Stochastic Search Equilibrium[1]

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We study equilibrium wage and employment dynamics in a class of popular search models with wage posting, in the presence of aggregate productivity shocks. Firms offer and commit to (Markov) contracts, which specify a wage contingent on all payoff-relevant states, but must pay equally all of their workers, who have limited commitment and are free to quit at any time. We find sufficient conditions for the existence and uniqueness of a stochastic search equilibrium in such contracts, which is Rank Preserving [RP]: larger and more productive firms offer more generous contracts to their workers in all states of the world. On the RP equilibrium path, turnover is always efficient as workers always move from less to more productive firms. The resulting stochastic dynamics of firm size provide an intuitive explanation for the empirical finding that large employers have more cyclical job creation (Moscarini and Postel-Vinay, 2012). Finally, computation of RP equilibrium contracts is tractable.

Keywords: Equilibrium Job Search, Dynamic Contracts, Stochastic Dynamics.

1. INTRODUCTION

The continuous reallocation of employment across firms, sectors and occupations, mediated by various kinds of frictions, is a powerful source of aggregate productivity growth.[1] Workers move in response to various reallocative shocks, and search on and off the job to take advantage of the large wage dispersion that they face. A popular class of search wage-posting models, originating with Burdett and Mortensen (1998, henceforth BM), aims to understand these phenomena. The BM model provides a coherent formalization of the hypothesis that cross-sectional wage dispersion is a consequence of labor market frictions, and started a fruitful line of research in the analysis of wage inequality and worker turnover, as the vibrant and empirically very successful literature building on that hypothesis continues to show (see Mortensen, 2003 for an overview). When allowing for heterogeneity in firm-level TFP, the BM model is a natural framework to study employment reallocation across firms.

This job search literature, however, is invariably cast in deterministic steady state. Ever since the first formulation of the BM model, job search scholars have regarded the

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1. See Foster, Haltiwanger and Krizan (2000) and Lentz and Mortensen (2008) for recent evidence.

characterization of its out-of-steady-state behavior as a daunting problem, essentially because one of the model's state variables, which is also the main object of interest, is the endogenous *distribution* of wage offers. This is an infinite-dimensional object, endogenously determined in equilibrium as the distribution across firms of offer strategies that are mutual best responses, which evolves stochastically with the aggregate impulse.

The restriction to steady state analysis is not costless. The ongoing reallocation of employment across firms has a cyclical pattern. Moscarini and Postel-Vinay (2012, henceforth MPV12) document that the net job creation of larger, higher-paying firms is more positively correlated with GDP, and more negatively with the unemployment rate, than at smaller firms, at business cycle frequencies. Essentially, the firm size/growth relationship "tilts" up and down with the business cycle.[2] Any theory of turnover and wage dispersion based on frictional worker reallocation among firms, and allowing for aggregate dynamics, speaks directly to these facts.

In this paper, we provide the first analysis of aggregate stochastic dynamics in wage-posting models with random search. We study a frictional labor market where firms offer and commit to employment contracts, workers search randomly on and off the job for those contracts, while aggregate productivity is subject to persistent shocks. In our economy, both in the constrained efficient allocation and in equilibrium, smaller firms contribute relatively more to net job creation when unemployment is high, consistently with MPV12's observations.

Our key contribution overcomes the technical hurdle that stunted progress of the job search literature beyond steady state analysis. We find sufficient conditions for a unique equilibrium, in which the distribution of wage contracts is easy to track: the workers' ranking of firms is the same in all aggregate states — what we call a *Rank-Preserving Equilibrium* (RPE). The sufficient conditions are simple. If firms are equally productive, no further restrictions are needed, and the unique equilibrium is RP and features, as in BM, dispersion in contracts and firm size. On the RPE path, initially larger firms always offer more and remain larger. If all firms have the same initial size, they randomize on the first offer, then diverge in size and we are back to the previous case. If firms differ in the permanent component of their productivity, then a sufficient (but not necessary) condition for the unique equilibrium to be a RPE is a restriction on initial conditions: more productive firms are initially (weakly) larger — for example, all firms start empty. More productive firms then offer a larger value and employ more workers at all points in time. Given a chance, a worker always moves from a less into a more productive firm, so that equilibrium reallocation of employment is constrained efficient. This parallels a similar property of BM's static equilibrium.

In our economy, infinitely lived and risk neutral firms and workers come in contact infrequently. Firms produce homogenous output with labor in a linear technology, which may permanently differ across firms. Aggregate multiplicative TFP shocks affect labor productivity as well as the job contact rates, on and off the job, the exogenous job destruction rate, and the value of leisure. A social planner constrained by search frictions and given job contact rates, when given the opportunity, moves an employed worker from a less productive to a more productive firm. This efficient turnover gives rise to a simple process for the evolution of the firm size distribution, which can be solved for analytically,

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2. Haltiwanger, Jarmin and Miranda (2010) present the most comprehensive study to date on the firm size/growth relationship, based on the full longitudinal census of US employers (Longitudinal Business Database, 1976-2005), the same data underlying MPV12's evidence. They find that a firm's growth is negatively related to its size, much less so when controlling for mean reversion, and not at all when controlling for firm age. They do not address business cycle patterns.

given any history of aggregate shocks. The solution replicates the MPV12 facts: larger firms grow relatively faster when aggregate TFP is high. If we shut down aggregate shocks, this process converges deterministically to BM's stationary size distribution.

To study equilibrium, we assume that firms offer and commit to a Markov contract, where the wage is allowed to depend on all four payoff-relevant states: two exogenous, firm-specific and aggregate productivity, one endogenous to the firm, its current size, and one endogenous to the economy but exogenous to the firm, the distribution of employment across all firms. We impose only one further restriction, in order to obtain a well-defined notion of a firm. Following BM, we define a firm as a wage policy, thus impose an *equal-treatment constraint*: the firm must pay the same wage in a given period to all of its employees, whether incumbent, newly hired from unemployment or from employment. Workers cannot commit not to quit to other jobs when the opportunity arises, or to unemployment whenever they please, so commitment is one-sided and firms face a standard moral hazard problem. We establish that at most one Markov contract-posting equilibrium exists, characterize it, and show that it decentralizes the constrained efficient allocation, thus is consistent with MPV12's evidence.

We then extend our analysis to allow for endogenous contact rates. We allow firms to post vacancies, at a convex cost, to meet job-seekers through a standard matching function. We prove that equilibrium turnover is RP for two reasons: more productive, larger firms both spend more effort to contact workers and offer more to each worker they contact. As before, workers always move up the productivity ladder, although not necessarily at the constrained efficient speed. We present an algorithm to compute RPE allocation and contracts.

Key to our analysis is the following comparative dynamics property of the best response contract offer: at any node in the game and for any distribution of offers made by other firms and values earned by employed workers, the more productive and/or larger a firm, the more generous the continuation value of the contract it offers to its existing and new workers. Therefore, if firms are homogeneous, or if more productive firms are initially no smaller, then no firm wants to break ranks in the distribution of competing offers, which then coincides with the given distribution of firm productivity or initial size. This immediately implies our main result that equilibrium, if it exists, is unique, and is also RP, thus constrained efficient.

The intuition behind this comparative dynamics property parallels and extends a single-crossing property of the static BM model. There, a more productive firm gains more from employing a worker, hence wants to (and can) pay a higher wage. In addition, under the equal treatment constraint, backloading incentives (i.e. reducing the current wage in exchange for a higher continuation value, while maintaining the current promised value) has several effects on profits. First, it increases hires and their future labor costs, both effects independent of current size. Second, it increases retention, proportionally to current size. Third, it shifts the labor cost of existing workers to the future: promising a higher continuation value to those workers among the existing workforce who will still be around tomorrow to receive the higher value allows the firm to cut today's wage, while still maintaining incentives. Both effects are proportional to initial size, and we show that they exactly cancel out. Thus, on net, the marginal effect of backloading on profits is increasing in size, through the retention effect only. As a consequence, ceteris paribus, a larger firm also wants to pay more from tomorrow on. The incentive effect of a firm's size on its wage offers has been, so far, overlooked in the literature following BM, because there, size is pinned down by the steady-state requirement. This effect clearly emerges in our dynamic stochastic setting, where firm size is an evolving state variable, with given

initial conditions.

For the same reason, the monotonic relationship between size, productivity and wages that holds on the equilibrium path in steady state does not immediately extend to the dynamic case. It does, provided that weakly more productive firms are initially weakly larger, because the initial size ranking self-perpetuates, so that more productive firms always pay and employ more. This intuitive and natural outcome is unique despite the strategic complementarity of a wage-posting game.

The restriction on initial conditions derives from the size-dependence of equilibrium contracts. As we discuss in Section 7, size-dependence limits the scope of our analysis in terms of firms' entry and idiosyncratic productivity shocks. Making contracts size-independent requires either relaxing equal treatment, which is a special case of our analysis, with similar results, but losing any notion of firm size — so that the model can no longer address our motivating empirical evidence from MPV12 — or assuming a specific functional form of the matching technology. We discuss these alternatives in detail.

As a by-product of our analysis, we offer a methodological contribution. We formulate the first (to the best of our knowledge) theory of Monotone Comparative Dynamics in a dynamic stochastic decision problem. In our setting, firms solve a fully dynamic problem in a changing environment, and make choices over an infinite sequence (a stochastic process). In the sequential formulation of this problem, the objective function of the one-step Bellman maximization contains the value function of the problem, whose properties are ex ante unknown. We show how the optimal policy changes with a parameter of the model (firm productivity) which affects initial conditions and current payoffs, but could also affect the law of motion of the state variables, a property not yet addressed by the theory of Monotone Comparative Statics (Topkis, 1998).[3]

The rest of the paper is organized as follows. In Section 2 we place our contribution in the context of the relevant literature. In Section 3 we lay out the basic environment. In Section 4 we characterize the constrained efficient allocation. In Section 5 we describe and formally define an equilibrium, introduce the notion of Rank Preserving Equilibrium, characterize RPE contracts, and present uniqueness and existence results. In Section 6 we extend the model to allow for endogenous hiring effort by firms, and establish again our main equilibrium characterization result, stating that every Markov equilibrium must be RP. In Section 7 we revisit our assumptions and discuss the robustness and interpretation of our results. Section 8 concludes and describes future research.

2. RELATED LITERATURE

Besides its intrinsic theoretical interest, our characterization of the dynamics of the BM model opens the analysis of aggregate labor market dynamics as a whole potential new field of application of search/wage-posting models, such as explaining the evidence in MPV12. More generally, we hope to contribute to a synthesis between the BM contract-posting approach and the "other", equally successful side of the search literature,

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3. The literature tackled the different question whether the optimal policy is monotonic in a state variable (Stokey and Lucas, 1989), sometimes using our same tools (Gonzalez and Shi, 2010). The logic of our comparative dynamics argument can be applied in many other settings. For example, one may ask in a stochastic growth model whether the socially optimal level of investment is decreasing at all states and dates in the initial stock of capital, in the labor share, or in risk aversion; in each case, one must track the effect of the parameter change on the endogenous state variable (capital) along the entire optimal path. In this analogy, the existing literature investigates instead when is investment monotonic in the current level of capital.

organized around the matching framework (Pissarides, 1990; Mortensen and Pissarides, 1994), initially designed for the understanding of labor market flows and equilibrium unemployment.

The analysis of equilibrium wage and employment dynamics in equilibrium search models with wage dispersion has recently become a subject of keen investigation. As a stepping stone to the present paper, Moscarini and Postel-Vinay (2009, MPV09) and its discussion by Shimer (2009) study the deterministic transitional dynamics of the BM model. The main result is that the allocation of BM's steady state solution is globally stable: under relatively weak conditions, a BM economy converges asymptotically to the stationary distribution of sizes and wages. The present paper extends the analysis to stochastic dynamics under aggregate uncertainty. This requires a conceptual and technical step, because the entire distribution of wage offers is a state variable that can no longer be simply summarized by calendar time.

Rudanko (2011) and Menzio and Shi (2011) analyze wage contract-posting models with aggregate productivity shocks, where job search is directed. This assumption greatly simplifies the analysis by severing the link between the individual firm's contract-posting problem and the distribution of contract offers. This is the main hurdle that we face, and that we resolve by exploiting the emergence of Rank-Preserving Equilibria, while maintaining BM's assumption of random search common to the majority of the search literature. While we see both programs as fruitful directions of theoretical exploration, from a quantitative viewpoint the directed search approach is focused on the response of the job-finding rate to aggregate shocks, and does not generate a well-defined notion of employer size. Hence, it does not speak to MPV12's facts, that we envision as central to our understanding of the propagation of aggregate shocks in labor markets. Kaas and Kircher (2011) extend Menzio and Shi's model, to allow for firm size. They obtain interesting and empirically accurate predictions on firm growth, pay and recruitment strategies, but do not allow for on-the-job search and do not address MPV12's business cycle facts.

Robin (2011) introduces aggregate productivity shocks in Postel-Vinay and Robin (2002)'s sequential auction model. Lack of commitment to offers, and renegotiation on receipt of an outside job offer, make the distribution of wages not a relevant state variable for equilibrium, and the framework quite tractable for business cycle analysis. Again, this model has no natural definition of firm size.

Finally, Coles and Mortensen (2011, henceforth CM) build on our notion of RPE to characterize the dynamic equilibrium of Coles' (2001) version of the BM model, which closely resembles ours, except that firms cannot explicitly commit to wage contracts, but do so through reputation. They endogenize firms' hiring behavior using a standard matching-function approach, only specifying a firm's recruitment cost *per current employee* as a function of new hires per current employee.[4] This assumption makes a firm's wage policy size-independent and guarantees a unique equilibrium within the RP class. In our setup, the RPE is unique among all (not just RP) equilibria. If firms are identical, this result always holds true and our analysis nests that of CM. When firms differ by productivity, to establish RPE we impose an additional restriction on initial conditions. CM's ingenious, albeit still knife-edge, assumption on the hiring technology allows them to dispose of initial restrictions and to accommodate firm entry anywhere in the productivity distribution, as well as specific idiosyncratic productivity processes.

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4. Formally, if a firm employs L workers and looks at hiring H new workers, it will incur a total cost of $Lc(H/L)$ where $c(\cdot)$ is an increasing and strictly convex function.

Because they only study deterministic transitional dynamics, their model cannot be confronted yet with MPV12's business cycle facts.

3. THE ECONOMY

Time $t = 0, 1, 2 \ldots$ is discrete. The labor market is populated by a unit-mass of workers, who can be either employed or unemployed, and by a unit measure of firms.[5] Workers and firms are risk neutral, infinitely lived, and maximize payoffs discounted with common factor $\beta \in (0, 1)$. Firms operate constant-return technologies with labor as the only input and with productivity scale $\omega_t p$, where $\omega_t \in \Omega$ is an aggregate component, evolving according to a stationary first-order Markov process $Q(d\omega_{t+1} \mid \omega_t)$, and p is a fixed, firm-specific component, distributed across firms p according to a c.d.f. Γ over some positive interval $[\underline{p}, \overline{p}]$.

The labor market is affected by search frictions in that unemployed workers can only sample job offers sequentially with some probability $\lambda_t \in (0, 1]$ at time t, and while searching enjoy a value of leisure b_t. Employed workers earn a wage and also sample job offers with probability $s\lambda_t \in (0, 1]$ each period, so that s is the search intensity of employed relative to unemployed job seekers. Workers can only send one job application, hence can never receive more than one offer per time period. All firms of equal productivity p start out with the same labor force. We denote by $N_t : [\underline{p}, \overline{p}] \to [0, 1]$ the cumulated population distribution of employment across firm types. So $N_0(p)$ is the (given) initial measure of employment at firms of productivity *at most* p, $N_t(\overline{p})$ is employment and $u_t = 1 - N_t(\overline{p})$ the unemployment (rate) at time t. Each employed worker is separated from his employer and enters unemployment every period with probability $\delta_t \in (0, 1]$. We maintain throughout the assumption that the destruction rate is exogenous and a function of the aggregate productivity state $\delta_t = \delta(\omega_t)$. Similarly for the flow value of non production $b_t = b(\omega_t)$. For much of the analysis we also assume that the job contact rate is an exogenous function of the aggregate state $\lambda_t = \lambda(\omega_t)$, and sampling of firms by workers is uniform, in that any worker receiving a job offer draws the type of the firm from which the offer emanates from the distribution $\Gamma(\cdot)$. In Section 6 we endogenize this map $\lambda(\cdot)$ and sampling weights as the result of optimal vacancy posting given a standard matching function, and we extend our main results to this case.

In each period, the timing is as follows. Given a current state ω_t of aggregate labor productivity and distribution of employed workers N_t:

1. production and payments take place at all firms in state ω_t; the flow benefit b_t accrues to unemployed workers;
2. the new state ω_{t+1} of aggregate labor productivity is realized;
3. employed workers can quit to unemployment;
4. jobs are destroyed exogenously with chance δ_{t+1};
5. the remaining employed workers receive an outside offer with chance $s\lambda_{t+1}$ and decide whether to accept it or to stay with the current employer;
6. each previously unemployed worker receives an offer with probability λ_{t+1}.

Finally, in order to avert unnecessary complications, and to simplify the illustration, we assume that the state space Ω is finite, the distribution of firm types, Γ, has continuous

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5. A firm can be inactive when its productivity is too low relative to the worker value of leisure. So the unit measure of firms includes all potential producers, active and inactive. We discuss in Section 7 alternative assumptions about entry. That the mass of firms and workers both have measure one is obviously innocuous and only there to simplify the notation.

and everywhere strictly positive density $\gamma = \Gamma'$ over $\left[\underline{p}, \overline{p}\right]$, and the initial measure of employment across firm types, N_0, is continuously differentiable in p. Therefore, the initial average size of a type-p firm, which is given by $L_0\left(p\right) = \frac{dN_0(p)/dp}{\gamma(p)}$, is a continuous function of p. The case of equally productive firms, where Γ is a mass point, is simpler and we will refer to it separately.

4. THE CONSTRAINED EFFICIENT ALLOCATION

A social planner constrained by the same search frictions as private agents only has to decide which transition opportunities to take up and which ones to ignore. Recall that opportunities to move from unemployment to employment or from job-to-job only arise infrequently due to search frictions, while the option to move workers into unemployment is always available.[6]

The constrained efficient allocation is then simple enough to characterize: the planner will take up any opportunity to move an unemployed worker into employment, and the unemployment rate evolves according to $u_{t+1} = \delta_{t+1}\left(1 - u_t\right) + \left(1 - \lambda_{t+1}\right) u_t$. Moreover, the planner always seeks to move employed workers from less productive toward more productive firms. This induces the following simple process for the measure $L_t^\star\left(p\right)$ of workers efficiently allocated to a typical type-p firm:

$$L_{t+1}^\star\left(p\right) = L_t^\star\left(p\right)\left(1 - \delta_{t+1}\right)\left(1 - s\lambda_{t+1}\overline{\Gamma}\left(p\right)\right) + \lambda_{t+1}u_t + s\lambda_{t+1}\left(1 - \delta_{t+1}\right) N_t^\star\left(p\right) \quad (4.1)$$

where

$$N_t^\star\left(p\right) = \int_{\underline{p}}^p L_t^\star\left(x\right)\gamma\left(x\right)dx.$$

Given new aggregate state ω_{t+1}, which determines δ_{t+1} and λ_{t+1}, of the $L_t^\star\left(p\right)$ workers initially employed by this firm, a fraction $\left(1 - \delta_{t+1}\right)$ are not separated exogenously into unemployment. Of these survivors, a fraction $s\lambda_{t+1}$ receive an opportunity to move to another firm. The planner exercises that option if and only if the new firm is more productive than p, which is the case with probability $\overline{\Gamma}\left(p\right) := 1 - \Gamma\left(p\right)$. The initially unemployed u_t find jobs with chance λ_{t+1}. Workers employed at other firms who have not lost their jobs draw with chance $s\lambda_{t+1}$ an opportunity to move to the type-p firm, that the planner exploits if and only if the firm they currently work at has productivity $x < p$. The measure of such workers in the optimal plan is $N_t^\star\left(p\right)$.

Equation (4.1) combines an ordinary differential equation and a first-order difference equation in $N_t^\star\left(p\right)$, a function of time t and p. Multiplying through by $\gamma\left(p\right)$ in (4.1) and integrating with respect to p yields:

$$N_{t+1}^\star\left(p\right) = \lambda_{t+1}u_t\Gamma\left(p\right) + \left(1 - \delta_{t+1}\right)\left(1 - s\lambda_{t+1}\overline{\Gamma}\left(p\right)\right) N_t^\star\left(p\right).$$

For any given initial condition $N_0^\star\left(p\right) = N_0\left(p\right)$ at some (renormalized) initial date 0 such that the aggregate state last switched to ω at time 0 and then remained at ω between 0 and t, the latter law of motion is a first-order difference equation which solves as:

$$N_t^\star\left(p\right) = \left[\left(1 - \delta_0\right)\left(1 - s\lambda_0\overline{\Gamma}\left(p\right)\right)\right]^t N_0\left(p\right) + \lambda_0\Gamma\left(p\right)\sum_{\tau=1}^t \left[\left(1 - \delta_0\right)\left(1 - s\lambda_0\overline{\Gamma}\left(p\right)\right)\right]^{\tau-1} u_{t-\tau}.$$

$$(4.2)$$

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6. Because the planner solution is mostly a benchmark to assist equilibrium analysis, which is our main focus, here we assume that productivity is always large enough, relative to the value of leisure, that the planner will not forgo any opportunity to move an unemployed worker into employment. So we concentrate on the optimal allocation of workers through on-the-job search.

By inspection, $N_t^\star (p)$ is differentiable in p at all dates t, and one obtains a closed-form expression for the workforce of any type-p firm:

$$
\begin{aligned}
L_t^\star (p) =& \frac{dN_t^\star (p) / dp}{\gamma (p)} \\
=& (1 - \delta_0)^t \left(1 - s\lambda_0 \overline{\Gamma} (p) \right)^{t-1} \left[\left(1 - s\lambda_0 \overline{\Gamma} (p) \right) L_0^\star (p) + ts\lambda_0 N_0 (p) \right] \\
& + \lambda_0 \left\{ u_{t-1} + \sum_{\tau=2}^t (1 - \delta_0)^{\tau-1} \left(1 - s\lambda_0 \overline{\Gamma} (p) \right)^{\tau-2} \left[1 - s\lambda_0 + s\lambda_0 \tau \Gamma (p) \right] u_{t-\tau} \right\},
\end{aligned}
$$
$$(4.3)$$

where $L_0^\star (p)$ was the value of this solution under state $\widehat{\omega}_0$ at the time of the last state switch from $\widehat{\omega}_0$ to the current ω.

If the aggregate productivity state forever stays at ω, so that transition rates δ and λ are constant over time, the solutions to (4.2) and (4.3) converge to:

$$
\begin{aligned}
N_\infty^\star (p) &= \frac{\delta\lambda}{\delta + \lambda} \cdot \frac{\Gamma (p)}{1 - (1 - \delta) \left(1 - s\lambda \overline{\Gamma} (p) \right)} \\
\text{and:} \quad L_\infty^\star (p) &= \frac{\delta\lambda}{\delta + \lambda} \cdot \frac{1 - (1 - \delta) (1 - s\lambda)}{\left[1 - (1 - \delta) \left(1 - s\lambda \overline{\Gamma} (p) \right) \right]^2}
\end{aligned}
$$
$$(4.4)$$

which are the familiar steady-state expressions found in the BM model. As is well known and immediately verifiable from (4.4), the (normalized) distribution of employment across firm types $L_\infty^\star (p) / N_\infty^\star (\bar{p})$ is increasing in λ and decreasing in δ in the sense of stochastic dominance. Intuitively, workers upgrade to higher-p firms in larger numbers if they receive more opportunities to do so (higher $s\lambda$) or if they get thrown off the job ladder into unemployment less often (lower δ). This comparative statics property is reflected in the dynamic behavior of the firm size distribution if we assume, as is consistent with empirical evidence on job-to-job quits and job separations, that $\lambda (\omega)$ is increasing and $\delta (\omega)$ decreasing in the state of aggregate productivity ω, and also that more productive firms initially employ more workers, as is suggested by empirical evidence on the size-productivity relationship (and as is necessarily the case in the model's steady state). Then, hitting the economy with a randomly drawn sequence of aggregate shocks, in Moscarini and Postel-Vinay (2010a, MPV10a) we find that large employers are more cyclically sensitive, because they gain workers faster over an aggregate expansion as job upgrading accelerates, and vice versa in a slump. This property of the efficient allocation replicates the new empirical evidence that we document in MPV12.

5. EQUILIBRIUM

5.1. *Definition*

Each firm chooses and commits to an employment contract, namely a state-contingent wage depending on some state variable, to maximize the present discounted value of profits, given other firms' contract offers. The firm is further subjected to an *equal treatment constraint*, whereby it must pay the same wage to all its workers. This is the sense in which we generalize the BM restrictions placed on the set of feasible wage contracts to a non-steady-state environment.[7] Under commitment, such a wage function

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7. We thus rule out, beyond contracts that condition wages on tenure (Burdett and Coles, 2003) and employment status (Carrillo-Tudela, 2009), also offer-matching and individual bargaining (Postel-

implies a value V for any worker to work for that firm. For reasons that will become clear shortly, we assume that a contract offered by a firm to its workers is observable only by the parties involved.[8]

Let $\mathcal{H}$ be the (Borel-)measurable set of all histories of play in the game, and $\mathcal{V}_{\mathcal{H}}$ the set of measurable functions $\mathcal{H} \times \left[\underline{p}, \overline{p}\right] \to \mathbb{R}$. A behavioral strategy of the contract-posting game is a function $V \in \mathcal{V}_{\mathcal{H}}$ such that, when the history of past play in the game at time t is $h^t \in \mathcal{H}$, each firm $p \in \left[\underline{p}, \overline{p}\right]$ offers value $V_t(p) = V(h^t, p)$ to all of its workers.[9]

As V is measurable, we can define the c.d.f. of *offered* values

$$F_t(W) := \int_{\underline{p}}^{\overline{p}} \mathbb{I}\{V_t(p) \leq W\}\, d\Gamma(p) \tag{5.5}$$

for every $h^t \in \mathcal{H}$, $W \in \mathbb{R}$, and where $\mathbb{I}$ is the indicator function. This is the fraction of firms that offer a value no greater than W, given history h^t and given that all firms follow strategy V. This is also the chance that a worker who receives a job offer draws a value no higher than W. Let $\overline{F}_t = 1 - F_t$ denote the survival function.

Let $N_t(p)$ be the measure of workers currently employed at all firms of productivity up to p, so $N_t(\overline{p})$ is total employment, both on the equilibrium path. For any history of play $h^t \in \mathcal{H}$, the c.d.f. of *earned* values

$$G_t(W) := \frac{1}{N_t(\overline{p})} \cdot \int_{\underline{p}}^{\overline{p}} \mathbb{I}\{V_t(p) \leq W\}\, dN_t(p) \tag{5.6}$$

is also well-defined. This is the probability that a randomly drawn worker is currently earning value no greater than W after history h^t. Due to aggregate shocks, which affect both the offered values directly and the distribution of employment N_t through turnover rates, the distributions F_t and G_t are themselves stochastic.

Given a strategy $V \in \mathcal{V}_{\mathcal{H}}$ followed by all firms and the resulting distribution of offers F_t, an unemployed worker earns a value U_t solving:

$$U_t = b_t + \beta \mathbf{E}_t \left[(1 - \lambda_{t+1}) U_{t+1} + \lambda_{t+1} \int \max \langle v, U_{t+1} \rangle\, dF_{t+1}(v) \right], \tag{5.7}$$

where the expectation is taken over the new state, conditional on the current history h^t (hence the time subscript in the expectation operator). The new state determines the stochastic contact rate λ_{t+1}, offer distribution F_{t+1}, and continuation value of unemployment U_{t+1}. The unemployed worker collects a flow value b_t and, next period, when aggregate productivity becomes ω_{t+1}, she draws with chance $\lambda_{t+1} = \lambda(\omega_{t+1})$ a job

Vinay and Robin, 2002; Dey and Flinn, 2005; Cahuc, Postel-Vinay and Robin, 2006). Note, however, that the model can be generalized to allow for time-varying individual heterogeneity under the assumption that firms offer the type of piece-rate contracts described in Barlevy (2008). In that sense experience and/or tenure effects can be introduced into the model.

8. More specifically, we assume that offered contracts are not publicly observable, but, in order to sustain commitment by the firm, contracts must be privately verifiable by a third party such as a court if the worker decides to make it so (for example, the firm states the compensation policy in a letter that the worker retains). Alternatively, we can assume that the contract is observable only to other workers at the same firm, in which case the source of commitment is reputation, as in Coles (2001). What matters is that a contract is not observable by other firms, who could use a publicly observable deviation to coordinate play. We thank a referee for pointing this out to us.

9. To simplify notation, a time superscript on a variable denotes its entire history to date, while a time subscript denotes its current value given the current state of aggregate productivity ω and history to date. Every variable dated $t+1$ has to be interpreted at time t as a random variable, with randomness generated by the evolution of the aggregate state and by history of play between t and $t+1$. We make dependence of the various value and policy functions on the relevant states explicit again in the appendix, which contains the proofs of our main results.

offer from the distribution of offered values F_{t+1}, that she accepts if its value exceeds that of staying unemployed.

A firm that observes state ω_{t+1} and decides to post a continuation value $W_{t+1} \leq U_{t+1}$ loses all workers, who quit to unemployment, so $L_{t+1} = 0$. Otherwise, by the Law of Large Numbers,[10] the size L_t of a firm which posts a value $W_{t+1} > U_{t+1}$ changes to:

$$L_{t+1} = L_t (1 - \delta_{t+1}) \left[1 - s\lambda_{t+1}\overline{F}_{t+1}(W_{t+1})\right]$$
$$+ \lambda_{t+1}[1 - N_t(\overline{p})] + s\lambda_{t+1}(1 - \delta_{t+1})N_t(\overline{p})G_{t+1}(W_{t+1}). \quad (5.8)$$

After the new aggregate state ω_{t+1} is realized, of the measure L_t of workers currently employed by this firm, a fraction $\delta_{t+1} = \delta(\omega_{t+1})$ are separated exogenously into unemployment. Of the $(1 - \delta_{t+1})$ fraction of survivors, a fraction $s\lambda_{t+1}\overline{F}_{t+1}(W_{t+1})$ quit because they draw from F_{t+1} an outside offer which gives them a value larger than W_{t+1}. The currently unemployed $1 - N_t(\overline{p})$ find jobs with chance $\lambda_{t+1} = \lambda(\omega_{t+1})$, and accept an offer $W_{t+1} > U_{t+1}$ from this firm. By random matching, each firm offering $W_{t+1} > U_{t+1}$ receives an inflow of workers from unemployment that is independent of W_{t+1}, and is equal for all firms due to the normalization to uniform sampling weights. The employed who have not lost their jobs $(1 - \delta_{t+1})N_t(\overline{p})$ receive an offer with chance $s\lambda_{t+1}$, and accept it if the value W_{t+1} they draw is larger than what they are already earning (probability $G_{t+1}(W_{t+1})$), in which case they quit to this firm offering W_{t+1}.

Let $L_t(p)$ be the size of a firm of productivity p on the equilibrium path, which follows:

$$L_{t+1}(p) = L_t(p)(1 - \delta_{t+1})\left[1 - s\lambda_{t+1}\overline{F}_{t+1}(V_{t+1}(p))\right] + \lambda_{t+1}[1 - N_t(\overline{p})]$$
$$+ s\lambda_{t+1}(1 - \delta_{t+1})N_t(\overline{p})G_{t+1}(V_{t+1}(p)) \quad (5.9)$$

so that

$$N_t(p) = \int_{\underline{p}}^{p} L_t(x)\gamma(x)\,dx. \quad (5.10)$$

The support of N_t is contained in that of firm types (of Γ), because no worker can be at a firm of type p if no such firm exists.

A value strategy $W \in \mathcal{V}_{\mathcal{H}}$ is implemented by a wage strategy $w \in \mathcal{V}_{\mathcal{H}}$ such that the worker's Bellman equation is solved by W given that all other firms play V: the worker receives the wage and, next period, the expected value of being either displaced, or retained at the same firm, or poached by a higher-paying firm.

$$W_t = w_t + \beta\mathbf{E}_t\left[\delta_{t+1}U_{t+1} + (1 - \delta_{t+1})W_{t+1}\right.$$
$$\left. + (1 - \delta_{t+1})s\lambda_{t+1}\int_{W_{t+1}}^{+\infty}(v - W_{t+1})\,dF_{t+1}(v)\right]. \quad (5.11)$$

Workers act sequentially, as they are always free to quit. Firms choose once and for all at time 0 a strategy V (a state-contingent value policy), and commit to it. The constraint of

10. The exogenous separation of worker $i \in [0, L_t]$ is a Bernoulli random variable $\Delta(i)$ equal to 1 with probability δ_t and to 0 with probability $1 - \delta_t$. Total separations equal the sum of the realizations of these Bernoulli events, $\int_0^{L_t}\Delta(i)\,di$. By assumption, $\Delta(i)$ and $\Delta(j)$ are independent for any two workers i, j, have common mean δ_t and finite variance $\delta_t(1 - \delta_t)$. By the Law of Large Numbers for a continuum of random variables in Uhlig (1996), Theorem 2, this integral equals $\delta_t L_t$ a.s.. Similarly for hires from unemployment and from other firms.

delivering the promised value to the workers, once hired, is binding because, after hiring a worker with a promise of W_{t+1}, the firm would like to renege and to squeeze the worker against the participation constraint $W_{t+1} = U_{t+1}$.

Our assumption that a contract offer is only observed by the firm and by the workers who receive it implies that any deviation by a firm to a different pre-committed contract will be observed at most by the workers that the firm will hire over the countable infinite horizon, and by the firms that will hire workers who worked in the past at the deviating firm. Both sets have measure zero. So the firm anticipates that any deviation will trigger no relevant change in other firms' behavior, and, when choosing its strategy, takes the distributions of offers F_t and earned contracts G_t as given at any future point in time and state. The economy is always on the equilibrium path, independently of individual deviations, and we only need to check that the latter are not profitable.

Our first task is to find the state space on which equilibrium strategies can be conditioned. By assumption, past play by other firms is unobservable, hence cannot be part of $\mathcal{H}$. By inspecting the equations, the strategically relevant history for a firm of type p is $h^t = \{\omega_0, \cdots, \omega_t, t, L_{t-1}\}$. Given initial conditions and a strategy V, knowing the value of this h^t is enough to compute the entire history of play by all firms to date t. The history of aggregate productivity $\omega^{t-1} := \{\omega_s\}_{s \leq t-1}$ is directly payoff-relevant at t only insofar as it determines the size distribution N_{t-1}, thus the value distribution G_{t-1} through (5.6). But ω^{t-1} can also be used by firms as a public randomization device to coordinate actions, hence N_{t-1} is not sufficient for ω^{t-1} in a strategic sense. Note that the relevant history h^t on which firms can condition their choices at time t includes ω_t and L_{t-1} but not L_t, because according to the timing we assumed current firm size L_t is determined by the value offered at time t, which depends on h^t.[11]

The equilibrium strategy V is a fixed point: if all firms precommit at time 0 to the value offer strategy V, a function of the history of play h^t, and workers act optimally at all points in time t, given the implied evolution of the cross-section distributions of values offered F_t and earned G_t and of the value of unemployment U_t, each firm's best response to maximize expected discounted profits at time 0 is to follow the same strategy V.

To further reduce the state space to a tractable dimensionality, we restrict attention to strategies that depend only on *current values of payoff-relevant variables*, namely own productivity p, own firm size L_{t-1}, the new state of aggregate productivity ω_t, and distribution of employment N_{t-1}. This restriction is in the spirit of Maskin and Tirole (2001)'s Markov Perfect Equilibrium, although that is defined for games of observable actions without commitment. Importantly, this restriction excludes calendar time, because not payoff-relevant, from the state space. In this sense, our Markov strategies are "stationary".

Definition 1. *A* Markov contract-posting equilibrium *is a measurable function V of own productivity p, own size L_{t-1}, the new state of aggregate productivity ω_t, and distribution of employment N_{t-1}, with the following property: for every firm type p, if all other firms of type x play V, so that (5.5), (5.6) and (5.9) hold with*

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11. Under this definition, part of h^t, specifically firm size L_t, is not publicly observed. When maximizing, each firm believes that other firms offer values according to the strategy evaluated at the the publicly observed history of aggregate productivity, ω^t, and their own size *on the equilibrium path*, $L_t(p)$, which the firm can calculate only based on known initial conditions and ω^t. As mentioned, when optimizing, each firm correctly believes that, irrespective of its own choices, the rest of the economy is always on the equilibrium path. The question is what is this path.

$V_t\left(x\right)=V\left(x,L_{t-1}\left(x\right),\omega_t,N_{t-1}\right)$, *the wage function that implements V (i.e. solves (5.11) with $W_t=V_t\left(p\right)$) also maximizes firm p's expected present discounted value of profits at time 0.*

Note that the Markov state variable contains only one endogenous (to the firm) state variable, its own size L_{t-1}, the rest being either a fixed parameter p or aggregate states, ω_t and N_{t-1}, that are independent of any individual firm's behavior, both on and off the equilibrium path. Making strategies independent of past values of aggregate productivity comes at the cost of introducing in the state the distribution of employment N_{t-1}. This is also an infinitely dimensional object, but it turns out to be much more tractable than the entire history of ω_t, as we will see next.

5.2. *The firm's contract-posting problem*

We fix the Markov strategy of other firms V and omit it from the notation for simplicity. The firm can always guarantee itself zero flow profits by making the participation constraint $W_t\geq U_t$ bind and dismissing all workers, so offering any value lower than U_t is equivalent to an offer $W_t=U_t$ which guarantees zero output and flow profits this period.

The firm maximizes, under commitment, the expected present discounted value of profits at time 0, Π_0. The problem can be formulated recursively (Spear and Srivastava, 1987) by introducing an additional, fictitious state variable, namely the continuation utility $\overline{V}$ that the firm promised at time $t-1$ to deliver to the worker from this period t on. While we subsume all other state variables $\{p,L_t,\omega_t,N_t\}$ in the time index of the firm's value/objective Π_t,[12] we make the dependence of these profits on promised value $\overline{V}$ explicit, because of its central strategic importance. So the firm solves

$$\Pi_t\left(\overline{V}\right)=\sup_{w_t,W_{t+1}\geq U_{t+1}}\left\langle\left(\omega_tp-w_t\right)L_t+\beta\mathbf{E}_t\left[\Pi_{t+1}\left(W_{t+1}\right)\right]\right\rangle \tag{5.12}$$

subject to a Promise-Keeping (PK) constraint to deliver the promised $\overline{V}$:

$$\overline{V}=w_t+\beta\mathbf{E}_t\left[\delta_{t+1}U_{t+1}+\left(1-\delta_{t+1}\right)\left(1-s\lambda_{t+1}\overline{F}_{t+1}\left(W_{t+1}\right)\right)W_{t+1}\right.$$
$$\left.+\left(1-\delta_{t+1}\right)s\lambda_{t+1}\int_{W_{t+1}}^{+\infty}vdF_{t+1}\left(v\right)\right]. \tag{5.13}$$

The expectations are taken with respect to the future realization ω_{t+1} of aggregate productivity conditional on the date-t state variable, while firm size L_t evolves according to (5.8), cumulated firm size N_t follows (5.10).

The continuation value on the RHS of (5.13) comes from (5.11) after a small algebraic manipulation. In (5.12), given the timing of events, the firm collects flow revenues, equal to per worker productivity ω_tp times firm size L_t, then chooses and pays the flow wage w_t

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12. Note that, given our assumed timing, the continuation value that the firm chooses to offer to the worker depends on $\{p,L_t,\omega_{t+1},N_t\}$ because the firm observes ω_{t+1} before quantifying the promised continuation value. But the present discounted value of profits of the firm at the beginning of time t, Π_t, is a function of the different state $\{p,L_t,\omega_t,N_t\}$, because ω_{t+1} is not known at the beginning of the period, when paying the wage w_t. So $\{p,L_t,\omega_{t+1},N_t\}$ is the strategically-relevant state variable, the argument of the optimal policy function, which determines the offered value, and $\{p,L_t,\omega_t,N_t\}$ is the state variable of the firm's optimization problem, the argument of the firm's value (profit) function.

to each worker, then observes the new state of aggregate productivity ω_{t+1}, and finally chooses the continuation contract (promised value) W_{t+1}, so that wage and continuation values deliver to the workers the current promised value $\overline{V}$, in expectation, as per (5.13).

To characterize the best response contract, we first describe an equivalent unconstrained recursive formulation of the contract-posting problem. We define the joint value of the firm and its *existing* workers:

$$S_t = \Pi_t + \overline{V} L_t.$$

Solving for the wage w_t from (5.13) and replacing it into the firm's Bellman equation (5.12) we see that the joint value function S_t solves:

$$S_t = \omega_t p L_t + \beta \mathbf{E}_t \left[\delta_{t+1} U_{t+1} L_t \right] + \beta \mathbf{E}_t \left[\sup_{W_{t+1} \geq U_{t+1}} \left\langle \left(1 - \delta_{t+1} \right) s \lambda_{t+1} L_t \int_{W_{t+1}}^{+\infty} v \, dF_{t+1}\left(v \right) \right. \right.$$

$$\left. \left. + S_{t+1} - W_{t+1} \left\{ \lambda_{t+1} \left(1 - N_t\left(\overline{p} \right) \right) + \left(1 - \delta_{t+1} \right) s \lambda_{t+1} N_t\left(\overline{p} \right) G_{t+1}\left(W_{t+1} \right) \right\} \right\rangle \right] \quad (5.14)$$

again subject to (5.8) and (5.10).[13] The joint value S_t to the firm and its existing workers equals the total flow output, $\omega_t p L_t$, plus the expected discounted continuation value. This includes (in order) the value of unemployment for those workers who are displaced exogenously, the value of a new job for those workers who are not displaced and find a better offer than the one extended by the current firm and, on the second line of (5.14), the joint continuation value of the firm and of its current (time t) workers. In turn, the latter equals the joint continuation value S_{t+1} of the firm and its future workforce — made up of stayers among the current (date-t) workforce plus next-period (date-$t+1$) hires — minus the value to be paid to new hires, either from unemployment or from other firms (respectively, the two terms in curly brackets in the second line of (5.14)). If we did not subtract this cost of employing new hires, this Bellman equation would generate the joint value of the firm and *all* of its workers, current *and* future. In this case, the firm would optimally offer its workers the maximum value, namely pay a wage equal to productivity (the proof, omitted, is available upon request). As is standard, the efficient solution to a moral hazard problem is to "sell the firm to the workers". In our economy, however, firms do not pursue efficiency, but maximize profits. Therefore, the optimal value-offer policy is an interior solution.

Crucially, the current promised value $\overline{V}$ does not appear in (5.14), either directly or in the law of motion of transition rates λ_{t+1}, δ_{t+1}, firm size L_{t+1} and value distributions F_{t+1}, G_{t+1}. So the DP problem in (5.14) and its solution are independent of $\overline{V}$. Along the optimal path, the level of current promised utility $\overline{V}$ only affects the distribution of payoffs between the firm and its existing workers, not its overall level S_t, nor the choice of tomorrow's promised values W_{t+1}. The intuition is clear. The workers' turnover decisions only depend on continuation values W_{t+1} promised by the firm, so the same applies to firm continuation profits Π_{t+1}. The firm thus offers state-contingent W_{t+1} to maximize Π_{t+1} independently of the currently promised value $\overline{V}$. Then, to deliver $\overline{V}$ as promised without distorting the optimally set future turnover, the firm adjusts the current wage w_t.

An issue arises with the initially promised value, before the firm chooses and commits to the contract at time 0. If the firm has some employees at time 0 and all

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13. A detailed proof of this claim is available in Section 1 of the online appendix to this paper.

the bargaining power, it optimally pays the lowest possible wage and extracts all rents from its initial workforce, by making them just indifferent between staying or quitting into unemployment through an offered value $U_0 = U(\omega_0, N_0)$. It cannot be always optimal, however, to pre-commit to offer again the same value of unemployment U_0 in the future, whenever the state of the firm returns to $\{p, L_0, \omega_0, N_0\}$, because that offer would maximize attrition, and in general the firm is willing to pay extra to retain more workers. So the best response by any firm in this state $\{p, L_0, \omega_0, N_0\}$ is different at time 0 and at a later date, thus is time-dependent, or non Markovian. This is true even if all other firms offer Markov contracts. Hence, if the firm has both employees before it has committed to the contracts *and* all the bargaining power, a Markov contract-posting equilibrium may or may not exist. This, however, turns out to be a technical issue of little economic substance, with multiple possible and equivalent resolutions. Assume all other firms offer a Markov contract. Then, as just shown, at each time $t = 1, 2..$ the best-response value that the firm promises in each state next period does not depend on the current value promised to its workers, inherited from commitment in the previous period $t - 1$. By induction, the best-response value that the firm promises at each future date and state does not depend on the value earned by its initial employees L_0; i.e., continuation play does not depend on the initial rent-sharing at time 0. Therefore, we can calculate a Markov best-response from time $t = 1$ by solving (5.14) and ignoring the time-0 problem. Then, we can fix arbitrarily the value paid to initial employees, subject to participation constraints of all parties. Just as one example, we can assume that initial workers bargain with the firm over the initial value, and each of them obtains exactly the same value $V_0(p)$ that the firm promises to deliver if the state returns later to the initial one $\{p, L_0, \omega_0, N_0\}$, so that the whole best response to a Markov contract is itself Markov. The continuation contract promised in each state from time $t = 1$ on, thus the equilibrium predictions of the model, are unaffected by this initial choice, so we do not delve into it.

The optimal policy solving the unconstrained DP problem (5.14) also solves (5.12) subject to (5.13). We therefore focus on the analysis of the simpler problem (5.14). An equilibrium is a fixed point, a solution V to this DP problem that coincides with the strategy followed by the other firms. To find the equilibrium, we proceed as follows. First, we show that the equilibrium distribution of values offered (F) and earned (G), whatever they are, must be atomless on a connected support. Next, under certain sufficient conditions a best response value to any strategy followed by all other firms must be strictly increasing in own productivity p and size L. Finally, in that smaller set of monotonic functions we construct the unique equilibrium.

5.3. *Properties of the equilibrium distributions of contracts*

The distributions of offered and accepted worker values, F_t and G_t, must satisfy certain general properties in equilibrium, which parallel similar properties of the corresponding wage distributions in the original BM model.

Proposition 1 (F_t and G_t are atomless). *In equilibrium F_t and G_t are atomless at all dates t and in all states, with their common support being compact and convex.*

To see why there can be no atom in either F_t or G_t, observe that, by the equal treatment constraint, if F_t had an atom at some value W_t, then so would G_t. But an

atom in G_t would open the way to a profitable deviation, as in BM. A firm that is part of the atom that offers the same W_t could marginally raise the wage w_t and leave the continuation value unchanged, increase the chance of beating at time $t-1$ all competitors who offer W_t, and thus poach at $t-1$ an additional measure of workers at a negligible marginal cost. This deviation is unprofitable only if the firm was already offering its workers so much as to break even in expected present discounted terms. But then a more drastic deviation towards offering, e.g., $W_t = U_t$ in all states is profitable as all unemployed workers accept this offer and stay for a while, generating strictly positive profits for all but the zero measure of firms with marginal productivity type that break even with $W_t = U_t$.

To see why the support of F_t and G_t is convex, observe that if there was a gap then the lower and upper bounds of this gap would generate the same hiring and retention, so the same firm size, but the upper bound would cost the firm more in terms of wages, so no firm would post such an upper bound. To see why the support is compact, observe that $\overline{W} = \overline{\omega p} / (1 - \beta)$ is a natural upper bound to the offered value: the firm can always do weakly better by offering less than $\overline{W}$, as it can hope to make some profits. So the support is a convex and bounded subset of $\mathbb{R}_+$, which we can therefore take to be compact WLOG.

5.4. *Rank-Preserving Equilibrium (RPE)*

While solving for equilibrium directly is an intractable problem because the size distribution of firms N_t is an infinitely-dimensional state variable, we can still define a tractable and natural class of equilibria, which have the following property. Recall that $L_t(p)$ denotes employment size of a type-p firm along the equilibrium path, i.e. the size attained by that firm given the initial size distribution at date 0 and given that all firms have played the equilibrium strategy from date 0 up to the current date t. Then:

Definition 2. *A* Rank-Preserving Equilibrium (RPE) *is a Markov equilibrium V where, on the equilibrium path, a more productive firm always offers its workers a higher continuation value: $V_{t+1}(p) = V(p, L_t(p), \omega_{t+1}, N_t)$ is increasing in p, including the effect of p on current firm size $L_t(p)$.*

As a direct consequence of the above definition, in a RPE, workers rank their preferences to work for different firms according to firm productivity at all dates. The following two properties thus hold true in any RPE: the proportion of firms that offer less than $V_{t+1}(p)$ is simply the proportion of firms that are less productive than p

$$F_t(V_t(p)) \equiv \Gamma(p), \tag{5.15}$$

and the fraction of employed workers who earn a value that is lower than that offered by p equals the share of employment at firms less productive than p:

$$G_t(V_t(p)) = \frac{N_{t-1}(p)}{N_{t-1}(\overline{p})}. \tag{5.16}$$

As we will see, these restrictions will drastically simplify the computation of equilibrium in the stochastic model. Moreover, the RP property is theoretically appealing for at least two more reasons. First, it parallels a well-known property of the unique static equilibrium characterized by BM, which is that workers rank firms according to productivity. Second, RPE feature constrained-efficient labor reallocation at all dates: if workers consistently

rank more productive firms higher than less productive ones, then job-to-job moves will always be up the productivity ladder. That is, if the value of leisure b_t is small enough, the unique RPE allocation is $L_t(p) = L_t^\star(p)$ and $N_t(p) = N_t^\star(p)$ at all points in time and states.

It is therefore natural to ask how general Rank-Preserving Equilibria are. We now show that under a sufficient condition on the initial size distribution of employment, all Markov equilibria must be Rank-Preserving. Because the employment allocation in a RPE is constrained efficient, it is also unique. This is the central result of the paper. It builds on the following, key technical result.

Proposition 2 (Increasing best response). *For any Markov strategy played by other firms, i.e. for any map V from payoff-relevant states $(p, L_t, \omega_{t+1}, N_t)$ to continuation values offered to workers, the best response of a given firm is Markov and increasing in the firm's productivity p and size L_t.*

As a Markov equilibrium strategy is a best response to itself, it must share the same monotonicity properties. Although its proof, in Appendix Appendix A, is technically quite involved, Proposition 2 has a simple economic intuition, which extends the logic of BM's steady-state model to our dynamic setting. Consider the first statement: offered values are increasing in firm productivity p. In BM, more productive firms offer higher wages due to a single-crossing property of their steady state profits, which in turn reflects two very basic economic forces. First, a higher wage implies a larger firm size, as a more generous offer makes it easier to poach workers and to fend off competition. Second, a larger firm size is more valuable to a more productive firm, because each worker produces more. Therefore, by a simple monotone comparative statics argument, it must be the case that more productive firms offer more, employ more workers, and earn higher profits. Simply put, a more productive firm can afford paying more, and is willing to do so to attract workers, because its opportunity cost of not producing is higher.

The intuition for the second statement of Proposition 2, that offered continuation values are increasing in current size, is slightly more involved, and can be clarified in a simple two-period, deterministic version of the model. Let F denote the distribution of second-period wages, rather than values, assumed to be differentiable for simplicity, with $F' = f$. Consider a firm inheriting a workforce size of L_0, who have been promised today a continuation value

$$\overline{V}_0 = w_0 + \beta \left[\delta U_1 + (1 - \delta)(1 - s\lambda \overline{F}(w_1)) w_1 + (1 - \delta) s\lambda \int_{w_1}^{+\infty} w \, dF(w) \right]. \quad (5.17)$$

Solve this PK constraint for the initial wage $w_0(w_1)$ as a function of the future wage w_1, given promised utility $\overline{V}_0$. The firm chooses the wage w_1 to maximize the PDV of profits

$$\Pi(w_1) = [p - w_0(w_1)] L_0 + \beta (p - w_1) L_1(w_1)$$

where next period's size is $L_1(w_1) = \left[(1 - \delta)(1 - s\lambda \overline{F}(w_1)) L_0 + H(w_1) \right]$ and $H(w_1)$ is the flow of hires from unemployment and from other firms (we keep this implicit). The marginal impact on profits of varying the future wage decomposes into three terms:

$$\Pi'(w_1) = \underbrace{-\frac{dw_0}{dw_1} L_0 - \beta L_1(w_1)}_{\text{Wage-bill effect}} + \underbrace{\beta (p - w_1) s\lambda f(w_1) L_0}_{\text{Retention effect}} + \underbrace{\beta (p - w_1) H'(w_1)}_{\text{Recruitment effect}}.$$

The last two effects are intuitive: by raising the wage, the firm retains more of its existing workers, an effect that is larger the bigger the firm to begin with, and attracts more workers, to an extent that does not depend on its current size. The wage-bill effect is less obvious. Raising the period-1 wage promise allows to reduce the period-0 wage by $\frac{dw_0}{dw_1}$ on the L_0 workers, and still deliver $\overline{V}_0$, but costs $\beta L_1(w_1)$ in present value, as future workers L_1 (survivors from period 0 plus new hires $H(w_1)$), will be paid more. Using (5.17) to calculate $\frac{dw_0}{dw_1}$ and the expression for $L_1(w)$, the wage bill effect is independent of initial size:

$$\beta(1-\delta)(1-s\lambda\overline{F}(w_1))L_0 - \beta[(1-\delta)(1-s\lambda\overline{F}(w_1))L_0 + H(w_1)] = -\beta H(w_1).$$

The intertemporal MRS between wages is exactly the same for the firm and for any worker in its period-0 workforce, namely, the discount factor β times the probability $(1-\delta)(1-s\lambda\overline{F}(w_1))$ that the worker will be around to collect a higher payment. So firms are indifferent about the timing of wages paid to deliver a certain promised utility to their initial workforce. The only (and negative) effect of a higher future promised wage on the wage bill is for new hires, whose measure is independent of initial size. So, overall, only the retention effect depends, positively, on initial size, and $\frac{d^2\Pi}{dw_1 dL_0} > 0$: the profit function is supermodular in initial size and future wage, so that initially larger firms offer higher future wages. Because the marginal trade-off between today's wages and tomorrow's promised values does not depend on the level of the current promised value, this reasoning extends to the infinite horizon model with uncertainty. In the proof of Proposition 2, we formalize this intuition, to establish single-crossing properties of the maximand in the Bellman equation of the value-posting problem (5.14),[14] which parallel the single-crossing property of steady-state profits in BM.

Translating this property of the best response into equilibrium characterization requires an extra induction step. In BM's stationary setup, firm size is an endogenous object, and BM look for an appropriate firm size distribution which guarantees a stationary allocation. In our dynamic model, firm size is a state variable, and its *initial* value is a parameter of the model, arbitrarily fixed, not an endogenous object. Therefore, in order to get a start on monotone comparative dynamics, it is sufficient (but not necessary) that the initial size distribution shares the key property of BM's steady state distribution; namely, it is weakly increasing in productivity.

To understand how restrictive these sufficient conditions are, consider first the case of completely homogeneous firms: they have the same productivity p (Γ is degenerate) and same initial size. By Proposition 1, in equilibrium firms must randomize on the value offered at time $t=0$. Firms that draw and offer a larger value pay higher wages but grow larger. By Proposition 2, from then on they pay a larger value, grow larger, and so on forever. So any Markov equilibrium is necessarily Rank-Preserving.

Next, assume that firms differ in their productivity but have the same initial size; for example, they all start empty. Then by Proposition 1 the more productive firms initially offer their workers a larger value, grow larger, thus in the following period again offer a larger value, and so on forever. Again, any Markov equilibrium is necessarily Rank-Preserving.

Finally, if firms differ both in their productivity and in their initial size, then for RPE it suffices that more productive firms start out larger. This is the only genuine restriction for a RPE, and is sufficient, but not necessary. It aligns two separate motives to pay

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14. In a way similar to that in which Caputo (2003) appeals to single-crossing properties of the Hamiltonian in his analysis of comparative dynamics for *deterministic* optimal control problems.

workers more, firm productivity and size, so there is some slack. The same induction logic applies.

We have proven the following:

Proposition 3 (Ranked initial firm size implies RPE). *Any Markov contract-posting Equilibrium is necessarily Rank-Preserving, and the ranking of firms' sizes is maintained on the equilibrium path, if either of the following two conditions is verified:*

  1 *firms have the same productivity p (Γ is degenerate); or*
  2 *firms differ in their productivity p (Γ has a continuous density), and more productive firms are initially weakly larger ($L_0(p)$ is non-decreasing in p).*

We stress that this is a characterization result, which neither establishes nor requires existence, let alone uniqueness, of a RPE. While the RPE allocation is unique, the RPE strategy needs not be unique. Our main result says that, if a Markov contract-posting Equilibrium V exists, then V can only be a best response to itself if it is increasing in p, including the effect of endogenous size on the posted value. So ours is a general monotonicity result, which does not require to either propose or calculate a particular value-offer strategy. In the next section, we show by construction existence and uniqueness of a RPE, which must then be the unique Markov equilibrium of the contract-posting game.

To characterize a RPE we need to describe how allocation and prices depend on exogenous states. The allocation is easy because constrained efficient. We already know from Section 4 how the size of each firm evolves in equilibrium. Indeed, the same logic applies to any job ladder model in which a similar concept of RPE can be defined. Nothing in the dynamics of $L_t^\star$ or $N_t^\star$ depends on the particulars of the wage setting mechanism, so long as this is such that employed job seekers move from lower-ranking into higher-ranking jobs in the sense of a time-invariant ranking. Therefore, this model's predictions about everything relating to firm sizes are in fact much more general than the wage- (or value-) posting assumption retained in the BM model. We now turn to supporting prices.

5.5. Existence and uniqueness of (Rank-Preserving) equilibrium

Our aim in this section is to characterize equilibrium contracts in a way that will provide a constructive proof of uniqueness and — subject to a sufficient condition — of existence of RP equilibrium contracts. We begin by establishing some important properties of optimal contracts. Equation eqreffirmsize combined with the assumption that initial firm size, $L_0(p)$, is a continuous function of p (see Section 3) ensures that $L_t^\star(p)$ is a continuous function of p at all dates t in a RPE. With that in mind, we can establish the following additional properties of the joint value function S_t and worker value function V_t in a RPE:

Proposition 4 (Differentiability of value functions in RPE). *The following properties hold in a RPE:*

  1 *the joint value $S(p, L, \omega_t, N_t^\star)$ of a firm of type p and of its L current employees is convex in L, differentiable in L at $L = L_t^\star(p)$, and such that $p \mapsto \frac{\partial S}{\partial L}(p, L_t^\star(p), \omega_t, N_t^\star)$ is continuous.*

2 *the value $V_t(p) = V\left(p, L^\star_{t-1}(p), \omega_t, N^\star_{t-1}\right)$ offered by a firm of type p is continuously totally differentiable in p.*

The proof is in Appendix Appendix B. While most of that proof is essentially technical, it begins by establishing continuity of $V_{t+1}(p)$ in p, which follows from Proposition 1: a jump in $V_{t+1}(p)$ would create a gap in the support of F_{t+1}, which Proposition 1 rules out.

The second statement in Proposition 4 allows us to differentiate (5.15) and (5.16) w.r.t. p. For all times t and states at time t:

$$f_t\left(V_t(p)\right) \frac{dV_t}{dp}(p) = \gamma(p) \qquad \text{and} \qquad g_t\left(V_t(p)\right) \frac{dV_t}{dp}(p) = \frac{L^\star_{t-1}(p)}{N^\star_{t-1}(\overline{p})} \gamma(p). \qquad (5.18)$$

This differentiability property allows the use in (5.14) of first-order conditions:

$$\lambda_{t+1}\left(1 - N^\star_t(\overline{p})\right) + s\lambda_{t+1}\left(1 - \delta_{t+1}\right) N^\star_t(\overline{p}) G\left(W_{t+1}\right)$$
$$= \left[S_L\left(p, L_{t+1}, \omega_{t+1}, N^\star_{t+1}\right) - W_{t+1}\right]$$
$$\times \left(1 - \delta_{t+1}\right) s\lambda_{t+1}\left[L^\star_t(p) f_{t+1}\left(W_{t+1}\right) + N^\star_t(\overline{p}) g_{t+1}\left(W_{t+1}\right)\right] - \xi_{t+1} \qquad (5.19)$$

where ξ_t is the Lagrange multiplier for the workers' participation constraint $W_t \geq U_t$, and where complementary slackness $\xi_t\left(W_t - U_t\right) = 0$ applies at all t. In a RPE, (5.19) is solved by $W_{t+1} = V_{t+1}(p) = V\left(p, L^\star_t(p), \omega_{t+1}, N^\star_t\right)$.

Define the costate variable

$$\mu_t(p) := \frac{\partial S}{\partial L}\left(p, L^\star_t(p), \omega_t, N^\star_t\right),$$

which measures the shadow value to the worker-firm collective of the marginal worker, given the aggregate state, along the equilibrium path. Combining (5.19) and the various restrictions (5.15), (5.16), and (5.18) that hold in a RPE, we obtain the RPE version of the FOC (5.19):

$$\lambda_{t+1}u_t + s\lambda_{t+1}\left(1 - \delta_{t+1}\right) N^\star_t(p) = \left[\mu_{t+1}(p) - V_{t+1}(p)\right] \frac{2s\lambda_{t+1}\left(1 - \delta_{t+1}\right) L^\star_t(p) \gamma(p)}{dV_{t+1}/dp} - \xi_t. \qquad (5.20)$$

Then writing the Envelope condition w.r.t. firm size in the firm's problem (5.14), we obtain:

$$\mu_t(p) = \omega_t p + \beta \mathbf{E}_t\left[\delta_{t+1}U_{t+1} + \left(1 - \delta_{t+1}\right) s\lambda_{t+1} \int_{W_{t+1}}^{+\infty} v dF_{t+1}(v)\right.$$
$$\left. + \mu_{t+1}(p)\left(1 - \delta_{t+1}\right)\left(1 - s\lambda_{t+1}\overline{F}_{t+1}\left(W_{t+1}\right)\right)\right]$$
$$= \omega_t p + \beta \mathbf{E}_t\left[\delta_{t+1}U_{t+1} + \left(1 - \delta_{t+1}\right) s\lambda_{t+1} \int_p^{+\infty} V_{t+1}(x) d\Gamma(x)\right.$$
$$\left. + \mu_{t+1}(p)\left(1 - \delta_{t+1}\right)\left(1 - s\lambda_{t+1}\overline{\Gamma}(p)\right)\right], \qquad (5.21)$$

where the second equality holds in RPE. Note from (5.21) that now the shadow marginal value $\mu_t(p)$ of an employee only depends on the distribution of employment $N^\star_t$ through

total employment in all firms of productivity up to p, $N_t^\star(p)$ and the corresponding density $L_t^\star(p)\gamma(p)$. Both are scalars, and the state reduces from $\{p, L_t, \omega_{t+1}, N_t\}$, which is infinite-dimensional due to the relevance of the entire firm size distribution N_t, to the four-dimensional vector $(p, L_t, \omega_{t+1}, N_t^\star(p))$: in order to make its decisions, firm p only needs to know the mass of employment at less productive firms $N_t^\star(p)$ and not the entire size distribution $N_t^\star$.

Equation (5.20) generalizes to a dynamic stochastic environment the NFOC in BM's steady state model with heterogeneous firms. To see why, let

$$H_{t+1}(p) := \lambda_{t+1}u_t + s\lambda_{t+1}(1 - \delta_{t+1})N_t^\star(p) \tag{5.22}$$

denote the flow of hires into a firm of type p after state ω_{t+1} is realized. For all firms in the market, except possibly one, at which the worker participation constraint is slack, rearranging (5.20), we can write it as:

$$\frac{dV_{t+1}}{dp} = [\mu_{t+1}(p) - V_{t+1}(p)]\, 2\frac{d\log H_{t+1}(p)}{dp} \tag{5.23}$$

Optimality demands that the additional value offered by firm p over its immediately lower competitor equals the net returns from additional labor (costate minus offered value) times the ratio between twice the marginal flow of hires when productivity p increases and the total flow of hires. The marginal flow is doubled because a higher productivity, thus a higher offered value, reduces attrition to other firms and also increases poaching from other firms, and the two effects are locally equal, and add up. In steady state, under a constant aggregate state ω, the shadow value of a marginal worker in (5.23) is simply the present value of a constant output flow of ωp over the infinite future discounted with factor $\beta(1-\delta)\left(1 - s\lambda\overline{\Gamma}(p)\right)$ (the agents' discount factor β times the survival probability of the match), the offered value is the present value of a constant wage $w(p)$ discounted with that same factor, and the flow of hires equals firm size times the total separation rate. Using these facts, equation (5.23) reduces to the static NFOC (14) in Bontemps et al. (2000).[15]

Finally, a Transversality Condition (TVC) requires that the discounted joint value of the marginal worker vanishes in expectation w.r. to the stochastic path of ω

$$\lim_{t\to\infty} \mathbf{E}\left[\beta^t \mu_t(p) L_t^\star(p) \mid L_0, \omega_0, N_0\right] = 0. \tag{5.24}$$

The assumption $b_t \geq 0$ for all ω_t guarantees that $U_t \geq 0$, because a worker has always the option of staying unemployed to collect positive payoffs. To guarantee existence of RPE, it suffices to assume that s is large enough that the worker participation constraint never binds in equilibrium.[16] A RPE is then a value V_t increasing in p, a shadow value of employment μ_t, and a value of unemployment U_t positive and smaller than V_t, obeying the boundary condition $V_t(\underline{p}) = U_t$ and solving the FOC (5.20), the Euler equation (5.21) and the unemployment Bellman equation (5.7) given the RPE employment dynamics (4.1), subject to the TVC (5.24).

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15. A detailed formal derivation of the equivalence between our model's steady state and Bontemps et al.'s (2000) model is available in Section 2 of the online appendix to this paper.

16. Notice that when s is large enough, the worker has no reason to decline any offer, and with $\omega_t \underline{p} > b_t$ even the least productive firm can hire some unemployed workers and obtain positive profits. None of our previous results depend on these restrictions. Our analysis allows for an occasionally binding participation constraint of the worker, hence for entry and exit of firms "at the bottom" of the productivity distribution. We return to entry and exit in Section 7.

We can easily solve the FOC, in its form (5.23), subject to the TVC (5.24), to find the RPE offered value:[17]

$$V_t(p) = U_t + \frac{1}{H_t(p)^2} \int_{\underline{p}}^{p} [\mu_t(x) - U_t] \cdot \frac{d}{dp}\left[H_t(x)^2\right] dx := \mathbf{T}_V[\mu_t, U_t](p \mid \omega_t).$$

where we make the dependence of the operator $\mathbf{T}_V$ on ω_t explicit to solve the system forward in time. Define the following linear maps:

$$\mathbf{T}_\mu[\mu_t, U_t](p \mid \omega_t) := \delta_t U_t + \mu_t(p)(1-\delta_t)\left(1 - s\lambda_t \overline{\Gamma}(p)\right)$$
$$+ (1-\delta_t)s\lambda_t \int_{\underline{p}}^{\overline{p}} \mathbf{T}_V[\mu_t, U_t](x \mid \omega_t) d\Gamma(x)$$

$$\mathbf{T}_U[\mu_t, U_t](\omega_t) := (1-\lambda_t)U_t + \lambda_t \int_{\underline{p}}^{\overline{p}} \mathbf{T}_V[\mu_t, U_t](x \mid \omega_t) d\Gamma(x)$$

and collect them in the operator

$$\mathbf{T}[\mu_t, U_t](p \mid \omega_t) = \begin{pmatrix} \mathbf{T}_\mu[\mu_t, U_t](p \mid \omega_t) \\ \mathbf{T}_U[\mu_t, U_t](\omega_t) \end{pmatrix}. \tag{5.25}$$

Then a RPE is a $\binom{\mu^\star}{U^\star}$ solving

$$\begin{pmatrix} \mu_t^\star \\ U_t^\star \end{pmatrix}(p) = \begin{pmatrix} \omega_t p \\ b(\omega_t) \end{pmatrix} + \beta \mathbf{E}_t\left\{\mathbf{T}\left[\mu_{t+1}^\star, U_{t+1}^\star\right](p \mid \omega_{t+1})\right\}, \tag{5.26}$$

which satisfies the TVC (5.24) and has $0 \le U^\star \le \mathbf{T}_V[\mu^\star, U^\star] \le \mu^\star$ and $\mathbf{T}_V[\mu^\star, U^\star](p)$ increasing in p. We are now in a position to prove the following result:

Proposition 5 (Uniqueness and Existence). *There exists at most one equilibrium, which is Rank-Preserving. If it exists, the optimal contract in this unique RPE is the wage policy that pays the worker a value $\mathbf{T}_V[\mu^\star, U^\star]$ where:*[18]

$$\begin{pmatrix} \mu_t^\star \\ U_t^\star \end{pmatrix}(p) := \lim_{n \to \infty} \sum_{j=0}^{n} \beta^j \mathbf{E}_t\left\{\mathbf{T}^j[\omega p, b(\omega)](p \mid \omega_{t+j})\right\}. \tag{5.27}$$

Existence is guaranteed under the sufficient condition $\forall \omega: \ \omega\underline{p} \ge b(\omega)$ and $s[1 - \delta(\omega)] \ge 1$.

The proof, in Appendix Appendix C, simply proceeds through forward substitution and induction, and establishes also that this limit exists. While we have not been able to derive conditions on parameters that are both necessary and sufficient for equilibrium existence (i.e. for $(\mu^\star, U^\star)$ to be a RPE), this is not an issue in applications. In fact, we proved that there is only one possible equilibrium set of contracts, that we can compute (see MPV10a) and then check ex post whether in fact it satisfies all equilibrium conditions.

17. Again using (5.22), this expression can be related directly to the solution for the wage in Bontemps et al. (2000) steady state model, Equation (15), where the offered value is a capitalized constant wage, the flow of hires equals firm size times the separation rate, and the costate simply equals productivity p.

18. Note the slight abuse of notation in (5.27), where p is used both as notation for the function $(\omega, p) \mapsto \omega p$ (itself an argument of $\mathbf{T}^j$ in the r.h.s.), and as the argument of $\binom{\mu_t^\star}{U_t^\star}$ in the l.h.s. and of $T^j[\omega p, b(\omega)]$ in the r.h.s.

6. ENDOGENOUS JOB CREATION

6.1. *Restating the firm's problem*

We have treated job-contact probabilities as exogenous, albeit state-dependent, objects. We now consider the natural extension of our model consisting of endogenizing them through a matching function.[19] Specifically, we now assume that, each period, before workers have a chance to search, a firm can post $a \geq 0$ job adverts (vacancies), or spend hiring effort $a \geq 0$, at a cost $c(a)$, with $c(\cdot)$ positive, strictly increasing and convex, continuously differentiable. Own hiring effort determines the firm's sampling weight in workers' job search, while total hiring effort determines the rate at which an advert returns contacts with workers. Specifically, consider a firm of current size L_t posting a value $W_{t+1} \geq U_{t+1}$ and a_{t+1} adverts in aggregate state (ω_{t+1}, N_t). The analogue of (5.8) characterizes that firm's size in the following period:

$$L_{t+1} = L_t (1 - \delta_{t+1}) \left(1 - s\lambda_{t+1} \overline{F}_{t+1} (W_{t+1})\right) + \eta_{t+1} a_{t+1} P_{t+1} \qquad (6.28)$$

where η_t is the rate at which adverts contact workers, λ_t is again the job contact rate, and

$$P_{t+1} = \frac{\lambda_{t+1} [1 - N_t (\overline{p})] + s\lambda_{t+1} (1 - \delta_{t+1}) N_t (\overline{p}) G_{t+1} (W_{t+1})}{\lambda_{t+1} [1 - N_t (\overline{p})] + s\lambda_{t+1} (1 - \delta_{t+1}) N_t (\overline{p})} \qquad (6.29)$$

is the chance that the offered value W_{t+1} is acceptable to a random job-seeker who makes contact with the adverts posted by this firm. In (6.29), the denominator is the measure of workers who make contact, and the numerator counts only those who accept the offer, namely all the unemployed and only the fraction of employed who will earn less than W_{t+1} by staying where they are.

All contact rates are now endogenous and related trough an aggregate matching function, as follows. First, let $a_t(p)$ denote the adverts posted on the equilibrium path by a firm of productivity p, size $L_{t-1}(p)$, in aggregate state ω_t, N_{t-1}, and define aggregate hiring effort A_t and aggregate search effort by workers Y_t as

$$A_t = \int_{\underline{p}}^{\overline{p}} a_t (p) \, d\Gamma (p)$$
$$Y_t = 1 - N_{t-1} (\overline{p}) + s (1 - \delta_t) N_{t-1} (\overline{p}) \qquad (6.30)$$

the latter adding the previously unemployed to the previously employed who are not displaced and draw a chance to search this period t. In each time period, employed and unemployed search simultaneously. Then:

$$\eta_t A_t = \lambda_t Y_t = m (A_t, Y_t) \qquad (6.31)$$

where $m(\cdot)$ is a constant-return-to-scale matching function, increasing and concave in both of its arguments, and such that $m(A, Y) \leq \min(A, Y)$. Finally, the distributions of offered and accepted worker values, F_t and G_t, are defined as before — see (5.5) and (5.6) — except that firm sampling is no longer uniform and each firm now has a sampling weight in F equal to its (normalized) hiring effort, a/A. Thus $F_t (W_t) = \frac{1}{A_t} \int_{\underline{p}}^{\overline{p}} \mathbb{I} \{V_t (p) \leq W\} a_t (p) \, d\Gamma (p)$.

As before, the best-response contract and hiring effort can be characterize as the solution to the unconstrained, recursive maximization of the joint value of the firm-worker

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19. We maintain the assumption of a exogenous, state-dependent job destruction rate, $\delta_t = \delta (\omega_t)$.

collective. That problem, formally expressed in (5.14) in the exogenous-contact-rate case, now becomes:

$$S_t = \omega_t p L_t + \beta \mathbf{E}_t \left[\delta_{t+1} U_{t+1} L_t + \sup_{W_{t+1} \geq U_{t+1}, a_{t+1}} \left\langle S_{t+1} - c(a_{t+1}) \right. \right.$$
$$\left. \left. + (1 - \delta_{t+1}) s \lambda_{t+1} L_t \int_{W_{t+1}}^{+\infty} v dF_{t+1}(v) - W_{t+1} \eta_{t+1} a_{t+1} P_{t+1} \right\rangle \right] \quad (6.32)$$

subject to (6.28), (6.29), (6.30), (6.31), as well as (5.7), (5.10) which are still valid.

In the next sub-section, we generalize Proposition 3, our main characterization result which states that all Markov equilibria of our model labor market must be Rank-Preserving. Before we do so, however, we note that Proposition 1, stating that F_t and G_t are atomless and have a common compact and convex support still holds, as its proof applies verbatim to the new environment considered in this section.

6.2. Rank-Preserving Equilibrium

Considering RPE, as defined in Definition 2, in our new environment, we see that, while G_t continues to satisfy (5.16), the implied restriction on F_t needs to be amended to take the new endogenous sampling weights into account. In a RPE:

$$F_t(V_t(p)) = \frac{1}{A_t} \int_{\underline{p}}^{p} a_t(x) d\Gamma(x). \quad (6.33)$$

The probability of drawing an offer worth $V_t(p)$ or less is equal to the fraction of adverts posted by firms of productivity p or less, a fraction which is now endogenous and responds to changes in the firms' and aggregate circumstances. Despite this complication, RPE still have the appealing property that, given the distribution of firm hiring effort, labor reallocation is efficient in the sense that workers always move up the productivity ladder. Yet, while the direction of worker turnover is efficient in RPE, its extent needs not be, because of standard externalities inherent in the matching function.

The main result of this section is the following generalization of our main characterization result to an environment with endogenous recruitment effort:

Proposition 6 (Ranked initial firm size implies RPE). *Assume each firm controls hiring effort a at convex cost $c(a)$. Then Proposition 3 holds. Furthermore, in any RPE, more productive firms spend more hiring effort: $a_t(p) = a(p, L_{t-1}(p), \omega_t, N_{t-1})$ is increasing in p.*

The proof of Proposition 6 follows the same lines as that of Proposition 2 and is available in Section 3 of the online appendix. As before, a best-response offered value increasing in own productivity p and size L_t then implies that any Markov equilibrium must be RP, under appropriate initial conditions. Propositions 6 and 2-3 also share the same economic intuition. The firm now has two tools at its disposal to recruit workers, promised value and hiring effort. The proof of Proposition 6 establishes that the dynamic single-crossing property that we uncovered in the value function of the optimal contract-posting problem implies that not only the value offered to the worker, but also the intensity of hiring effort, increase with firm productivity and size. This result only reinforces the mechanism that gives rise to RPE.

We now return to a major motivation of our exercise, namely MPV12's facts. To explain them in the extended model with endogenous hiring effort, we cannot appeal to the planner solution as we did in the simpler model. Because of standard, and well-understood, congestion effects, equilibrium hiring effort is not generally constrained efficient. So equilibrium turnover is efficient in its direction — workers always move up the productivity ladder — but not necessarily in its pace. The exogenous contact rate model, however, is a special case of the extended model. To see why, choose a matching functions linear in aggregate search effort and an appropriate piece-wise linear cost of hiring effort, both easily accommodated by our proofs. Then in RPE all firms will choose to spend the same level of hiring effort that only depends on the level of aggregate productivity. With a linear matching function, the contact rate is independent of unemployment, and we are back to the previous model. Therefore, the extended model is flexible enough to be qualitatively consistent with MPV12's facts just like the simpler model, and quantitatively it can only do better.

6.3. *Further characterization of RPE with endogenous job creation*

Before we conclude, let us re-emphasize that Proposition 6 is, just like Proposition 3, only a characterization result, which neither requires nor proves existence or uniqueness of equilibrium. We learn that, in order to understand the predictions of equilibrium for observed behavior, we need look no further than monotonic allocations, such that any firm's offered value and hiring effort are increasing in productivity and size. From an operational viewpoint, Proposition 6 also provides the grounds on which to build a simulation algorithm for our model. We provide a sketch in this sub-section: solving the resulting dynamic system is a difficult computational challenge, which goes beyond the scope of this paper, and we will tackle it in future research.

We establish in Section 3 of the online appendix that the value offered to workers along the RPE path, $p \mapsto V_t(p) := V(p, L_{t-1}^\star(p), \omega_t, N_{t-1}^\star)$, is continuous in p, and that the joint value $S_t = S(p, L, \omega_t, N_t^\star)$ of a firm of type p and of its L employees is differentiable in L along the RPE path. The latter property allows us to define the costate variable $\mu_t(p) = \frac{\partial S}{\partial L}(p, L^\star(p), \omega_t, N_t^\star)$ as in the exogenous contact rate case, and to write the following Envelope condition w.r.t. firm size:

$$\mu_t(p) = \omega_t p + \beta \mathbf{E}_t \left[\delta_{t+1} U_{t+1} + (1 - \delta_{t+1}) \left(1 - \frac{s \lambda_{t+1}}{A_{t+1}} \int_p^{\overline{p}} a_{t+1}(x) \, d\Gamma(x) \right) \mu_{t+1}(p) \right.$$

$$\left. + (1 - \delta_{t+1}) \frac{s \lambda_{t+1}}{A_{t+1}} \int_p^{\overline{p}} a_{t+1}(x) V_{t+1}(x) \, d\Gamma(x) \right]. \quad (6.34)$$

Next, we show in Section 3 of the online appendix that, because $V_t(p)$ is increasing, thus a.e. differentiable, the same NFOC (5.23) as in the case of exogenous contact rates must hold outside of a zero-measure set of values of p on the RPE path, with the flow of hires defined as (5.22) but now with contact rate λ_t endogenized through a matching function and the new RPE employment distribution N_t.[20]

Finally, the firm's optimal choice of hiring effort implies a second NFOC:

$$c'(a_t(p)) = [\mu_t(p) - V_t(p)] \cdot \frac{\lambda_t}{A_t} H_t(p). \quad (6.35)$$

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20. The slight loss compared to the exogenous contact rate case is that the NFOC for V now holds a.e., instead of holding for all $p \in \left[\underline{p}, \overline{p}\right]$. The reason is that $V_t(p)$ may not be everywhere differentiable.

Together with the law of motion of employment, equations (6.34), (5.23) and (6.35) are the backbone of a simulation algorithm, the principle of which is as follows:

1. guess a path for $a_t(p)$ for all t and p, increasing in p for each t;
2. use the matching technology (6.30) and (6.31) to compute A_t and λ_t for all t;
3. use the law of motion of employment (6.28)-(6.29), specialized to RPE using (5.16) and (6.33), to compute $N_t(p)$, and then use (5.22) to compute $H_t(p)$, for all t and p;
4. use (5.23), (6.34) and the Bellman equation for the value of unemployment[21] to compute $V_t(p)$, $\mu_t(p)$ and U_t;
5. use (6.35) to update the guess for $a_t(p)$.

7. DISCUSSION

7.1. *Contracts*

There is no unique way to generalize the steady-state BM model to an environment that is subjected to aggregate shocks. Our proposed extension of BM's model features contracts that implement the efficient allocation (at given job contact rates) and generate equilibrium dynamics that preserves all desirable properties from BM, but also explain MPV12's facts. We now revisit some of our assumptions, both to explore the robustness of our theoretical results and to prepare the ground for future research.

While our proposed contracts are "general"[22] within the bounds imposed by commitment and the equal treatment constraint, the RP property also emerges in more restricted contract-posting games. For example, without equal treatment, by constant returns to scale in production the firm treats each job independently of the others, as in the standard search-and-matching model, and maximizes the profits that each job produces. Then, equilibrium contracts are simpler, size-independent. The offered value does not determine the dynamics of firm size, but rather the probability that the job is filled. Our proofs can be easily adapted to this special case, to establish that offered values are increasing in firm productivity. Thus any Markov equilibrium is RP. In this case, however, the model loses any meaningful notion of firm size: as just stated, without equal treatment, there is nothing to tie up jobs together, and the distribution of employment into 'firms' of equal productivity p is irrelevant. (In equilibrium, of course, jobs with higher p will offer more, and employ more workers in total, but the size-wage relationship will only result from a correlation due to a common latent factor, productivity). Under equal treatment, a firm is defined not only by its productivity p, but also by a wage policy. If a firm of type p deviates and accumulates a different size than another firm of the same type p, then offered values will differ between the two firms. Thus, the model yields a well-defined notion of firm size, and can generate predictions about the cyclical dynamics of the firm size distribution, that we document empirically in MPV12.

If we dispose of both the Markov and the equal treatment requirements, equilibrium contracts under commitment are even simpler. As is standard, to solve the moral hazard problem of the worker, who cannot commit not to accept outside offers, the firm maximizes the joint value by 'selling the job to the worker', i.e. setting the wage equal to output after the first period, in exchange for a very low, possibly negative,

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21. $U_t = b(\omega_t) + \beta \mathbf{E}_t \left[(1 - \lambda_{t+1}) U_{t+1} + \frac{\lambda_{t+1}}{A_{t+1}} \int_{\underline{p}}^{\overline{p}} a_{t+1}(x) V_{t+1}(x) d\Gamma(x) \right]$.

22. In the sense that they are conditioned on the largest possible state space consistent with fairly standard assumptions about observability and the Markov requirement.

initial wage, or lump-sum transfer. Note, however, that this moral hazard problem has additional ramifications in equilibrium: outside offers promise less than their own expected discounted output (because of the initial low wage required to 'buy' the firm), so transferring the whole output to the worker from the second period onward does not give rise to efficient turnover. A worker, after 'buying' a job and enjoying its full flow productivity, will not be willing to quit and buy a new, slightly more productive job, because of the non-negligible upfront cost. In this case, privately efficient contracts are socially inefficient.

Finally, while the assumption of commitment to state-contingent wages is standard in the literature, where it is well understood that commitment may both be beneficial to the firm and sustainable in long term employment relationships, that assumption could nonetheless be relaxed in many different ways (for example, the firm may be allowed a choice of whether or not commit to a specific strategy as in Postel-Vinay and Robin, 2004, or it may only be committed to end-of-period payments with an equal-treatment constraint as in CM, or without), opening as many interesting avenues for future research.

7.2. *Entry, exit and productivity shocks*

The conditions for uniqueness and RP property of equilibrium are only sufficient and can therefore be relaxed to an extent. However it is also possible to modify the economy so that those conditions fail and our unique, efficient RPE breaks down. One of the key restrictions here is that we have treated firm productivity as a fixed, time-invariant parameter. In turn, this allows for entry and exit of firms only "at the bottom", whereas the least productive firms may be temporarily inactive, when aggregate productivity is low. Shocks to firm productivity create obvious issues for RPE, as a very large and productive firm may suddenly become unproductive, and then face contrasting incentives to offer its employees a high value, its sheer size and retention needs against low productivity. Similarly, highly profitable business opportunities may arise and cause entry of highly productive firms, which by definition start out with a size of zero.

Clearly a similar phenomenon occurs when the productivity of *existing* firms is subject to idiosyncratic shocks. The relevant empirical questions, then, are (i) how much do entry and exit of firms and establishments contribute to aggregate employment fluctuations, and (ii) how variable is a typical firm's productivity at business cycle frequencies. To answer the first question, it is well known that the contribution of entry and exit to the cyclical volatility of employment is modest. The Business Employment Dynamics from the Bureau of Labor Statistics provide the relevant data on job flows in the US at quarterly frequency for 1992:Q3-2011:Q3. The standard deviation of the HP-filtered net employment growth rate is 0.57. The analogous statistic equals 0.1 for: the net percentage increase in employment due to opening establishments, the net percentage decrease due to closing establishments, and the difference between the two (entry minus exit). For the second question, the available empirical evidence using longitudinal business microdata is mostly limited to the manufacturing sector. Summarizing the results of the early literature, Bartelsman and Doms (2000) conclude that firm-level productivity is best characterized as a unit-root process. Haltiwanger et al. (2008, Table 3) find that establishment-level TFP is very persistent, about as much as aggregate TFP. MPV12 show with data from a few countries that several correlated features of a firm, such as its size, the average wages it pays, and its revenue-based productivity, when measured at one point in time strongly predict how job creation by the same firm responds to business cycle shocks that hit it *over two decades later*. This striking phenomenon suggests that

our assumption of fixed firm productivity might be a reasonable approximation for our purposes, at least at business cycle frequencies. We note that the assumption of fixed and heterogeneous firm-level TFP has become commonplace in the International Trade literature (Eaton and Kortum 2002, Melitz 2003).

From a theoretical standpoint, however, the question remains whether one can analyze aggregate dynamics in wage-posting models in the presence of idiosyncratic shocks to firm productivity. As discussed in Section 2, to answer this question, CM assume a specific hiring technology, which makes contracts size-independent. In contrast, models that allow ex-post competition between employers for employed workers always implement efficient turnover, as those models all share the auction-flavored property that the employer with the highest valuation of the worker's services (i.e., the most productive employer) always succeeds in hiring/retaining the worker.[23] These models, however, do not have a well-defined notion of firm size.

7.3. *Other determinants of firm size*

Finally, we are aware that multiple factors, beyond employment frictions, contribute to determine the size of a firm, most notably capital adjustment costs, including financial frictions, and diminishing marginal revenues from hiring, due to either technology (decreasing returns to labor), span of control frictions, or price-making power. Diminishing returns in wage-posting models have been partially explored in a steady state context (Manning, 1992), and can invalidate some of the equilibrium properties, such as the absence of atoms in the offer distribution. We cannot identify, though, obvious reasons why they would overturn the main result that equilibrium must be RP. To violate this property, a more productive firm would have to optimally hire so many more workers as to drive its marginal revenue of labor below that of a less productive firm. This is another avenue for future investigation.

8. CONCLUSION

This paper is the first to characterize stochastic equilibrium of an economy where the Law of One Price fails due to random search frictions and monopsony power, a problem that was long held to be intractable. Specifically, we introduce aggregate productivity shocks in a wage-posting model a la Burdett and Mortensen (1998), and we allow for rich state-contingent employment contracts. By extending the theory of Monotone Comparative Statics to a Dynamic Programming environment, we identify sufficient conditions under which the equilibrium is unique, constrained efficient, and very tractable. The second best is decentralized by contracts that do not respond to outside offers. The equilibrium stochastic dynamics of this model economy exhibit qualitative properties that are in line with the new business cycle facts that we illustrate in Moscarini and Postel-Vinay (2009, 2010b, 2012): small firms as a group exhibit less cyclical net job creation and returns to capital than large firms.

Future research will pursue a full quantitative analysis of this model, to illustrate its practicality as a tool for business cycle analysis. In MPV10a we illustrate an algorithm to solve quickly and efficiently for equilibrium with exogenous contact rates, and we present some preliminary quantitative results. The constrained efficient allocation in the

23. Postel-Vinay and Robin (2002), Dey and Flinn (2005), Cahuc, Postel-Vinay and Robin (2006) — see also Postel-Vinay and Turon (2010) for an example with idiosyncratic productivity shocks.

stochastic economy already naturally explains why larger firms have more cyclical net job creation. We believe that the slow propagation of aggregate shocks to average labor productivity and wages, due to the slow upgrading of labor through job-to-job quits, is an important feature of actual business cycles which is missing altogether from existing quantitative business cycle models. The extensions mentioned above, as well as possibly others, are bound to help the quantitative performance of the model.

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APPENDIX

In order to prove the main statements, we heavily rely on the recursive formulation of the contracting problem. Therefore, we need to spell out the notation, including all arguments of the value and policy functions. We drop the time index and use primes to denote next period's values. We also make the dependence on state variables p, L, ω, N explicit, so in the candidate equilibrium a firm of type p offers a value $V(p, L(p), \omega', N)$. A subscript of a function now denotes the variable of partial differentiation of that function. Let $\lambda^\omega \in (0,1)$ denote the chance of contact with a hiring firm in current state of aggregate productivity ω, $\delta^\omega \in (0,1)$ the chance of exogenous separation, b^ω the flow unemployment benefit, $F(W \mid \omega', N)$ the offer distribution given that all firms follow the Markov offer strategy V, $G(W \mid \omega', N)$ the value distribution. In fully recursive notation, the laws of motion of firm size (5.8) and employment distribution (5.10) become

$$L' = \mathscr{L}(L, W, \omega', N) := L\left(1 - \delta^{\omega'}\right)\left(1 - s\lambda^{\omega'}\overline{F}(W \mid \omega', N)\right)$$
$$+ \lambda^{\omega'}[1 - N(\overline{p})]\,\mathbb{I}\{W \geq U(\omega, N)\} + s\lambda^{\omega'}(1 - \delta^{\omega'})N(\overline{p})G(W \mid \omega', N)$$

$$N(p \mid \omega', N)' = \int_{\underline{p}}^{p} \mathscr{L}(L(x), V(x, L(x), \omega', N), \omega', N)\,d\Gamma(x) \Rightarrow N(\cdot \mid \omega', N)' := \mathscr{N}(\omega', N)$$

the Bellman equation of the unemployed worker (5.7) reads

$$U(\omega, N) = b^\omega + \beta \int_\Omega \Bigg[\left(1 - \lambda^{\omega'}\right) U\left(\omega', \mathscr{N}(\omega', N)\right)$$
$$+ \lambda^{\omega'} \int \max\left\langle v, U\left(\omega', \mathscr{N}(\omega', N)\right)\right\rangle dF\left(v \mid \omega', N\right) \Bigg] H\left(d\omega' \mid \omega\right)$$

and the key DP problem (5.14) reads

$$S(p, L, \omega, N) = \omega p L + \beta \int_\Omega \Bigg\{ \delta^{\omega'} U\left(\omega', \mathscr{N}(\omega', N)\right) L$$
$$+ \sup_{W(\omega') \geq U(\omega', \mathscr{N}(\omega', N))} \Bigg\langle S\left(p, \mathscr{L}(L, W(\omega'), \omega', N), \omega', \mathscr{N}(\omega', N)\right)$$
$$+ L\left(1 - \delta^{\omega'}\right) s\lambda^{\omega'} \int_{W(\omega')}^{+\infty} v\,dF\left(v \mid \omega', N\right)$$
$$- W(\omega')\left[\lambda^{\omega'}\left(1 - N(\overline{p})\right) + s\lambda^{\omega'}\left(1 - \delta^{\omega'}\right)N(\overline{p})G\left(W(\omega') \mid \omega', N\right)\right]\Bigg\rangle \Bigg\} H\left(d\omega' \mid \omega\right). \quad (36)$$

APPENDIX A. PROOF OF PROPOSITION 2

The claim is: if problem (36) has a solution, then any measurable selection $V(p, L, \omega, N)$ from the optimal correspondence is such that $V(p, L, \omega, N)$ is increasing in p and L.

Our proof strategy is as follows. First, we define certain supermodularity properties SM of a value function that imply that the maximizer V in (36) is increasing in p. Then, we fix an arbitrary N and show that the Bellman operator in (36) for the restricted problem with fixed N is a contraction mapping from the space of SM functions into itself, and that this space is Banach and closed under the sup norm. Therefore, for any fixed N (36) has a unique solution. Finally, *if* there exists a solution S to (36) when N is not fixed, then S must also solve the restricted problem (36) for any fixed N. By uniqueness and SM of the solution to the restricted problem any solution to the unrestricted problem must also have the SM properties. We cannot extend the same logic to show existence of S with variable N because Blackwell's sufficient conditions for a contraction mapping apply only to functions over $\mathbb{R}^n$.

We introduce the following notation:

$$\varphi\left(p, L, \omega, N\right) := \omega p L + \beta \int_{\Omega} \delta^{\omega'} U\left(\omega', \mathcal{N}\left(\omega', N\right)\right) L Q\left(d\omega' \mid \omega\right),$$

$$\Phi\left(L, W\left(\omega'\right), \omega', N\right) := L\left(1 - \delta^{\omega'}\right) s \lambda^{\omega'} \int_{W(\omega')}^{+\infty} v dF\left(v \mid \omega', N\right)$$
$$- W\left(\omega'\right)\left[\lambda^{\omega'}\left(1 - N\left(\overline{p}\right)\right) + s\lambda^{\omega'}\left(1 - \delta^{\omega'}\right) N\left(\overline{p}\right) G\left(W\left(\omega'\right) \mid \omega', N\right)\right].$$

So fix N to be some given positive and increasing function over $\left[\underline{p}, \overline{p}\right]$ with $N\left(\overline{p}\right) = 0$. Then, for any function $\mathscr{S}\left(p, L, \omega\right)$, we define the following operator $\mathbf{M^N}$:

$$\mathbf{M^N}\mathscr{S}\left(p, L, \omega\right) := \varphi\left(p, L, \omega, N\right)$$
$$+ \beta \int_{\Omega} \max_{W(\omega')} \left\langle \mathscr{S}\left[p, \mathscr{L}\left(L, W\left(\omega'\right), \omega', N\right), \omega'\right] + \Phi\left(L, W\left(\omega'\right), \omega', N\right)\right\rangle Q\left(d\omega' \mid \omega\right). \quad \text{(A37)}$$

The worker participation constraint $W \geq U$ can be ignored in the proof. To see why, observe the following. Once we establish that an interior solution is weakly increasing in p, we can conclude that any set of firms that offers a corner solution $W = U$ and shuts down must be the set of the least productive firms. But then, the global solution, including the corner, is weakly increasing in p, strictly so above the corner by Proposition 1, as claimed. Incidentally, if all firms offered U, from the previous reasoning (and barring the trivial case where all firms are too unproductive to operate) the most productive firms would deviate and profitably offer more, so there exist always some firms that have an interior solution where the worker participation constraint does not bind.

Lemma A1. *Let $\mathscr{S}\left(p, L, \omega\right)$ be bounded, continuous in p and L, increasing and convex in L and with increasing differences in $\left(p, L\right)$ over $\left(\underline{p}, \overline{p}\right) \times \left(0, 1\right)$. Then:*

1. *$\mathbf{M^N}\mathscr{S}$ is bounded and continuous in p and L;*

2. *There exists a measurable selection $V\left(p, L, \omega, N\right)$ from the maximizing correspondence associated with $\mathbf{M^N}\mathscr{S}$;*

3. *Any such measurable selection V is increasing in p and L;*

4. *$\mathbf{M^N}\mathscr{S}$ is increasing and convex in L and with increasing differences in $\left(p, L\right)$ over $\left(\underline{p}, \overline{p}\right) \times \left(0, 1\right)$.*

Proof. In this proof, wherever possible without causing confusion, we will make the dependence of all functions on aggregate state variables ω and N implicit to streamline the notation.

Points 1 and 2 of this lemma are immediate: continuity of $\mathbf{M^N}\mathscr{S}$ follows from Berge's Theorem (Stokey and Lucas, 1989, Theorem 3.6). Boundedness of $\mathbf{M^N}\mathscr{S}$ is obvious by construction. Existence of a measurable selection from the maximizing correspondence associated with $\mathbf{M^N}\mathscr{S}$ is a direct consequence of the Measurable Selection Theorem (Stokey and Lucas, 1989, Theorem 7.6).

To prove point 3, we first establish that the maximand in (A37) has increasing differences in $\left(p, W\right)$ and $\left(L, W\right)$. Monotonicity of V in p and L will then

follow from standard monotone comparative statics arguments.[24] Proving that the maximand in (A37) has increasing differences in (p, W) is immediate as Φ is independent of p: letting $\tau > 0$, differences in p of the maximand equal $\mathscr{S}(p + \tau, \mathscr{L}(L, W)) - \mathscr{S}(p, \mathscr{L}(L, W))$ which is increasing in W because $\mathscr{L}$ is increasing in W by construction and $\mathscr{S}$ has increasing differences in (p, L) by assumption. We thus now fix p and focus on establishing that the maximand in (A37) has increasing differences in (L, W). To this end, first note that, since $\mathscr{S}$ is assumed to be continuous and convex in L, it has left and right derivatives everywhere (and those two can at most differ at countably many points). Now take L and $\varepsilon > 0$ and define the difference in L of the maximand in (A37):

$$
\mathscr{D}(W) := \mathscr{S}(p, \mathscr{L}(L + \varepsilon, W)) - \mathscr{S}(p, \mathscr{L}(L, W)) \\
+ \varepsilon (1 - \delta^{\omega'}) s \lambda^{\omega'} \int_W^{+\infty} v \, dF(v \mid \omega').
$$

(The dependence of $\mathscr{D}$ on p is kept implicit.) We want to establish that $\mathscr{D}(W)$ is increasing in W. We do not know whether F and G, thus $\mathscr{D}$, are differentiable, so we proceed by showing that the upper-right Dini derivative of $\mathscr{D}(W)$, which we denote as $D^+ \mathscr{D}(W)$ and which exists everywhere (although possibly equalling $\pm\infty$), is everywhere positive. Take $x > 0$:

$$
\mathscr{D}(W + x) - \mathscr{D}(W) \\
= \mathscr{S}(p, \mathscr{L}(L + \varepsilon, W + x)) - \mathscr{S}(p, \mathscr{L}(L + \varepsilon, W)) \\
- [\mathscr{S}(p, \mathscr{L}(L, W + x)) - \mathscr{S}(p, \mathscr{L}(L, W))] \\
- \varepsilon(1 - \delta^{\omega'}) s \lambda^{\omega'} \int_W^{W+x} v \, dF(v \mid \omega').
$$

Majorizing the last integral:

$$
\mathscr{D}(W + x) - \mathscr{D}(W) \geq \mathscr{S}(p, \mathscr{L}(L + \varepsilon, W + x)) - \mathscr{S}(p, \mathscr{L}(L + \varepsilon, W)) \\
- [\mathscr{S}(p, \mathscr{L}(L, W + x)) - \mathscr{S}(p, \mathscr{L}(L, W))] \\
- \varepsilon(1 - \delta^{\omega'}) s \lambda^{\omega'} (W + x) [F(W + x \mid \omega') - F(W \mid \omega')].
$$

Dividing through by x and taking the limit superior as $x \to 0^+$ (using the definition of $\mathscr{L}$, continuity of F and G, and some basic properties of Dini

24. See Section 4 of the online appendix for details.

derivatives), we obtain:

$$D^+ \mathscr{D}(W) \geq$$

$$\mathscr{S}_{L,r}(p, \mathscr{L}(L+\varepsilon, W)) \cdot s\lambda^{\omega'}(1-\delta^{\omega'}) \left\{ (L+\varepsilon) D^+ F(W \mid \omega') + N(\overline{p}) D^+ G(W \mid \omega') \right\}$$

$$- \mathscr{S}_{L,r}(p, \mathscr{L}(L, W)) \cdot s\lambda^{\omega'}(1-\delta^{\omega'}) \left\{ L D_+ F(W \mid \omega') + N(\overline{p}) D_+ G(W \mid \omega') \right\}$$

$$- \varepsilon(1-\delta^{\omega'}) s\lambda^{\omega'} W D_+ F(W \mid \omega'),$$

where, in standard fashion, $f_{x,\ell}$ [$f_{x,r}$] is used to designate the left [right] partial derivative of any function f w.r.t. x, and $D_+ f$ denotes the lower-right Dini derivative of f. Because F and G are increasing, their Dini derivatives are such that $D^+ F \geq D_+ F \geq 0$ (and likewise for G). Because $\mathscr{S}$ is convex in L by assumption, $\mathscr{S}_{L,r}$ is increasing in L. Combining all those properties, the latter inequality implies:

$$D^+ \mathscr{D}(W) \geq [\mathscr{S}_{L,r}(p, \mathscr{L}(L, W)) - W] \cdot (1-\delta^{\omega'}) s\lambda^{\omega'} \varepsilon D_+ F(W \mid \omega'). \quad \text{(A38)}$$

The only way the RHS in this last inequality can be negative is if $\mathscr{S}_{L,r}(p, \mathscr{L}(L, W)) - W < 0$. We now show that this cannot be if W is an optimal selection. Let V be an optimal selection and let $x > 0$. Optimality requires that:

$$0 \geq \mathscr{S}(p, \mathscr{L}(L, V-x)) + L(1-\delta^{\omega'}) s\lambda^{\omega'} \int_{V-x}^{+\infty} v dF(v \mid \omega')$$

$$- (V-x) \left(\lambda^{\omega'}(1-N(\overline{p})) + s\lambda^{\omega'}(1-\delta^{\omega'}) N(\overline{p}) G(V-x \mid \omega') \right)$$

$$- \mathscr{S}(p, \mathscr{L}(L, V)) - L(1-\delta^{\omega'}) s\lambda^{\omega'} \int_{V}^{+\infty} v dF(v \mid \omega')$$

$$+ V \left(\lambda^{\omega'}(1-N(\overline{p})) + s\lambda^{\omega'}(1-\delta^{\omega'}) N(\overline{p}) G(V \mid \omega') \right).$$

Collecting terms and majorizing the integral terms as we did for $\mathscr{D}$:

$$0 \geq S(p, \mathscr{L}(L, V-x)) - \mathscr{S}(p, \mathscr{L}(L, V))$$

$$+ L(1-\delta^{\omega'}) s\lambda^{\omega'}(V-x) \left[F(V-x \mid \omega') - F(V \mid \omega') \right]$$

$$- V s\lambda^{\omega'}(1-\delta^{\omega'}) N(\overline{p}) \left[G(V-x \mid \omega') - G(V \mid \omega') \right]$$

$$+ x \left(\lambda^{\omega'}(1-N(\overline{p})) + s\lambda^{\omega'}(1-\delta^{\omega'}) N(\overline{p}) G(V \mid \omega') \right).$$

Now again taking the limit superior as $x \to 0^+$ (in what follows $D^- F$ and $D_- F$ designate the upper and lower left Dini derivative of F, respectively, and likewise

for G):[25]

$$0 \geq -\mathscr{S}_{L,\ell}\left(p, \mathscr{L}\left(L, V\right)\right) \cdot s\lambda^{\omega'}(1-\delta^{\omega'})\left\{LD_-F\left(V \mid \omega'\right) + N\left(\overline{p}\right)D_-G\left(V \mid \omega'\right)\right\}$$
$$+ V s\lambda^{\omega'}(1-\delta^{\omega'})\left\{LD^-F\left(V \mid \omega'\right) + N\left(\overline{p}\right)D^-G\left(V \mid \omega'\right)\right\}$$
$$+ \lambda^{\omega'}\left(1 - N\left(\overline{p}\right)\right) + s\lambda^{\omega'}(1-\delta^{\omega'})N\left(\overline{p}\right)G\left(V \mid \omega'\right).$$

Finally recalling that $D^-F \geq D_-F \geq 0$ (and likewise for G), the latter inequality implies:

$$\mathscr{S}_{L,\ell}\left(p, \mathscr{L}\left(L, V\right)\right) - V$$
$$\geq \frac{\lambda^{\omega'}\left(1 - N\left(\overline{p}\right)\right) + s\lambda^{\omega'}(1-\delta^{\omega'})N\left(\overline{p}\right)G\left(V \mid \omega'\right)}{s\lambda^{\omega'}(1-\delta^{\omega'})\left\{LD_-F\left(V \mid \omega'\right) + N\left(\overline{p}\right)D_-G\left(V \mid \omega'\right)\right\}} \geq 0. \quad \text{(A39)}$$

This, together with (A38), shows that $D^+\mathscr{D}\left(V\right) \geq 0$ at all V which is an optimal selection, i.e. at all V in the support of F. To finally establish that $\mathscr{D}$ is increasing over the support of F, recall that, as F and G are continuous by Proposition 1, so is $W \mapsto \mathscr{L}\left(L, W\right)$. Moreover, as $\mathscr{S}$ is convex in L (by assumption), it is continuous w.r.t. L. Thus by inspection, $\mathscr{D}$ is a continuous function of W. Continuity plus the fact that $D^+\mathscr{D}\left(V\right) \geq 0$ are sufficient to ensure that $\mathscr{D}$ is strictly increasing (see, e.g., Proposition 2 p99 in Royden, 1988). Point 3 of the lemma is thus proven.

Now on to point 4. Take $(p_0, L_0) \in \left(\underline{p}, \overline{p}\right) \times (0,1)$ and $\varepsilon > 0$ such that $(p_0 + \varepsilon, L_0 + \varepsilon)$ are still in $\left(\underline{p}, \overline{p}\right) \times (0,1)$. We first consider right-differentiability of $\mathbf{M^N}\mathscr{S}$ w.r.t. L at L_0. Fixing an arbitrary selection V from the optimal policy correspondence, we note that, while V may have a discontinuity at (p_0, L_0), the fact that it is increasing in L ensures that $V\left(p_0, L_0^+, \omega'\right) := \lim_{\varepsilon \to 0^+} V\left(p_0, L_0 + \varepsilon, \omega'\right)$ exists everywhere (and likewise for $V\left(p_0^+, L_0, \omega'\right)$). By

---

25. This uses the facts that $\mathscr{S}_{L,\ell} \geq 0$, that F and G are continuous, and that $D^-\left[-f\right] = -D^-f$ for any function f.

point 3, $V\left(p_0, L_0^+, \omega'\right)$ is increasing in L_0. Then:

$$
\begin{aligned}
\mathbf{M^N}\mathscr{S}\left(p_0, L_0 + \varepsilon\right) &- \mathbf{M^N}\mathscr{S}\left(p_0, L_0^+\right) = \varphi\left(p_0, L_0 + \varepsilon\right) - \varphi\left(p_0, L_0\right) \\
&+ \beta \int_\Omega \Big\langle \mathscr{S}\left[p_0, \mathscr{L}\left(L_0 + \varepsilon, V\left(p_0, L_0 + \varepsilon, \omega'\right)\right)\right] - \mathscr{S}\left[p_0, \mathscr{L}\left(L_0, V\left(p_0, L_0^+, \omega'\right)\right)\right] \\
&\quad + \Phi\left(L_0 + \varepsilon; V\left(p_0, L_0 + \varepsilon, \omega'\right)\right) - \Phi\left(L_0; V\left(p_0, L_0^+, \omega'\right)\right) \Big\rangle Q\left(d\omega'|\omega\right) \\
\geq \varphi\left(p_0, L_0 + \varepsilon\right) &- \varphi\left(p_0, L_0\right) \\
&+ \beta \int_\Omega \Big\langle \mathscr{S}\left[p_0, \mathscr{L}\left(L_0 + \varepsilon, V\left(p_0, L_0^+, \omega'\right)\right)\right] - \mathscr{S}\left[p_0, \mathscr{L}\left(L_0, V\left(p_0, L_0^+, \omega'\right)\right)\right] \\
&\quad + \Phi\left(L_0 + \varepsilon; V\left(p_0, L_0^+, \omega'\right)\right) - \Phi\left(L_0; V\left(p_0, L_0^+, \omega'\right)\right) \Big\rangle Q\left(d\omega'|\omega\right) \\
= \left(\omega p_0 \right. &\left. + \beta \int_\Omega \delta^{\omega'} U\left(\omega'\right) Q\left(d\omega'|\omega\right)\right) \cdot \varepsilon \\
&+ \beta \int_\Omega \Big\langle \mathscr{S}\left[p_0, \mathscr{L}\left(L_0 + \varepsilon, V\left(p_0, L_0^+, \omega'\right)\right)\right] - \mathscr{S}\left[p_0, \mathscr{L}\left(L_0, V\left(p_0, L_0^+, \omega'\right)\right)\right] \\
&\quad + \varepsilon \cdot (1 - \delta^{\omega'}) s\lambda^{\omega'} \int_{V\left(p_0, L_0^+, \omega'\right)}^{+\infty} v dF\left(v \mid \omega'\right) \Big\rangle Q\left(d\omega'|\omega\right),
\end{aligned}
$$

$$\text{(A40)}$$

where the last equality follows from the definitions of φ and Φ. Symmetrically:

$$
\begin{aligned}
\mathbf{M^N}\mathscr{S}\left(p_0, L_0 + \varepsilon\right) &- \mathbf{M^N}\mathscr{S}\left(p_0, L_0^+\right) = \varphi\left(p_0, L_0 + \varepsilon\right) - \varphi\left(p_0, L_0\right) \\
&+ \beta \int_\Omega \Big\langle \mathscr{S}\left[p_0, \mathscr{L}\left(L_0 + \varepsilon, V\left(p_0, L_0 + \varepsilon, \omega'\right)\right)\right] - \mathscr{S}\left[p_0, \mathscr{L}\left(L_0, V\left(p_0, L_0^+, \omega'\right)\right)\right] \\
&\quad + \Phi\left(L_0 + \varepsilon; V\left(p_0, L_0 + \varepsilon, \omega'\right)\right) - \Phi\left(L_0; V\left(p_0, L_0^+, \omega'\right)\right) \Big\rangle Q\left(d\omega'|\omega\right) \\
\leq \left(\omega p_0 \right. &\left. + \beta \int_\Omega \delta^{\omega'} U\left(\omega'\right) Q\left(d\omega'|\omega\right)\right) \cdot \varepsilon \\
&+ \beta \int_\Omega \Big\langle \mathscr{S}\left[p_0, \mathscr{L}\left(L_0 + \varepsilon, V\left(p_0, L_0 + \varepsilon, \omega'\right)\right)\right] - \mathscr{S}\left[p_0, \mathscr{L}\left(L_0, V\left(p_0, L_0 + \varepsilon, \omega'\right)\right)\right] \\
&\quad + \varepsilon \cdot (1 - \delta^{\omega'}) s\lambda^{\omega'} \int_{V\left(p_0, L_0 + \varepsilon, \omega'\right)}^{+\infty} v dF\left(v \mid \omega'\right) \Big\rangle Q\left(d\omega'|\omega\right).
\end{aligned}
$$

$$\text{(A41)}$$

Now dividing through by ε in (A40) and (A41), and invoking continuity w.r.t. V of $\mathscr{L}_L\left(L, V\right) = \left(1 - \delta^{\omega'}\right)\left(1 - s\lambda^{\omega'}\overline{F}\left(V\right)\right)$ (by continuity of F), everywhere right-differentiability of $\mathscr{S}$ w.r.t. L (by convexity of $\mathscr{S}$), and existence of a right limit of V at any L_0 (by monotonicity of V established in point 1 of this lemma), we see that the lower and upper bounds of $\frac{1}{\varepsilon}\left[\mathbf{M^N}\mathscr{S}\left(p_0, L_0 + \varepsilon\right) - \mathbf{M^N}\mathscr{S}\left(p_0, L_0^+\right)\right]$ exhibited in (A40) and (A41) both converge to the same limit as $\varepsilon \to 0^+$, which, together with continuity of $\mathbf{M^N}\mathscr{S}$ in L at L_0 which implies $\mathbf{M^N}\mathscr{S}\left(p_0, L_0^+\right) = \mathbf{M^N}\mathscr{S}\left(p_0, L_0\right)$, establishes right-differentiability of $\mathbf{M^N}\mathscr{S}$ w.r.t L with the

following expression for $\left[\mathbf{M^N}\mathscr{S}\right]_{L,r}(p,L)$

$$\left[\mathbf{M^N}\mathscr{S}\right]_{L,r}(p,L) = \omega p + \beta \int_\Omega \delta^{\omega'} U\left(\omega'\right) Q\left(d\omega'|\omega\right)$$

$$+ \beta \int_\Omega \Bigg\langle \mathscr{S}_{L,r}\left[p, \mathscr{L}\left(L, V\left(p, L^+, \omega'\right)\right)\right] \cdot \mathscr{L}_L\left(L, V\left(p, L^+, \omega'\right)\right)$$

$$+ \left(1-\delta^{\omega'}\right)s\lambda^{\omega'} \int_{V\left(p,L^+,\omega'\right)}^{+\infty} v dF\left(v \mid \omega'\right) \Bigg\rangle Q\left(d\omega' \mid \omega\right). \quad \text{(A42)}$$

Straightforward inspection shows that $\left[\mathbf{M^N}\mathscr{S}\right]_{L,r}(p,L) > 0$, so that $\mathbf{M^N}\mathscr{S}$ is increasing in L. We now show that $\left[\mathbf{M^N}\mathscr{S}\right]_{L,r}(p,L)$ is increasing in L and p. It is sufficient to show that the term under the $\int$ in (A42) is increasing in L and p for all $\omega' \in \Omega$. We begin with L. Let $L_1 < L_2 \in [0,1]^2$. To lighten the notation, let $V_k = V\left(p, L_k^+, \omega'\right)$ for $k = 1,2$. Because V is increasing in L, $V_2 \geq V_1$. Then:

$$\mathscr{S}_{L,r}\left[p, \mathscr{L}\left(L_2, V_2\right)\right] \cdot \mathscr{L}_L\left(L_2, V_2\right) - \mathscr{S}_{L,r}\left[p, \mathscr{L}\left(L_1, V_1\right)\right] \cdot \mathscr{L}_L\left(L_1, V_1\right)$$
$$- \left(1-\delta^{\omega'}\right)s\lambda^{\omega'} \int_{V_1}^{V_2} v dF\left(v \mid \omega'\right)$$

$$= \left[\mathscr{L}_L\left(L_2, V_2\right) - \mathscr{L}_L\left(L_1, V_1\right)\right] \cdot \mathscr{S}_{L,r}\left[p, \mathscr{L}\left(L_2, V_2\right)\right]$$
$$+ \mathscr{L}_L\left(L_1, V_1\right) \cdot \left(\mathscr{S}_{L,r}\left[p, \mathscr{L}\left(L_2, V_2\right)\right] - \mathscr{S}_{L,r}\left[p, \mathscr{L}\left(L_1, V_1\right)\right]\right)$$
$$- \left(1-\delta^{\omega'}\right)s\lambda^{\omega'} \int_{V_1}^{V_2} v dF\left(v \mid \omega'\right)$$

$$= \mathscr{L}_L\left(L_1, V_1\right) \cdot \left(\mathscr{S}_{L,r}\left[p, \mathscr{L}\left(L_2, V_2\right)\right] - \mathscr{S}_{L,r}\left[p, \mathscr{L}\left(L_1, V_1\right)\right]\right)$$
$$+ \left(1-\delta^{\omega'}\right)s\lambda^{\omega'} \int_{V_1}^{V_2} \left(\mathscr{S}_{L,r}\left[p, \mathscr{L}\left(L_2, V_2\right)\right] - v\right) dF\left(v \mid \omega'\right),$$

where the last equality stems from the definition of $\mathscr{L}_L$. Because $\mathscr{S}_{L,r}$ and $\mathscr{L}$ are both increasing in L, and because $\mathscr{L}$ is also increasing in V, the first term in the r.h.s. of the last equality above is positive. Finally, convexity of $\mathscr{S}$ combined with the first-order condition (A39) implies that $\mathscr{S}_{L,r}\left[p, \mathscr{L}\left(L_2, V_2\right)\right] \geq \mathscr{S}_{L,\ell}\left[p, \mathscr{L}\left(L_2, V_2\right)\right] \geq V_2$, so that $\mathscr{S}_{L,r}\left[p, \mathscr{L}\left(L_2, V_2\right)\right] \geq v$ for all $v \leq V_2$, implying that the integral term is nonnegative. This shows that $\left[\mathbf{M^N}\mathscr{S}\right]_{L,r}$ is (strictly) increasing in L. The proof that $\left[\mathbf{M^N}\mathscr{S}\right]_{L,r}$ is strictly increasing in p proceeds along similar lines (details available upon request). Thus $\mathbf{M^N}\mathscr{S}$ is a continuous function whose right partial derivative w.r.t. L exists everywhere, is increasing in L — which proves convexity w.r.t. L —, and increasing in p — which proves increasing differences in (p, L). $\qquad\square$

Now consider the set of functions defined over $\left[\underline{p}, \overline{p}\right] \times [0,1] \times \Omega$ that are continuous in (p, L) and

call it $\mathcal{C}_{[\underline{p},\overline{p}]\times[0,1]\times\Omega}$. That set is a Banach space when endowed with the sup norm. As Lemma A1 suggests we will be interested in the properties of a subset $\mathcal{C}'_{[\underline{p},\overline{p}]\times[0,1]\times\Omega} \subset \mathcal{C}_{[\underline{p},\overline{p}]\times[0,1]\times\Omega}$ of functions that are increasing and convex in L and have increasing differences in (p, L). We next prove two ancillary lemmas, which will establish as a corollary (Corollary A1) that $\mathcal{C}'_{[\underline{p},\overline{p}]\times[0,1]\times\Omega}$ is closed in $\mathcal{C}_{[\underline{p},\overline{p}]\times[0,1]\times\Omega}$ under the sup norm.[26]

Lemma A2. *Let X be an interval in $\mathbb{R}$ and $f_n : X \to \mathbb{R}$, $N \in \mathbb{N}$ such that $\{f_n\}$ converges uniformly to f. Then:*

1 *if f_n is nondecreasing for all n, so is f;*

2 *if f_n is convex for all n, so is f.*

Proof. For point 1, take $(x_1, x_2) \in X^2$ such that $x_2 > x_1$. Fix $k \in \mathbb{N}$. By uniform convergence, $\exists n_k \in \mathbb{N} : \forall n \geq n_k, \forall x \in X, |f_n(x) - f(x)| < \frac{1}{2k}$. Then:

$$f(x_2) - f(x_1) = \underbrace{f(x_2) - f_{n_k}(x_2)}_{>-1/2k} + \underbrace{f_{n_k}(x_2) - f_{n_k}(x_1)}_{\geq 0 \text{ by monotonicity of } f_{n_k}} + \underbrace{f_{n_k}(x_1) - f(x_1)}_{>-1/2k}$$
$$> -\frac{1}{k}.$$

As the above is valid for an arbitrary choice of $k \in \mathbb{N}$ and $(x_1, x_2) \in X^2$, it establishes that f is nondecreasing. For point 2, uniform convergence of $\{f_n\}$ to f implies pointwise convergence, so that Theorem 6.2.35 p282 in Corbae, Stinchcombe and Zeman (2009) can be applied. $\square$

Lemma A3. *Let $X \subset \mathbb{R}^2$ be a convex set and $f_n : X \to \mathbb{R}$, $N \in \mathbb{N}$ be functions with increasing differences such that $\{f_n\}$ converges uniformly to f. Then f has increasing differences.*

Proof. Let $\{(x_1, y_1), (x_2, y_2)\} \in X^2$ such that $x_2 > x_1$ and $y_2 > y_1$. Fix $k \in \mathbb{N}$. By uniform convergence, $\exists n_k \in \mathbb{N} : \forall n \geq n_k, \forall (x, y) \in X, |f_n(x, y) - f(x, y)| < \frac{1}{4k}$.

---

26. While for the purposes of this proof (which is concerned with closedness under the sup norm) both lemmas are stated for sequences that converge uniformly, it is straightforward to extend them to the case of pointwise convergent sequences.

Then:

$$f\left(x_2, y_2\right) - f\left(x_1, y_2\right)$$

$$= \underbrace{f\left(x_2, y_2\right) - f_{n_k}\left(x_2, y_2\right)}_{>-1/4k} + \underbrace{f_{n_k}\left(x_2, y_2\right) - f_{n_k}\left(x_1, y_2\right)}_{>f_{n_k}\left(x_2, y_1\right) - f_{n_k}\left(x_1, y_1\right) \text{ by ID of } f_{n_k}} + \underbrace{f_{n_k}\left(x_1, y_2\right) - f\left(x_1, y_2\right)}_{>-1/4k}$$

$$> -\frac{1}{2k} + f_{n_k}\left(x_2, y_1\right) - f_{n_k}\left(x_1, y_1\right)$$

$$= -\frac{1}{2k} + \underbrace{f_{n_k}\left(x_2, y_1\right) - f\left(x_2, y_1\right)}_{>-1/4k} + f\left(x_2, y_1\right) - f\left(x_1, y_1\right) + \underbrace{f\left(x_1, y_1\right) - f_{n_k}\left(x_1, y_1\right)}_{>-1/4k}$$

$$> -\frac{1}{k} + f\left(x_2, y_1\right) - f\left(x_1, y_1\right).$$

As the above is valid for an arbitrary choice of $k \in \mathbb{N}$ and $\{(x_1, y_1), (x_2, y_2)\} \in X^2$,
it establishes that f has increasing differences. $\qquad\square$

Corollary A1. *The set* $\mathcal{C}'_{\left[\underline{p}, \overline{p}\right] \times [0,1] \times \Omega}$ *of functions defined over* $\left[\underline{p}, \overline{p}\right] \times [0,1] \times \Omega$ *that are increasing and convex in L and have increasing differences in (p, L) is a closed subset of* $\mathcal{C}_{\left[\underline{p}, \overline{p}\right] \times [0,1] \times \Omega}$ *under the sup norm.*

The latter corollary establishes that, given a fixed N, the set of functions that are relevant to Lemma A1 is a closed subset of a Banach space of functions under the sup norm. The following lemma shows that the operator considered in Lemma A1 is a contraction under that same norm.

Lemma A4. *The operator* $\mathbf{M}^N$ *defined in* (A37) *maps* $\mathcal{C}'_{\left[\underline{p}, \overline{p}\right] \times [0,1] \times \Omega}$ *into itself and is a contraction of modulus β under the sup norm.*

Proof. That $\mathbf{M}^N$ maps $\mathcal{C}'_{\left[\underline{p}, \overline{p}\right] \times [0,1] \times \Omega}$ into itself flows directly from part of the proof of Lemma A1. To prove that $\mathbf{M}$ is a contraction, it is straightforward to check using (A37) that $\mathbf{M^N}$ satisfies Blackwell's sufficient conditions with modulus β. $\qquad\square$

We are now in a position to prove the proposition. Given the initially fixed N, the operator $\mathbf{M^N}$, which by Lemma A4 is a contraction from $\mathcal{C}_{\left[\underline{p}, \overline{p}\right] \times [0,1] \times \Omega}$ into itself, and has a unique fixed point $\mathscr{S}_N$ in that set (by the Contraction Mapping Theorem). Moreover, since $\mathcal{C}'_{\left[\underline{p}, \overline{p}\right] \times [0,1] \times \Omega}$ is a closed subset of $\mathcal{C}_{\left[\underline{p}, \overline{p}\right] \times [0,1] \times \Omega}$ (Lemma A2) and since $\mathbf{M^N}$ also maps $\mathcal{C}'_{\left[\underline{p}, \overline{p}\right] \times [0,1] \times \Omega}$ into itself (Lemma A1), that fixed point $\mathscr{S}_N$ belongs to $\mathcal{C}'_{\left[\underline{p}, \overline{p}\right] \times [0,1] \times \Omega}$.

Summing up, what we have established thus far is that for any fixed $N \in \mathcal{C}_{[\underline{p}, \overline{p}]}$, the operator $\mathbf{M^N}$ over functions of (p, L, ω) has a unique, bounded and continuous fixed point $\mathscr{S}_N^\star = \mathbf{M^N} \mathscr{S}_N^\star \in \mathcal{C}'_{[\underline{p}, \overline{p}] \times [0,1] \times \Omega} \subset \mathcal{C}_{[\underline{p}, \overline{p}] \times [0,1] \times \Omega}.$

We finally turn to the Bellman operator $\mathbf{M}$ which is relevant to the firm's problem. That operator $\mathbf{M}$ applies to functions $\overline{\mathscr{S}}$ defined on $\left[\underline{p}, \overline{p}\right] \times [0,1] \times \Omega \times \mathcal{C}_{[\underline{p}, \overline{p}]}$ and is defined as the following "extension" of $\mathbf{M^N}$:

$$
\begin{aligned}
\mathbf{M}\overline{\mathscr{S}}\,(p, L, \omega, N) :=\ & \varphi\,(p, L, \omega, N) \\
& + \beta \int_\Omega \max_{W(\omega')} \Big\langle \overline{\mathscr{S}}\left[p, \mathscr{L}\left(L, W(\omega'), \omega', N\right), \omega', \mathscr{N}\left(\omega', N\right)\right] + \Phi\left(L, W\left(\omega'\right), \omega', N\right) \Big\rangle Q\left(d\omega' \mid \omega\right).
\end{aligned}
$$

If an equilibrium exists, then a firm has a best response and a value S which solves $S = \mathbf{M}S$. For every $N \in \mathcal{C}_{[\underline{p}, \overline{p}]}$, by definition of $\mathbf{M}$ and $\mathbf{M^N}$ this implies $S = \mathbf{M^N}S$. Since the fixed point of $\mathbf{M^N}$ is unique, if $S = \mathbf{M}S$ exists then for every fixed $N \in \mathcal{C}_{[\underline{p}, \overline{p}]}$ we have for all $(p, L, \omega) \in \left[\underline{p}, \overline{p}\right] \times [0,1] \times \Omega$: $S\,(p, L, \omega, N) = \mathscr{S}_N^\star\,(p, L, \omega)$. Therefore, if the value function S and an equilibrium of the contract-posting game exist, then $S \in \mathcal{C}'_{[\underline{p}, \overline{p}] \times [0,1] \times \Omega}$: the typical firm's value function is continuous in p and L, increasing and convex in L and has increasing differences in (p, L). By the same standard monotone comparative statics arguments that we invoked in the proof of Lemma A1, the maximizing correspondence is increasing in p and L in the strong set sense, hence all of its measurable selections are weakly increasing in p and L.

APPENDIX B. PROOF OF PROPOSITION 4

In an attempt to simplify the notation without causing confusion, we define the value offered by firm p in equilibrium:

$$
V^\star\,(p, \omega) := V\,(p, L^\star\,(p), \omega, N^\star)
$$

for use throughout this proof. This notation keeps the dependence of V on N implicit.

The main purpose of Proposition 4 is actually to establish claim 2, continuous differentiability of $V^\star$. Our proof strategy is as follows. We know from Proposition 2 that the optimal policy $V^\star$ is increasing in p, hence differentiable a.e. It remains to show that it is continuously differentiable everywhere. To do so, first, we establish continuity properties of $V\,(p, L, \omega, N^\star)$ in p, both for fixed L and for $L = L^\star\,(p)$, and in L at $L = L^\star\,(p)$ for fixed p. Using these properties, we show that any solution to the Bellman equation defining $S\,(p, L, \omega, N)$ when all other firms are playing a RPE is differentiable in L at $L = L^\star\,(p)$ and such that $p \mapsto S\,(p, L^\star\,(p), \omega, N^\star)$ is continuous; that is, on the equilibrium path the shadow marginal value of one worker always exists and is continuous in firm productivity. Next, we exploit this property

and the implications of RPE to show that the optimal policy $V^\star$ is indeed continuously differentiable everywhere.

We begin with an ancillary lemma, which is interesting in its own right.

Lemma B5.   *V has the following continuity properties along the (RP) Equilibrium path:*

1 $p \mapsto V\left(p, L^\star\left(p\right), \omega, N^\star\right) = V^\star\left(p, \omega\right)$ is continuous;

2 $L \mapsto V\left(p, L, \omega, N^\star\right)$ is continuous at $L = L^\star\left(p\right)$;

3 $q \mapsto V\left(q, L^\star\left(p\right), \omega, N^\star\right)$ is continuous at $q = p$.

Proof. $p \mapsto V^\star\left(p, \omega\right)$ is increasing by Proposition 2, so $V^\star$ can only have (countably many) jump discontinuities. But then a jump discontinuity in $V^\star$ would imply a gap in the support of F, which is inconsistent with equilibrium as argued in Appendix Appendix A. This proves claim 1 of the lemma.

For claim 2, fix p and $\varepsilon > 0$. Then by continuity of $V^\star$ (point 1 of this lemma), $\exists \alpha > 0 : \forall \sigma \in (0, \alpha]$, $V^\star\left(p, \omega\right) \leq V^\star\left(p + \sigma, \omega\right) \leq V^\star\left(p, \omega\right) + \varepsilon$. But then monotonicity of V in L and in p (see Appendix Appendix A) further implies: $V^\star\left(p, \omega\right) \leq V\left(p, L^\star\left(p + \sigma\right), \omega, N^\star\right) \leq V^\star\left(p + \sigma, \omega\right) \leq V^\star\left(p, \omega\right) + \varepsilon$, so that $\forall L \in \left[L^\star\left(p\right), L^\star\left(p + \sigma\right)\right]$, $V\left(p, L, \omega, N^\star\right) - V\left(p, L^\star\left(p\right), \omega, N^\star\right) \leq \varepsilon$, which establishes right-continuity of V in L at $L^\star\left(p\right)$. Left-continuity is established in the same way, and so is claim 3. $\square$

We now go on to establish point 1 of the proposition. In so doing, to avoid notational overload, we will keep the dependence of all value functions and laws of motion on $N^\star$ implicit. Now first, convexity of S w.r.t. L was established as a by-product of Proposition 2 (see Appendix Appendix A), and implies that S is everywhere left-and right-differentiable w.r.t. L, and that the right and left derivatives $S_{L,r}$ and $S_{L,\ell}$ are both increasing functions of L. As such they have right and left limits everywhere. We can thus define $S_{L,r}\left(p, L^+, \omega\right) = \lim_{\varepsilon \to 0+} S_{L,r}\left(p, L + \varepsilon, \omega\right)$, and symmetrically $S_{L,\ell}\left(p, L^-, \omega\right) = \lim_{\varepsilon \to 0+} S_{L,\ell}\left(p, L - \varepsilon, \omega\right)$. Now following exactly the same steps as in (A40) and (A41)

(see the proof of Lemma A1 in Appendix Appendix A), only applied to S, we establish:

$$S_{L,r}\left(p, L^+, \omega\right) = \omega p + \beta \int_\Omega \delta^{\omega'} U\left(\omega'\right) Q\left(d\omega' \mid \omega\right)$$
$$+ \beta \int_\Omega \Bigg\langle S_{L,r}\left[p, \mathscr{L}\left(L, V\left(p, L^+, \omega'\right), \omega'\right), \omega'\right] \cdot \mathscr{L}_L\left(L, V\left(p, L^+, \omega'\right), \omega'\right)$$
$$+ \left(1 - \delta^{\omega'}\right) s\lambda^{\omega'} \int_{V\left(p, L^+, \omega'\right)}^{+\infty} v dF\left(v \mid \omega'\right) \Bigg\rangle Q\left(d\omega' \mid \omega\right).$$

Next, the facts that V is increasing in L (see the proof of Proposition 2) and continuous in L at $L = L^\star\left(p\right)$ (from Lemma B5), combined with continuity of $\mathscr{L}$ and $\mathscr{L}_L$ w.r.t. V (by continuity of F), imply that $\mathscr{L}_L\left(L, V\left(p, L^+, \omega'\right), \omega'\right) = \mathscr{L}_L\left(L, V\left(p, L, \omega'\right), \omega'\right)$ and $S_{L,r}\left[p, \mathscr{L}\left(L, V\left(p, L^+, \omega'\right), \omega'\right), \omega'\right] = S_{L,r}\left[p, \mathscr{L}\left(L, V\left(p, L, \omega'\right), \omega'\right)^+, \omega'\right]$ at $L = L^\star\left(p\right)$. As a further consequence:

$$S_{L,r}\left(p, L^\star\left(p\right)^+, \omega\right) = \omega p + \beta \int_\Omega \delta^{\omega'} U\left(\omega'\right) Q\left(d\omega' \mid \omega\right)$$
$$+ \beta \int_\Omega \Bigg\langle S_{L,r}\left[p, \mathscr{L}\left(L^\star\left(p\right), V^\star\left(p, \omega'\right), \omega'\right)^+, \omega'\right] \cdot \mathscr{L}_L\left(L^\star\left(p\right), V^\star\left(p, \omega'\right), \omega'\right)$$
$$+ \left(1 - \delta^{\omega'}\right) s\lambda^{\omega'} \int_{V^\star\left(p, \omega'\right)}^{+\infty} v dF\left(v \mid \omega'\right) \Bigg\rangle Q\left(d\omega' \mid \omega\right). \quad \text{(B43)}$$

A symmetric expression can be arrived at in the same way for $S_{L,\ell}\left(p, L^\star\left(p\right)^-, \omega\right)$.

Let $\mathscr{D}_{S_L}^L\left(p, L, \omega\right) := S_{L,r}\left(p, L^+, \omega\right) - S_{L,\ell}\left(p, L^-, \omega\right)$, positive by convexity of S in L. Moreover, because $S_{L,\ell} \geq 0$, it follows that $\mathscr{D}_{S_L}^L \leq S_{L,r}$. But then clearly, $S_{L,r}\left(p, L^\star\left(p\right), \omega\right)$ is bounded above by the maximum feasible output per worker in the economy, $\max_\Omega \omega \overline{p}/\left(1 - \beta\right)$. This proves that $\mathscr{D}_{S_L}^L$ is uniformly bounded above and below. Now:

$$\mathscr{D}_{S_L}^L\left(p, L^\star\left(p\right), \omega\right)$$
$$= \beta \int_\Omega \mathscr{D}_{S_L}^L\left[p, \mathscr{L}\left(L^\star\left(p\right), V^\star\left(p, \omega'\right), \omega'\right), \omega'\right] \cdot \mathscr{L}_L\left(L^\star\left(p\right), V^\star\left(p, \omega'\right), \omega'\right) Q\left(d\omega' \mid \omega\right)$$
$$< \beta \int_\Omega \mathscr{D}_{S_L}^L\left[p, \mathscr{L}\left(L^\star\left(p\right), V^\star\left(p, \omega'\right), \omega'\right), \omega'\right] Q\left(d\omega' \mid \omega\right).$$

Iterating the last inequality shows that $0 \leq \mathscr{D}_{S_L}^L\left(p, L^\star\left(p\right), \omega\right) < \beta^n \max_\Omega \omega \overline{p}/\left(1 - \beta\right)$ for all $n \in \mathbb{N}$, which implies that $\mathscr{D}_{S_L}^L\left(p, L^\star\left(p\right), \omega\right) = 0$ for all $\left(p, \omega\right)$ and that S_L exists everywhere. Since S is convex, S_L is increasing, hence it can only have jumps up. But we just concluded that its right and left limit are equal everywhere, so S_L is continuous for all $L \in [0, 1)$.

We finally prove continuity of $x \mapsto S_L\left(x, L^\star\left(p\right), \omega\right)$ at $x = p$. Because S has increasing differences in $\left(p, L\right)$ (as established in Appendix Appendix A), $x \mapsto S_L\left(x, L^\star\left(p\right), \omega\right)$ is increasing, therefore continuous except for at most countably many upward jumps. Now defining $\mathscr{D}_{S_L}^p\left(p, L, \omega\right) :=

$S_L\left(p^+, L, \omega\right) - S_L\left(p, L, \omega\right):$

$$\mathscr{D}^p_{S_L}\left(p, L^\star\left(p\right), \omega\right)$$
$$= \beta \int_\Omega \mathscr{D}^p_{S_L}\left[p, \mathscr{L}\left(L^\star\left(p\right), V^\star\left(p, \omega'\right), \omega'\right), \omega'\right] \cdot \mathscr{L}_L\left(L^\star\left(p\right), V^\star\left(p, \omega'\right), \omega'\right) Q\left(d\omega' \mid \omega\right)$$
$$< \beta \int_\Omega \mathscr{D}^p_{S_L}\left[p, \mathscr{L}\left(L^\star\left(p\right), V^\star\left(p, \omega'\right), \omega'\right), \omega'\right] Q\left(d\omega' \mid \omega\right).$$

(This uses continuity of $V^\star$ w.r.t. p, of $\mathscr{L}$ w.r.t. V, and continuity of $S_L\left(p, L, \omega\right)$ in L at $L = L^\star\left(p\right)$.) Moreover, $\mathscr{D}^p_{S_L} \geq 0$ since S_L is increasing in p, and $\mathscr{D}^p_{S_L}$ is uniformly bounded above by $\max_\Omega \omega \overline{p}/\left(1 - \beta\right)$ for the same reasons as $\mathscr{D}^L_{S_L}$. Iterating the above inequality establishes that $\mathscr{D}^p_{S_L}\left(p, L^\star\left(p\right), \omega\right) = 0$ for all $\left(p, \omega\right)$, so that $x \mapsto S_L\left(x, L^\star\left(p\right), \omega\right)$ is continuous at $x = p$. Combined with continuity of $S_L\left(p, L, \omega\right)$ in L at $L = L^\star\left(p\right)$ (proven above), and continuity of $L^\star\left(p\right)$ (from the assumption that L_0 is continuous), this established point 1 in the proposition.

We now prove point 2 of the proposition, namely that in any RPE, $V^\star\left(p\right)$ is continuously differentiable. Consider the problem of a firm choosing W to best-respond to all other firms playing a RPE. By a simple improvement argument, $W \in \left[V^\star\left(\underline{p}, \omega'\right), V^\star\left(\overline{p}, \omega'\right)\right]$. Since $V^\star$ is continuous and increasing, offering any such best response W is equivalent to choosing a type q to imitate such that $W = V^\star\left(q, \omega'\right)$. In any RPE, by Proposition 2, the best response by a firm p of current size $L^\star\left(p\right)$ is 'truthful revelation', $q = p$, which solves

$$S\left(p, L^\star\left(p\right), \omega\right) = \varphi\left(p, L^\star\left(p\right), \omega\right)$$
$$+ \beta \int_\Omega \max_{q\left(\omega'\right)} \left\langle S\left[p, \mathscr{L}\left(L^\star\left(p\right), \omega', q\left(\omega'\right)\right), \omega'\right] + \Phi\left(L^\star\left(p\right), \omega', q\left(\omega'\right)\right)\right\rangle Q\left(d\omega' \mid \omega\right) \quad \text{(B44)}$$

where, with a slight abuse of notation:

$$\mathscr{L}\left(L, \omega', q\right) = L\left(1 - \delta^{\omega'}\right)\left(1 - s\lambda^{\omega'}\overline{F}\left(V^\star\left(q, \omega'\right) \mid \omega'\right)\right)$$
$$+ \lambda^{\omega'}\left(1 - N\left(\overline{p}\right)\right) + s\lambda^{\omega'}\left(1 - \delta^{\omega'}\right) N\left(\overline{p}\right) G\left(V^\star\left(q, \omega'\right) \mid \omega'\right)$$

and

$$\Phi\left(L, \omega', q\right) = L\left(1 - \delta^{\omega'}\right) s\lambda^{\omega'} \int_{V^\star\left(q, \omega'\right)}^{+\infty} v dF\left(v \mid \omega'\right)$$
$$- V^\star\left(q, \omega'\right)\left(\lambda^{\omega'}\left(1 - N\left(\overline{p}\right)\right) + s\lambda^{\omega'}\left(1 - \delta^{\omega'}\right) N\left(\overline{p}\right) G\left(V^\star\left(q, \omega'\right) \mid \omega'\right)\right).$$

Using the RP property

$$\mathscr{L}\left(L, \omega', q\right) = L^\star\left(p\right)\left(1 - \delta^{\omega'}\right)\left(1 - s\lambda^{\omega'}\overline{\Gamma}\left(q\right)\right) + \lambda^{\omega'}\left(1 - N^\star\left(\overline{p}\right)\right) + s\lambda^{\omega'}\left(1 - \delta^{\omega'}\right) N^\star\left(q\right)$$
$$\Phi\left(L, \omega', q\right) = L\left(1 - \delta^{\omega'}\right) s\lambda^{\omega'} \int_q^{\overline{p}} V^\star\left(x, \omega'\right) d\Gamma\left(x\right) \quad \text{(B45)}$$
$$- V^\star\left(q, \omega'\right)\left(\lambda^{\omega'}\left(1 - N^\star\left(\overline{p}\right)\right) + s\lambda^{\omega'}\left(1 - \delta^{\omega'}\right) N^\star\left(q\right)\right).$$

Because $V^\star$ is increasing, we know that it is differentiable almost everywhere, i.e. $V_p^\star$ exists outside of a null set (say N_V), and that for all $p \in \left[\underline{p}, \overline{p}\right] \setminus N_V$, $V^\star(p, \omega') = V^\star\left(\underline{p}, \omega'\right) + \int_{\underline{p}}^p V_p^\star(x, \omega') \, dx$. Thus, for all $p \in \left[\underline{p}, \overline{p}\right] \setminus N_V$ we can write a NFOC for the maximization problem in (B44), using (B45), $V_p^\star(p, \omega') = v(p, \omega')$ where the function

$$v(p, \omega') := 2s\lambda^{\omega'}\left(1 - \delta^{\omega'}\right) L^\star(p) \gamma(p) \frac{S_L\left[p, \mathscr{L}\left(L^\star(p), V^\star(p, \omega'), \omega'\right), \omega'\right] - V^\star(p, \omega')}{\lambda^{\omega'}(1 - N(\overline{p})) + s\lambda^{\omega'}\left(1 - \delta^{\omega'}\right) N^\star(p)}$$

is, by Lemma B5, continuous in p over the interval $\left[\underline{p}, \overline{p}\right]$ (recall that $L^\star$ is continuous by the assumption that L_0 is). Then,

$$V^\star(p, \omega') = V^\star\left(\underline{p}, \omega'\right) + \int_{\underline{p}}^p V_p^\star(x, \omega') \, dx = V^\star\left(\underline{p}, \omega'\right) + \int_{\underline{p}}^p v(x, \omega') \, dx,$$

where the second equality follows from the fact that $V_p^\star(p, \omega') \neq v(p, \omega')$ only on a null set. So $V^\star(p, \omega')$ is the integral of a continuous function v, hence it is continuously differentiable with $V_p^\star(p, \omega') = v(p, \omega')$ everywhere (i.e., the FOC $V_p^\star(p, \omega') = v(p, \omega')$ holds everywhere). Point 2 of the proposition is thus proven. $\square$

APPENDIX C. PROOF OF PROPOSITION 5

Let $\mathcal{E}_{[\underline{p}, \bar{p}]}$ be the space of continuous c.d.f.'s over $\left[\underline{p}, \bar{p}\right]$, $\mathcal{F}_{[\underline{p}, \overline{p}] \times \Omega \times \mathcal{E}_{[\underline{p}, \bar{p}]}}$ be the space of positive functions $\left[\underline{p}, \bar{p}\right] \times \Omega \times \mathcal{E}_{[\underline{p}, \bar{p}]} \to \mathbb{R}_+^2$ such that the first component is p-integrable and the second component does not depend on p. Then the operator $\mathbf{T}$ defined in (5.25) is a linear function on $\mathcal{F}_{[\underline{p}, \overline{p}] \times \Omega \times \mathcal{E}_{[\underline{p}, \overline{p}]}}$ which, by definition, preserves positivity of its arguments and is such that the second component is independent of p, so that $\mathbf{T}$ maps $\mathcal{F}_{[\underline{p}, \overline{p}] \times \Omega \times \mathcal{E}_{[\underline{p}, \overline{p}]}}$ into itself, whenever the function is well defined (the integrals exist).

We first show that the limit in (5.27) exists, is positive and uniformly bounded above. By assumption, $\omega_t p$ and $b(\omega_t)$ are positive and uniformly bounded above by some $K < +\infty$, therefore by the definition of $\mathbf{T}$, $\mathbf{E}_t \{\mathbf{T}[\omega p, b(\omega)](p \mid \omega_{t+1})\}$ is also positive and uniformly bounded above by K, and by induction the same is true of $\mathbf{E}_t \{\mathbf{T}^j[\omega p, b(\omega)](p \mid \omega_{t+j})\}$. Hence the sequence $\sum_{j=0}^{n-1} \beta^j \mathbf{E}_t \{\mathbf{T}^j[\omega p, b(\omega)](p \mid \omega_{t+j})\}$ is increasing and uniformly bounded above by $K/(1 - \beta)$, so each of the two sums in this sequence must converge and the limit exists and is positive and bounded above by $K/(1 - \beta)$.

We next show that, if there exists a RPE, then it is given by (5.27). Suppose there exists a RPE $\left(\begin{smallmatrix}\mu \\ U\end{smallmatrix}\right)$. By definition of a RPE, $\left(\begin{smallmatrix}\mu \\ U\end{smallmatrix}\right)$ must solve (5.26). Substituting forward in (5.26), we find for all $n \in \mathbb{N}$:

$$\begin{pmatrix}\mu_t \\ U_t\end{pmatrix}(p) := \sum_{j=0}^n \beta^j \mathbf{E}_t \{\mathbf{T}^j[\omega p, b(\omega)](p \mid \omega_{t+j})\} + \beta^{n+1} \mathbf{E}_t \{\mathbf{T}^n[\mu_{t+n}, U_{t+n}](p \mid \omega_{t+n})\}. \qquad (C46)$$

The proof of Proposition 2 further shows that S has increasing differences in (p, L), so that $\mu = S_L$ is increasing in p. Then, by inspection of $\mathbf{T}_\mu$: $\mu_t(\overline{p}) = \omega_t \overline{p} + \beta \mathbf{E}_t \{\mathbf{T}_\mu[\mu_{t+1}, U_{t+1}](\overline{p} \mid \omega_{t+1})\} \leq$

$K + \beta \mathbf{E}_t \mu_{t+1}(\overline{p})$. Iterating forward, this establishes that $\mu_t(\overline{p}) \leq K/(1-\beta)$. But by definition of RPE and the fact that μ_t is increasing in p, $0 \leq U_t \leq \mu_t(p) \leq K/(1-\beta)$, namely, $\binom{\mu}{U}$ is uniformly bounded above. Therefore, the last term in (C46) is such that:

$$0 \leq \beta^{n+1} \mathbf{E}_t \left\{ \mathbf{T}^n \left[\mu_{t+n}, U_{t+n} \right] (p \mid \omega_{t+n}) \right\} \leq \beta^{n+1} K/(1-\beta) \xrightarrow[n \to +\infty]{} 0,$$

and $\binom{\mu}{U}$ is given by (5.27) as claimed.

We finally turn to existence. We prove it by construction, i.e. by checking that the candidate $\binom{\mu^\star}{U^\star}$ defined in (5.27) satisfies all the properties of a RPE. By construction, $\binom{\mu^\star}{U^\star}$ solves (5.26). Moreover, since $\binom{\mu^\star}{U^\star}$ is uniformly bounded above, it satisfies the TVC. So we only have to show that it satisfies $0 \leq U^\star \leq \mathbf{T}_V[\mu^\star, U^\star]$ and $\mathbf{T}_V[\mu^\star, U^\star]$ increasing in p. By definition of $\mathbf{T}_V$:

$$\frac{\partial \mathbf{T}_V[\mu_t^\star, U_t^\star]}{\partial p}(p \mid \omega) = \frac{2\frac{dH_t}{dp}(p)}{H_t(p)} [\mu_t^\star(p) - \mathbf{T}_V[\mu_t^\star, U_t^\star](p \mid \omega)]$$

$$= \frac{2\frac{dH_t}{dp}(p)}{H_t(p)} \left(\mu_t^\star(p) - U_t^\star - \int_{\underline{p}}^p [\mu_t^\star(x) - U_t^\star] \frac{\frac{d}{dp}\left[H_t(x)^2 \right]}{H_t(p)^2} dx \right)$$

so it suffices to prove that $\mu_t^\star\left(\underline{p}\right) \geq U_t$ and $\mu_t^\star$ is increasing in p at all dates. For this, consider an increasing function $\widetilde{\mu}(p)$ and a constant $\widetilde{U} \leq \widetilde{\mu}\left(\underline{p}\right)$. It is then straightforward to establish that $\mathbf{T}_\mu\left[\widetilde{\mu}, \widetilde{U}\right](p \mid \omega)$ is increasing in p. Moreover:

$$\mathbf{T}_\mu\left[\widetilde{\mu}, \widetilde{U}\right]\left(\underline{p} \mid \omega\right) - \mathbf{T}_U\left[\widetilde{\mu}, \widetilde{U}\right] = (1 - \delta(\omega))(1 - s\lambda(\omega))\left[\widetilde{\mu}\left(\underline{p}\right) - \widetilde{U}\right]$$

$$+ \lambda(\omega)[1 - s(1 - \delta(\omega))] \int_{\underline{p}}^{\overline{p}} \frac{1}{H(x)^2} \int_{\underline{p}}^x \left[\widetilde{\mu}(z) - \widetilde{U}\right] \frac{d}{dp}\left[H(z)^2\right] dz d\Gamma(x),$$

which is positive if $1 - s(1 - \delta(\omega)) \geq 0$ (the second condition in the proposition). Applying the above to $\widetilde{\mu}(p) = \omega p$ and $\widetilde{U} = b(\omega)$ (which is less than $\omega \underline{p}$ under the first condition in the proposition) establishes that $\mathbf{T}_\mu[\omega p, b(\omega)](p \mid \omega)$ is increasing in p and greater than $\mathbf{T}_U[\omega p, b(\omega)]$ for all p and ω. Repeating with $\widetilde{\mu}(p) = \mathbf{T}_\mu[\omega p, b(\omega)](p \mid \omega)$ and $\widetilde{U} = \mathbf{T}_U[\omega p, b(\omega)]$, and iterating ad infinitum shows that for all j, $\mathbf{T}_\mu^j[\omega p, b(\omega)](p \mid \omega)$ is increasing in p and greater than $\mathbf{T}_U^j[\omega p, b(\omega)]$ for all p and ω, which proves that $\mu_t^\star\left(\underline{p}\right) \geq U_t$ and $\mu_t^\star$ is increasing in p. $\square$