Supplement to Appendix B: The Firm's Innovation Policy

Denote the optimal innovation policy of the firm as $I(n)$. It is convenient to express the optimal policy in terms of the optimal innovation intensity $\lambda(n) = I(n)/n$. The value of a firm with n products as of date 0 is thus

$$V(n) = E\left[\int_0^\infty e^{-rt}[\overline{\pi} - c(\lambda(N_t))]N_t dt \mid N_0 = n\right].$$

The stochastic process $\{N_t\}$ for the size of the firm then evolves on the integers, starting at $N_0 = n$, in the following manner. From some state m there is a Poisson hazard $\lambda(m)m$ of transiting to state $m+1$ and an independent Poisson hazard μm of transiting to state $m-1$. The current flow return to the firm in state m is $[\overline{\pi} - c(\lambda(m))]m$.

Now consider stochastic processes $\left\{N_t^k\right\}$ for $k = 1, 2, \ldots, n$. Suppose at date t that $N_t^1 = m^1, N_t^2 = m^2, \ldots, N_t^n = m^n$. If we think of process k as the size of the kth firm, then there is a Poisson hazard $\lambda(m^k)m^k$ of this process transiting to state m^k+1 and an independent Poisson hazard μm^k of it transiting to state $m^k - 1$. The processes of n such firms will evolve independently of each other. The current flow return to the kth firm in state m^k will be $[\overline{\pi} - c(\lambda(m^k))]m^k$.

Alternatively, suppose we think of $\sum_{k=1}^{n} N_t^k$ as the size of one firm at date t, and define $m = \sum_{k=1}^n m^k$. Under the control of a single firm, this sum will evolve as described earlier, that is, from some state m there is a Poisson hazard $\lambda(m)m$ of transiting to state $m+1$ and an independent Poisson hazard μm of transiting to state $m-1$. We can also imagine this one firm controlling each of the individual processes $\left\{N_t^k\right\}$ separately. From state m^k there is a Poisson hazard $\lambda(m)m^k$ for process k to transit to state m^k+1 and a hazard μm^k of it transiting to state $m^k - 1$. By the summability of Poisson hazards, it is clear that it makes no difference whether the firm controls the the n individual processes or just their sum. Similarly the flow return is the same since $\sum_{k=1}^n[\overline{\pi} - c(\lambda(m))]m^k = [\overline{\pi} - c(\lambda(m))]\sum_{k=1}^n m^k = [\overline{\pi} - c(\lambda(m))]m$.

We want to use these results to derive properties of the optimal policy and the value function. First we show that a size n firm does at least as well as if it followed the policies of n separate size 1 firms. Hence, $V(n) \geq nV(1)$:

$$\begin{aligned}
V(n) &= E\left[\int_0^\infty e^{-rt}[\overline{\pi} - c(\lambda(\sum_{k=1}^n N_t^k))]\left(\sum_{k=1}^n N_t^k\right) dt \mid N_0^1 = 1, N_0^2 = 1, \ldots, N_0^n = 1\right] \\
&= E\left[\sum_{k=1}^n \left(\int_0^\infty e^{-rt}[\overline{\pi} - c(\lambda(\sum_{k'=1}^n N_t^{k'}))]N_t^k dt\right) \mid N_0^1 = 1, N_0^2 = 1, \ldots, N_0^n = 1\right] \\
&= \sum_{k=1}^n E\left[\int_0^\infty e^{-rt}[\overline{\pi} - c(\lambda(\sum_{k'=1}^n N_t^{k'}))]N_t^k dt \mid N_0^1 = 1, N_0^2 = 1, \ldots, N_0^n = 1\right] \\
&\geq \sum_{k=1}^n E\left[\int_0^\infty e^{-rt}[\overline{\pi} - c(\lambda(N_t^k))]N_t^k dt \mid N_0^1 = 1, N_0^2 = 1, \ldots, N_0^n = 1\right] \\
&= \sum_{k=1}^n E\left[\int_0^\infty e^{-rt}[\overline{\pi} - c(\lambda(N_t^k))]N_t^k dt \mid N_0^k = 1\right] \\
&= \sum_{k=1}^n V(1) \\
&= nV(1).
\end{aligned}$$

Next we show that n size 1 firms do at least as well as if they joined together and followed the policy of one size n firm. Hence $nV(1) \geq V(n)$:

$$\begin{aligned}
nV(1) &= \sum_{k=1}^n E\left[\int_0^\infty e^{-rt}[\overline{\pi} - c(\lambda(N_t^k))]N_t^k dt \mid N_0^k = 1\right] \\
&= \sum_{k=1}^n E\left[\int_0^\infty e^{-rt}[\overline{\pi} - c(\lambda(N_t^k))]N_t^k dt \mid N_0^1 = 1, N_0^2 = 1, \ldots, N_0^n = 1\right] \\
&= E\left[\sum_{k=1}^n \left(\int_0^\infty e^{-rt}[\overline{\pi} - c(\lambda(N_t^k))]N_t^k dt\right) \mid N_0^1 = 1, N_0^2 = 1, \ldots, N_0^n = 1\right] \\
&\geq E\left[\sum_{k=1}^n \left(\int_0^\infty e^{-rt}[\overline{\pi} - c(\lambda(\sum_{k'=1}^n N_t^{k'}))]N_t^k dt\right) \mid N_0^1 = 1, N_0^2 = 1, \ldots, N_0^n = 1\right] \\
&= E\left[\int_0^\infty e^{-rt}[\overline{\pi} - c(\lambda(\sum_{k=1}^n N_t^k))]\left(\sum_{k=1}^n N_t^k\right) dt \mid N_0^1 = 1, N_0^2 = 1, \ldots, N_0^n = 1\right] \\
&= V(n)
\end{aligned}$$

For both inequalities to hold it must be that $V(n) = nV(1)$. It follows that $\lambda(n)$ is a constant and so the optimal policy is $I(n) = \lambda n$.