrow \exists$ Discs Q & Q'

Q & Q' add upto $S^5 \Rightarrow \pi_5(SU(3)) = \mathbb{Z} \Rightarrow n\Gamma$

— WITTEN ANOMALY : WITTEN 1982

SU(2) GAUGE THEORY OF LEFT HANDED WEYL FERMIONS

QUESTION: HOW COULD FERMIONS REMAIN MASSLESS IN THE PRESENCE OF STRONG SU(2) GAUGE FORCES AT LONG DISTANCES?

PARTIAL ANSWER: EVEN NUMBER OF DOUBLETS CAN PAIR UP, BECOME DIRAC FERMIONS, AND GET MASS VIA CHIRAL SSB.

PUZZLE: HOW ABOUT ODD NUMBER OF DOUBLETS?

FULL ANSWER: SU(2) GAUGE THEORIES WITH ODD NUMBER OF ELEMENTARY DOUBLETS ARE MATHEMATICALLY INCONSISTENCE! (OF TOPOLOGICAL NATURE)

— SOURCE OF INCONSISTENCY:

- Analyze group manifold of SU(2).

2×2 complex matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ → Since unitary columns should be orthogonal $\begin{pmatrix} a & -b^+ \\ b & a^+ \end{pmatrix}$ → Since special determinat should be 1 $\det\begin{pmatrix} a & -b^+ \\ b & a^+ \end{pmatrix} = 1 \Rightarrow |a|^2 + |b|^2 = 1$

$SU(2) \sim S^3 \leftarrow a_R^2 + a_I^2 + b_R^2 + b_I^2 = 1 = S^3$

— In 4d, we are mapping from E^4 to $SU(2)$, hence we should check: $\boxed{\pi_4(S^3) = \mathbb{Z}_2}$. This means

$\exists\ U(x) \to 1$ as $|x| \to \infty$ which wraps around gauge group such that it can NOT be continuously deformed back to identity.

As $\pi_4(S^3) = \mathbb{Z}_2$, it can be " " " " " if wraps twice.

— RESULT: A_μ and $\boxed{A_\mu^U = U^{-1}(A_\mu - i\partial_\mu)U}$ contributes same!

$\int [DA] \exp\left[-\frac{1}{2g^2}\int d^4x\ tr(F^2)\right]$ double counts!

— can NOT be eliminated!

— its effects cancel in VEV calculations, in the absence of fermions!

— PROBLEM IN THE PRESENCE OF FERMIONS:

$$\boxed{Z = \int [D\psi][D\bar{\psi}][dA] \exp\left(-\int d^4x \left(\frac{1}{2g^2} tr(F^2) + \bar{\psi} i\not{D}\psi\right)\right)}$$

single doublet of left handed Weyl fermion

— We want to integrate out fermions and analyze resultant EFT

We know

$$\boxed{\int [D\psi]_{Dirac}[D\bar{\psi}]_{Dirac} \exp\left(\bar{\psi} i\not{D}\psi\right) = \det i\not{D}}$$

- As SU(2) Dirac Doublet = Two left handed Weyl Doublet:

$$\int [\not{D}\psi]_{Weyl} [\not{D}\bar{\psi}]_{Weyl} \exp(\bar{\psi} i \not{D} \psi) = \underbrace{\mp}_{\text{free to choose}} (\det i\not{D})^{1/2}$$

- PROBLEM: $(\det i\not{D}(A_\mu))^{1/2} = -(\det i\not{D}(A^U_\mu))^{1/2}$

Therefore

$$Z_{eff} = \int [DA] (\det i\not{D})^{1/2} \exp\left(-\frac{1}{2g^2}\int d^4x \, tr(F^2)\right) = 0$$

This is true with insertions as well, hence

$$\langle X \rangle = \frac{Z_X}{Z} = \frac{0}{0} : \text{Theory is ill defined!}$$

- CONCLUSION: The illness of theory is
  - present for any n=odd left-handed fermion doublets
  - present regardless of additional gauge or Yukawa couplings
- STANDARD MODEL WITH SU(3)×SU(2)×U(1) SYMMETRY WOULD BE INCONSISTENT IF NUMBER OF FERMION DOUBLETS WERE ODD!

ASIDE: WEAK INTERACTION GAUGE THEORY CAN BE CONSISTENTLY COMBINED WITH QED ONLY IF THEORY CONTAINS EQUAL NUMBER OF QUARK AND LEPTON DOUBLETS! OTHERWISE THERE WOULD BE ANOMALIES! (PESKIN, CHAPTER 19, P680)

$\Downarrow$

AXIAL VECTOR ANOMALY MATCHING ENSURES WE WILL NOT GET WITTEN ANOMALY!

ASIDE: WHAT HAPPENS IF WE HAVE GAUGE ANOMALY?

- Divergent gauge boson masses will be generated
- Unphysical states may not cancel: We lose unitarity of S-matrix

Gross, Jackiw, PRD6 477 (1972): One needs to insist that

$$\boxed{A^{abc} = \mathrm{tr}\left(t^a\{t^b, t^c\}\right) = 0}$$

in chiral gauge theories.

DERIVATION OF EOM FOR (1) of WITTEN '83

$$\boxed{\mathcal{L} = \frac{1}{16} F_\pi^2 \, \mathrm{Tr}\left(\partial_\mu U \, \partial^\mu U^{-1}\right)}$$

$$UU^{-1} = 1 \Rightarrow \partial_\mu (UU^{-1}) = 0 \Rightarrow (\partial_\mu U) U^{-1} + U \partial_\mu U^{-1} = 0 \Rightarrow \boxed{\partial_\mu U^{-1} = -U^{-1} (\partial_\mu U) U^{-1}}$$

$$UU^{-1} = 1 \Rightarrow \frac{\delta (U_{ab} U^{-1}_{bc})}{\delta U_{ij}} = 0 \Rightarrow \delta_{ai}\delta_{bj} U^{-1}_{bc} + U_{ab} \frac{\delta U^{-1}_{bc}}{\delta U_{ij}} = 0$$

$$U^{-1}_{ki} \delta_{bj} U^{-1}_{bc} + \delta_{kb} \frac{\delta U^{-1}_{bc}}{\delta U_{ij}} \Rightarrow \boxed{\frac{\delta U^{-1}_{ab}}{\delta U_{ij}} = -U^{-1}_{ai} U^{-1}_{jb}}$$

$$\mathcal{L} = -\frac{1}{16} F_\pi^2 \, \mathrm{Tr}\left((\partial U) U^{-1} (\partial U) U^{-1}\right) \sim (U^{-1}\partial U)_{ab} (U^{-1}\partial U)_{ba}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu U_{ij})} = (U^{-1}\partial U)_{ab} (U^{-1})_{bc} \delta_{ci}\delta_{ja} + (U^{-1})_{ac}\delta_{ci}\delta_{jb}(U^{-1}\partial U)_{ba}$$

$$= 2\left[U^{-1}(\partial U) U^{-1}\right]_{ji}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial U_{ij})} = (U^{-1}\partial U)_{ab} \frac{\delta U^{-1}_{bc}}{\delta U_{ij}} (\partial U)_{ca} + \frac{\delta U^{-1}_{ac}}{\delta U_{ij}} (\partial U)_{cb} (U^{-1}\partial U)_{ba}$$

$$= -(\partial U) U^{-1} \partial U)_{cb} U^{-1}_{bi} U^{-1}_{jc} - U^{-1}_{ai} U^{-1}_{jc} ((\partial U) U^{-1} \partial U)_{ca}$$

$$= -2\left[U^{-1}(\partial U) U^{-1} (\partial U) U^{-1}\right]_{ji}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu U_{ij})} \right) - \frac{\partial \mathcal{L}}{\partial U_{ij}} = 0$$

$$\partial_\mu \left[U^{-1}(\partial U) U^{-1} \right] + \left[U^{-1}(\partial U) U^{-1}(\partial U) U^{-1} \right] = 0$$

$$\left[\partial_\mu (U^{-1}\partial U)\right] U^{-1} + \left[U^{-1}(\partial U)(\partial U^{-1}) \right] + \left[U^{-1}(\partial U) U^{-1}(\partial U) U^{-1} \right] = 0$$

$$\partial U^{-1} = -U^{-1}(\partial U) U^{-1}$$

$$\left(\partial_\mu [U^{-1}\partial U]\right) U^{-1} = 0 \Rightarrow \boxed{\partial_\mu (U^{-1}\partial_\mu U) = 0}$$