Electrostatics of edge channels

D. B. Chklovskii
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

B. I. Shklovskii and L. I. Glazman
Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455
(Received 19 February 1992; revised manuscript received 6 April 1992)

We propose a quantitative electrostatic theory of the gate-induced confinement of two-dimensional electron gas (2DEG) in the quantum Hall regime. The self-consistent electrostatic potential in the region occupied by 2DEG changes in a steplike manner due to the formation of alternating strips of compressible and incompressible electron liquids. We obtain the dependence of positions and widths of these strips on the filling factor. Incompressible strips are shown to be much more narrow than the compressible ones. The relationship between the widths of the adjacent compressible and incompressible strips is found to be universal: It does not depend on the strip number, magnetic field, or gate voltage. Our theory enables us to explain results obtained in experimental studies of edge-state equilibration.

I. INTRODUCTION

Magnetotransport along edge states that are formed in high-mobility two-dimensional electron gas (2DEG) has attracted significant attention in recent years (see, e.g., Ref. 1). The concept of such transport allows one to interpret a number of experiments performed in the integer quantum Hall effect (IQHE) regime as well as in the fractional quantum Hall effect (FQHE). Particularly, it allows one to analyze the impact of leads and relaxation processes between the electron states quantized by a strong magnetic field on conductance measurements.[2] The overlap between electron wave functions belonging to different edge states is exponentially small. This makes the geometry of these states and separation between adjacent ones crucial factors in the understanding of all transport phenomena. This geometry depends strongly on the shape of the potential confining electrons. The bare potential (formed by an external metallic electrode gate, or by an etching process) is usually smooth on a scale determined by the magnetic length $\lambda=(c\hbar/eH)^{1/2}$, and one can use the quasiclassical approach in the description of edge-state geometry.

A naive one-electron picture is based on the assumption that the bare potential bends Landau levels, and the positions of the edge states are given by their intersection with the constant Fermi level; see Figs. 1(a)–1(c). This picture has a serious drawback from the experimental point of view. A strong separation between the edge states due to the smoothness of the confining potential reduces the relaxation rate between edge states too strongly: it is difficult to obtain equilibration length that does not exceed significantly the sample size of any realistic experimental parameters. From the theoretical point of view, the one-electron picture fails to account for screening and its modification in a strong magnetic field. The latter modulates the electron density of states, making the screening highly dependent on the filling factor, which changes in the region of interest from its bulk value to zero at the boundary of the 2DEG.

The effect of screening in the presence of the magnetic field was included in a qualitative picture of edge states by Beenakker and Chang.[3] They divide the electron gas into alternating strips of incompressible and compressible states, the former originating from the discontinuities of the chemical-potential dependence on filling factor $\mu(\nu)$. Existing treatments[3] lack a quantitative approach that could yield the geometric dimensions and positions of those strips. Knowledge of them is necessary for the explanation of transport experiments involving edge states, in particular selective population of the edge states by the point contacts and relaxation between them (see, e.g., Refs. 4 and 5).

In this paper we present a quantitative electrostatic treatment of the edge states in the case of gate-induced depletion that is self-consistent and free of unjustified assumptions about the external potential. We obtain the dependence of the widths of compressible and incompressible liquid strips on the filling factor. These widths scale with the width of the depletion layer l that separates the gate and the boundary of the 2DEG as l (compressible) and as $(a_B l)^{1/2}$ for IQHE and as $(\lambda l)^{1/2}$ for FQHE (incompressible). Here a_B is the Bohr radius $a_B=\hbar^2\epsilon/m_{\rm eff}e^2$ for a semiconductor with a dielectric constant ϵ and effective electron mass $m_{\rm eff}$. Length l is controlled by the gate voltage V_g and is usually very large (several thousand Å). Therefore, we find the incompressible strips to be parametrically more narrow than the adjacent compressible ones, the innermost being the widest [see Figs. 1(d)–1(f)]. This can serve to explain the high equilibration rate of all the states but the innermost one.[4] Our results provide an explanation for the experimentally observed equilibration length dependences on magnetic field in the IQHE regime. Difference in equilibration rates for $\frac{1}{3}$ and $\frac{2}{3}$ FQHE edge states[5] is also discussed.

The paper begins (Sec. II) with the formulation and

solution of the electrostatics problem at the gate-induced 2DEG edge in the absence of magnetic field. In Sec. III we study the magnetic-field-induced redistribution of charge in the vicinity of the incompressible strip forming the so-called dipolar strip. We generalize our treatment to the case when several dipolar strips are formed in Sec IV. In Sec. V we discuss tunneling through the incompressible strip and the relation of our theory to experiment, including the influence of disorder. Section VI contains our major conclusions.

II. ELECTROSTATICS OF GATE-INDUCED DEPLETION IN ZERO MAGNETIC FIELD

The 2DEG density in $GaAs/Al_xGa_{1-x}As$ heterostructures is defined by the concentration of donors located behind a spacer layer. In our model[6] we neglect the donor concentration fluctuations and the discreteness of their charge. This means that far from the boundaries, the electron density is homogeneous and equal to that of the positive background (n_0). The boundary of the 2DEG is created by applying a negative voltage $-V_g$ to a metal half-plane serving as a gate. We neglect the distance from the gate to the 2DEG plane and the spacer layer thickness. Thus the positive background, the gate, and the 2DEG all belong to the same plane ($z=0$) (see Fig. 2). The validity of this assumption will be discussed below. The half-space $z<0$ is occupied by the semiconductor with dielectric constant $\epsilon \gg 1$. As the system is invariant in the y direction, the problem becomes effectively two dimensional.

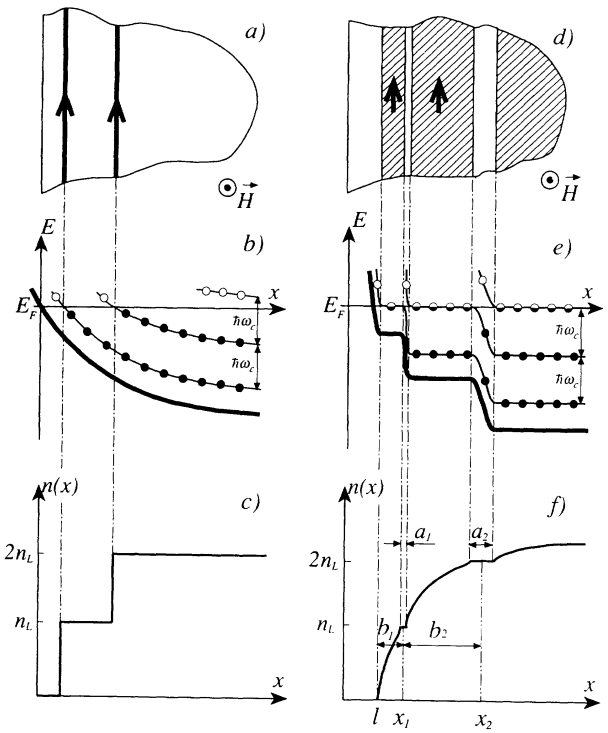

FIG. 1. Structure of spinless edge states in the IQHE regime. (a)–(c) One-electron picture of edge states. (a) Top view on the 2DEG plane near the edge. Arrows designate electron flow direction in the two edge channels. (b) Adiabatic bending of Landau levels along the increasing potential energy near the edge. Energy is measured from the Fermi level. Circles represent local filling of the Landau levels: ●, occupied; ○, empty. (c) Electron density as a function of the distance to the boundary. (d)–(f) Self-consistent electrostatic picture. (d) Top view of the 2DEG near the edge. Shaded strips represent regions with noninteger filling factor (compressible liquid), unshaded strips represent integer filling factor regions (incompressible liquid). Arrows show the direction of electron flow. (e) Bending of the electrostatic potential energy and the Landau levels. Circles represent local filling of the Landau levels: ●, occupied; ○, partially occupied; ○, empty. (f) Electron density as a function of distance to the middle of the depletion region.

Let us first discuss what happens qualitatively. At zero gate voltage (or some cutoff voltage in real devices), the electron density (being zero under the gate) reaches its bulk value n_0 right at the gate edge. By applying a negative potential to the gate, electrons are repelled from it, leaving a depleted strip behind. The width of this strip, $2l$, is determined by V_g. Also, one may expect the density of the 2DEG to grow gradually from 0 at the end of the depletion to n_0 in the bulk, where it compensates for the positive background. In our treatment we rely on the smallness of the parameter $a_B/l \sim E_F/V_g$, which gives the ratio of the screening length r_s to the characteristic length scale. This means that the x component of the electric field is screened out completely. Hence the potential is constant in the area occupied by electrons. Then the problem is the one of a capacitor, both metal plates of which are in the same plane. And there is a uniformly charged insulator of width $2l$ filling in the slit between the plates. Following Ref. 6, we solve the electrostatic problem for given V_g, l, and n_0. Then we find l from the condition that the electron-gas boundary should be in mechanical equilibrium. This means that electric field E_x ($x=l$) should be zero both to the left and to the right of the boundary ($\lim_{x\to l-0}E_x = \lim_{x\to l+0}E_x = 0$).

A high value of the dielectric constant in semiconductors ($\epsilon \gg 1$) allows us to solve the Laplace equation in the half-space $z<0$ using the simplified boundary conditions

$$\phi(x,z=0)=\begin{cases}-V_g, & x<-l\\ 0, & x>l\ ,\end{cases} \tag{1}$$

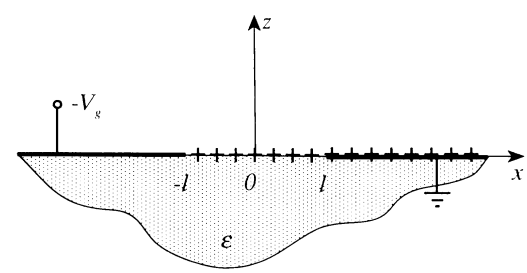

FIG. 2. Two-dimensional capacitor formed at the 2DEG edge. Thick lines represent two conductors: the gate at potential $-V_g$ on the left, and the grounded 2DEG on the right. Plusses represent a uniform positive background due to donors. The dotted area is occupied by a semiconductor with high dielectric constant ϵ.

$$\left.\frac{d\phi(x,z)}{dz}\right|_{z\to -0}=\frac{4\pi e n_0}{\epsilon}\ ,\quad |x|<l\ . \tag{2}$$

The solution can be given as a sum $\phi=\phi_1+\phi_2$ of harmonic functions ϕ_1 and ϕ_2 that satisfy separately the conditions

$$\phi_1(x,z=0)=\begin{cases}-V_g, & x<-l\\ 0, & x>l\ ,\end{cases} \tag{3}$$

$$\left.\frac{d\phi_1(x,z)}{dz}\right|_{z\to -0}=0\ ,\quad |x|<l\ , \tag{4}$$

$$\phi_2(x,z=0)=0\ ,\quad |x|>l \tag{5}$$

$$\left.\frac{d\phi_2(x,z)}{dz}\right|_{z\to -0}=\frac{4\pi e n_0}{\epsilon}\ ,\quad |x|<l\ . \tag{6}$$

Both functions can be found using the theory of complex variables. At $z=0$ the first one is given by[7]

$$\phi_1(x,z=0)=\frac{-V_g}{2}+\frac{V_g}{\pi}\arcsin(x/l),\quad |x|<l\ . \tag{7}$$

The derivation of the second function is reproduced in the Appendix:

$$\phi_2(x,z=0)=\frac{4\pi e n_0}{\epsilon}(l^2-x^2)^{1/2}\ ,\quad |x|<l\ . \tag{8}$$

Both solutions have square-root singularities in the electric field ($E_x=-d\phi/dx$) at $x=l$ which can be cancelled out only if[6]

$$l=\frac{V_g\epsilon}{4\pi^2 n_0 e}\ . \tag{9}$$

The singularity at $x=-l$ remains but should not cause any problem because this boundary is fixed and electrons are confined due to the existence of the work function. The density of the 2DEG for l defined in Eq. (9) is given by (see Fig. 3)

$$n(x)=\left[\frac{x-l}{x+l}\right]^{1/2} n_0\ ,\quad x>l\ . \tag{10}$$

These results deserve some discussion. It is important to mention that l is the only scale in the electrostatic solution. It defines the electron density variation as well as the width of the depletion strip. l is proportional to the gate voltage. Its numerical value for $V_g=1$ V, $n_0=10^{11}$ cm^{-2}, and $\epsilon=12.5$ is $l=2200$ Å. We would like to emphasize that this is a very large length. The typical spacer thickness is about 500 Å. The gate to the 2DEG distance is usually of the same order, ≈ 800 Å. This justifies bringing all the charges into one plane. Also, for a typical value $a_B=100$ Å, condition $a_B/l \ll 1$ is satisfied. In a real system we do not expect our solution to be accurate on the scale less than a_B. At a large distance from the gate $x \gg l$, Eq. (10) yields approximately

$$n(x)\approx(1-l/x)n_0\ . \tag{11}$$

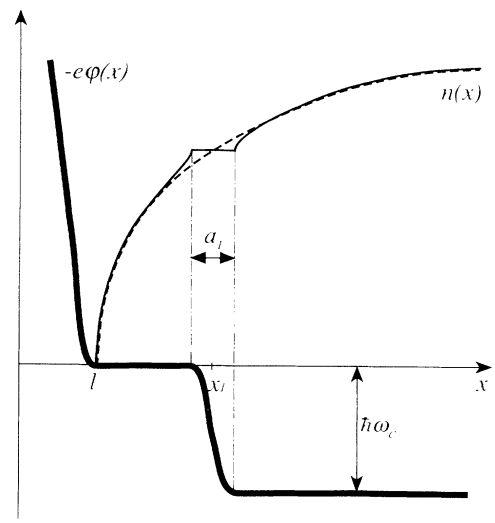

FIG. 3. 2DEG edge at magnetic field corresponding to the bulk filling factor $\nu_0=1.5$. The dashed line is the electron density at zero magnetic field. The solid line is the electron density, and the thick line is the electrostatic energy at $\nu_0=1.5$.

Despite the fact that the width of the depletion strip has been found for the gate-confined 2DEG, we believe that our result can be also applied to the etched structures. In that case, the half-width of the forbidden gap takes the place of the gate voltage due to the pinning of the Fermi level by the surface states. The width of the edge depletion ($2l$ in our notation) has been studied experimentally by Choi, Tsui, and Alavi.[8] They obtained the value 5000±2000 Å for a 2DEG density $n_0=1.2\times10^{11}$ cm^{-2}, which is in a reasonable agreement with our estimate.

III. DIPOLAR STRIP FORMATION IN HIGH MAGNETIC FIELD

Our next step is to consider the effect of strong magnetic field H in the IQHE regime. Here we ignore the electron spin. Due to the smallness of the parameter $\hbar\omega_c/eV_g$ (where $\omega_c=eH/m_{\rm eff}c$ is a cyclotron frequency) at any reasonable magnetic field, we expect that the width of the depletion region given by Eq. (9) will remain practically unchanged. Also, one might anticipate that the electron density distribution (10) obtained from electrostatics will not be altered significantly. This is because of the huge amount of work needed to be performed against electrostatic forces in order to produce a variation.

The only effect of the magnetic field from the electrostatic point of view is the periodic dependence of screening properties of the 2DEG on the filling factor ν, caused by the oscillations in the density of states. Screening at integer filling factors ($\nu=k$, $k=1,2,3,\ldots$) is absent, while at noninteger ν it is very strong. This should lead to the formation of the alternating compressible and incompressible liquid regions. The latter ones are characterized by different integer filling factors $\nu=k$.[3] Near the boundary, these regions should take the form of strips parallel to the gate edge. The location of the kth in-

compressible liquid strip x_k (measured from the middle of the depletion strip) can be found by substituting $n(x)=k/2\pi\lambda^2$ in Eq. (10) and solving it for x:

$$x_k = l\frac{n_0^2+k^2n_L^2}{n_0^2-k^2n_L^2} = l\frac{\nu_0^2+k^2}{\nu_0^2-k^2}\ , \tag{12}$$

where we use the notation $n_L=1/2\pi\lambda^2$ for the electron density corresponding to one completely filled Landau level, and $\nu_0=n_0/n_L$ is the filling factor far away from the boundary.

Let us ignore the effects of disorder for a while. Then the density of states is given by a set of δ functions centered at $(k-1/2)\hbar\omega_c$, and the screening length r_s as a function of the filling factor takes the following form:

$$r_s = \begin{cases} \infty, & \nu=k \\ 0, & \nu\neq k\ . \end{cases} \tag{13}$$

This means that the electrostatic potential is constant throughout any one compressible strip, just as in a metal. The electric field in the incompressible ones is unscreened. Our model is essentially similar to the one proposed in Ref. 9 for a Coulomb island.

For simplicity we consider, initially, the 2DEG edge at such magnetic fields that ν_0 satisfies the inequality $1<\nu_0<2$. This means that only one incompressible strip is formed. An example of such a situation corresponding to $\nu_0=1.5$ is shown in Fig. 3. The electrostatic solution (10) does not give the minimum energy state anymore. This is due to the fact that there is an additional energy cost $\hbar\omega_c$ involved in creating electron density exceeding $\nu=1$. Clearly, we could gain in energy if we relocate some of the electrons from the second Landau level to the first one in the close vicinity of x_1. This would create a flat region in the density distribution with the density corresponding to $\nu=1$ (see Fig. 3). This region is an incompressible strip, as discussed above. The drop of the potential between its edges should be $\hbar\omega_c/e$ in order to bring the second Landau level to the Fermi level. On both sides of the incompressible strip we have a compressible liquid, where the electric field is completely screened out. The discussed charge distribution can be thought of as an electrostatic solution (10) plus some additional charge creating the voltage drop. Because of its similarity to the three-dimensional dipole layer, we call this additional charge pileup a *dipolar strip.*

In order to find the width of the incompressible strip a_1, we need to solve an electrostatic problem similar to the one considered above. We solve the Laplace equation in the half-space $z<0$ for the given boundary conditions including a_1. Then we find the strip width a_1 from the requirement for the electric field to be zero at $x=x_1\pm a_1/2$. The boundary conditions for this problem are the following:

$$\phi'(x,z=0) = \begin{cases} -V_g\ , & x<-l \\ 0\ , & l<x<x_1-a_1/2 \\ \hbar\omega_c\ , & x>x_1+a_1/2\ , \end{cases} \tag{14}$$

$$\left.\frac{d\phi'(x,z)}{dz}\right|_{z\to-0} = \begin{cases} \dfrac{4\pi e n_0}{\epsilon}\ , & -l<x<l \\ \dfrac{4\pi e}{\epsilon}(n_0-n_L)\ , & |x-x_1|<a_1/2\ . \end{cases} \tag{15}$$

The solution of the Laplace equation can be found as a sum $\phi'=\phi_1'+\phi_2'+\phi$ of harmonic functions ϕ_1' ϕ_2', and the zero-magnetic-field solution ϕ. The first two satisfy the conditions

$$\phi_1'(x,z=0) = \begin{cases} 0\ , & x<x_1-a_1/2 \\ \hbar\omega_c\ , & x>x_1+a_1/2\ , \end{cases} \tag{16}$$

$$\left.\frac{d\phi_1'(x,z)}{dz}\right|_{z\to-0} = 0\ ,\quad |x-x_1|<a_1/2\ , \tag{17}$$

$$\phi_2'(x,z=0)=0\ ,\quad |x-x_1|>a_1/2\ , \tag{18}$$

$$\left.\frac{d\phi_2'(x,z)}{dz}\right|_{z\to-0} = \frac{4\pi e}{\epsilon}[n(x)-n_L] = \frac{4\pi e}{\epsilon}\left.\frac{dn(x)}{dx}\right|_{x=x_1}(x-x_1)\ ,\quad |x-x_1|<a_1/2\ . \tag{19}$$

Here we make two approximations based on the smallness of the parameter a_1/x_1, which is confirmed below. First, we extend conditions (16) and (18) to include $x<l$. In so doing, we neglect the charge distribution tail from the dipolar strip at the distances of order of x_1 away from this strip. Second, we substitute the exact $n(x)$ (10) by the first two terms in its Taylor series around x_1 in Eq. (19). Function ϕ_1' can be obtained from ϕ_1 by making the following substitutions: $V_g\to\hbar\omega_c$, $l\to a_1/2$, and $x\to x-x_1$. ϕ_2' is obtained in the Appendix. Just as in the absence of magnetic field, both solutions display singularities in electric field E_x at the incompressible strip edges ($x=x_1\pm a_1/2$). However, due to the symmetry of this problem, they cancel out on both sides at the same value of a_1 given by

$$a_1^2 = \frac{2\epsilon\hbar\omega_c}{\pi^2e^2dn/dx|_{x=x_1}}\ . \tag{20}$$

This is the equation defining the dipolar strip width. The magnetic-field-induced electron density in the dipolar strip is given by (see Fig. 4)

$$\Delta n = \frac{a_1}{2}\left.\frac{dn}{dx}\right|_{x=x_1}\times\begin{cases} t\ , & |t|<1 \\ t[1-(1-t^{-2})^{1/2}]\ , & |t|>1\ , \end{cases}$$

in terms of the normalized coordinate $t=2(x-x_1)/a_1$.

Equation (20) can be obtained from a simple qualitative consideration. On one hand, the drop of electrostatic potential across a dipolar strip is $\hbar\omega_c/e$. On the other hand, it is equal to the characteristic electric field E_x inside the strip $e(dn/dx)|_{x=x_1}a_1/\epsilon$ times its width a_1, thus yielding

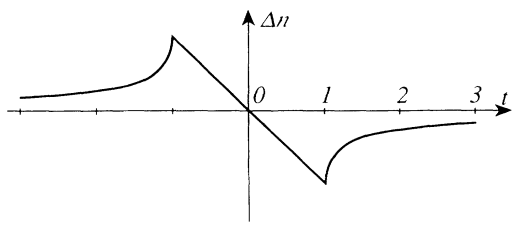

FIG. 4. Magnetic-field-induced additional electron density in the dipolar strip.

$$\frac{e^2 dn/dx|_{x=x_1} a_1^2}{\epsilon} \approx \hbar\omega_c \ . \tag{21}$$

Equation (21) gives an estimate for a_1 that coincides with Eq. (20) up to a numerical factor.

From both the qualitative and the quantitative derivations of Eq. (20), it should be clear that the appearance of the dipolar strip is a property of the 2DEG more general than the particular electrostatic problem under study. It happens in any situation when in a zero magnetic field the 2DEG has a small gradient of concentration. This gradient can be caused by the potentials of inhomogeneously distributed ionized donors as well as by gate-induced confinement of the 2DEG. The former case was studied by several authors.[10–12] Efros[13] gave the foregoing qualitative argument leading to Eq. (20). His estimate of the width of the regions occupied by incompressible liquid, being expressed in terms of disorder-induced ∇n, agrees qualitatively with our Eq. (20).

Let us rewrite Eq. (20) in terms of filling factor ν_0. From Eq. (10),

$$dn/dx|_{x=x_1} = n_0 \frac{l}{(x_1+l)(x_1^2-l^2)^{1/2}} \ . \tag{22}$$

Substituting x_1 from Eq. (12) in Eq. (22), we use it in Eq. (20) to express a_1^2 in terms of ν_0:

$$a_1^2 = \frac{8\epsilon\hbar\omega_c l}{\pi^2 e^2 n_L} \frac{\nu_0^2}{(\nu_0^2-1)^2} = a_B l f(\nu_0) \ . \tag{23}$$

Here we took into account that the combination $\epsilon\hbar\omega_c/2\pi e^2 n_L$ is nothing else but the Bohr radius $a_B = \hbar^2\epsilon/m_{\rm eff}e^2$ and $f(\nu_0)$ is a dimensionless function $(16/\pi)[\nu_0^2/(\nu_0^2-1)^2]$. Now we are in a position to check the assumption we made about the smallness of a_1. Making use of Eqs. (12) and (23), we find

$$\frac{a_1}{x_1} = \frac{4}{\pi^{1/2}} \frac{\nu_0}{\nu_0^2+1} \left[\frac{a_B}{l}\right]^{1/2} \ . \tag{24}$$

Equation (24) justifies the crucial assumption regarding the smallness of the dipolar strip width because $a_B \ll l$.

IV. ALTERNATING STRIPS OF COMPRESSIBLE AND INCOMPRESSIBLE LIQUID: QUANTITATIVE DESCRIPTION

Now we generalize the above consideration to the case when M Landau levels are completely filled far from the edge (M is the integer part of ν_0). Then M dipolar strips should form at the edge. Their positions are given by Eq. (12). The dipolar strip widths are defined by Eq. (20), easily generalized for any number $k=1,2,\ldots,M$:

$$a_k^2 = \frac{2\hbar\omega_c\epsilon}{\pi^2 e^2 dn/dx|_{x=x_k}} \ . \tag{25}$$

It is helpful to introduce $b_k = x_k - x_{k-1}$, which is essentially the width of a compressible strip to the left from x_k. At $\nu_0, k \gg 1$ it can be found from

$$b_k = \frac{n_L}{dn/dx|_{x=x_k}} \ . \tag{26}$$

Combining Eqs. (25) and (26), we find

$$a_k^2 = \frac{4}{\pi} b_k a_B \ , \quad \nu_0, k \gg 1 \ . \tag{27}$$

A key assumption in the derivation of Eq. (27) is the existence of the concentration gradient dn/dx in the zero-magnetic-field solution, which, however, did not enter the final expression. Hence the area of applicability of this relation is more general than just the solution for the gate-depleted 2DEG boundary. For example, it can be applied to etched structures.

Going back to the original problem, we can rewrite Eqs. (26) and (27) using Eqs. (10) and (12) in terms of the filling factor ν_0:

$$a_k = \frac{4}{\pi^{1/2}} (a_B l)^{1/2} \frac{\nu_0 k^{1/2}}{\nu_0^2 - k^2} \ , \tag{28}$$

$$\frac{a_k}{b_k} = \frac{1}{(\pi)^{1/2}} \left[\frac{a_B}{l}\right]^{1/2} \frac{\nu_0^2-k^2}{\nu_0 k^{1/2}} \ , \quad \nu_0, k \gg 1 \ . \tag{29}$$

We would like to mention here that the boundary conditions (19) should be altered for the case when a_k is much smaller than the distance to the surface of the semiconductor. Then we have

$$\left.\frac{d\phi_2'(x,z)}{dz}\right|_{z\to -0} = \frac{2\pi e}{\epsilon}[n(x)-n_L] = \frac{2\pi e}{\epsilon} \left.\frac{dn(x)}{dx}\right|_{x=x_1} (x-x_1) \ , \quad |x-x_1| < a_1/2 \tag{19'}$$

instead of (19). This leads to the value of a_k larger than in Eq. (28) by factor of $2^{1/2}$. We think that this correction is relevant only for the outer states.

For the inner edge states ($\nu_0 - k \ll \nu_0$) one gets

$$a_k = \left[\frac{4}{\pi}\right]^{1/2} (a_B l)^{1/2} \frac{\nu_0^{1/2}}{\nu_0 - k} \ , \tag{30}$$

$$\frac{a_k}{b_k} = \left[\frac{4}{\pi}\right]^{1/2} \left[\frac{a_B}{l}\right]^{1/2} \frac{\nu_0 - k}{\nu_0^{1/2}} \ . \tag{31}$$

Using $a_B = 100$ Å and $l = 2200$ Å, we see that for inner edge states ($\nu_0 - k \approx 1$), although a_k is large, a very

strong inequality $a_k/b_k \ll 1$ holds. It means that the approximation of independent and noninteracting dipolar strips used above works very well. When we move towards outer edge states, inequality $a_k/b_k \ll 1$ becomes weaker and eventually fails at small enough k.

Another important condition of validity of our theory is that the compressible liquids on both sides of incompressible strip k should screen well on the scale of a_k, i.e., behave like a good metal. We see two conditions for such behavior.

(i) The electron (hole) concentration on the $(k+1)$st (kth) Landau level at the distance a_k from the kth incompressible liquid strip is larger than a_k^{-2}:

$$\left[a_k \frac{dn}{dx}\bigg|_{x=x_k}\right] a_k^2 \gg 1 \ . \tag{32}$$

(ii) The size $\lambda k^{1/2}$ of electron wave functions for the kth Landau level satisfies the inequality

$$\lambda k^{1/2} \ll a_k \ . \tag{33}$$

One can show that at $na_B^2 > 1$ and $k > \nu_0/2$, condition (33) is violated earlier than (32) with decreasing magnetic field. Using Eq. (30), we can rewrite (33) in the form

$$\lambda \ll \frac{(a_B l)^{1/2}}{\nu_0 - k} \quad \text{for } \nu_0 - k \ll \nu_0 \ . \tag{34}$$

Because of the large value of l, inequality (34) for the inner channels ($\nu_0 - k \approx 1$) does not lead to substantial restrictions. With decreasing k, inequality (34) becomes more critical. We do not think that the violation of inequality (33) leads to a collapse of the dipolar strip, though our theory is not applicable in this case.

So far we have considered the IQHE regime ignoring the spin splitting of Landau levels. The crucial thing in our theory was the presence of a discontinuity in the chemical potential (equal to $\hbar\omega_c$), which led to the formation of incompressible liquid strips. Thus our theory can be generalized to include electron spin by substituting spin-splitting energy instead of the cyclotron energy. In a similar way, the edge states in the fractional quantum Hall effect regime are formed. Then the quasiparticle energy gap Δ_f takes the place of $\hbar\omega_c$ in our theory.[3] Positions of the incompressible strips (x_f) with filling factors f ($f = \frac{1}{3}, \frac{2}{3}, \frac{1}{5}, \ldots$) are given by the slightly modified Eq. (12):

$$x_f = l \frac{\nu_0^2 + f^2}{\nu_0^2 - f^2} \ . \tag{35}$$

Their widths can be found from Eq. (20), with an extra factor $2^{1/2}$. We use boundary condition (19′) because we anticipate the narrowness of fractional strips:

$$a_f = \left[\frac{4\epsilon\Delta_f}{\pi^2 e^2 dn/dx|_{x=x_f}}\right]^{1/2} = \frac{4\sqrt{2}c_f^{1/2}}{\pi^{1/2}} \frac{\nu_0 f^{1/2}}{\nu_0^2 - f^2} (\lambda l)^{1/2} \ . \tag{36}$$

Here we used expression $\Delta_f = c_f e^2/\lambda\epsilon$ for the fractional energy gap. If Eq. (36) yields $a_k < \lambda$, then the incompressible strip does not form.

V. TUNNELING THROUGH THE INCOMPRESSIBLE LIQUID STRIP: COMPARISON WITH EXPERIMENT

In this section we discuss our theory in relation to two experiments (one in the IQHE regime, the other in the FQHE regime). We start with the IQHE regime. Alphenaar *et al.*[4] studied equilibration among edge channels using a technique due to van Wees.[14] Current was injected only in the outermost channel, and its redistribution among the remaining channels was measured. It was shown that when filling factor ν_0 decreases in the vicinity of integer occupation ($N - 0.3 < \nu_0 < N + 0.3$), equilibration length $L_{N-1,N}$ between the $(N-1)$ and Nth channel grows rapidly and becomes too large to be measured at $\nu_0 \approx N - 0.3$. One of the most surprising results was the fact that in spite of a strong dependence of $L_{N-1,N}$ on ν_0, it is a periodic function with the period 1 (for ν_0 varying from 5 to 12). Indeed, functions $L_{N-1,N}(\nu_0)$ for various N collapsed on one curve if presented as functions of $\Delta\nu$. We would like to concentrate on this fact and show that, according to our theory, $L_{N-1,N}$ depends on magnetic field only through $\Delta\nu = \nu_0 - N$ for $N \gg 1$.

Tunneling between adjacent edge states is determined by overlap of the corresponding wave functions. Therefore, equilibration length $L_{N-1,N}$ depends crucially on the ratio a_{N-1}/λ. Substituting $k = N - 1 \gg 1$ in Eq. (28), we find

$$a_{N-1}/\lambda = \frac{(8a_B l n_0)^{1/2}}{\Delta\nu + 1} = \left[\frac{eV_g}{\pi^2 E_B}\right]^{1/2} \frac{1}{\Delta\nu + 1} \ , \tag{37}$$

where $E_B = e^2/2a_B\epsilon$ is the Bohr energy of the hydrogen-like impurity in GaAs. This result proves that the equilibration length is a function of $\Delta\nu$ independent of N, and explains the striking behavior of $L_{N-1,N}$ versus ν observed in experiment.[4] Substituting $V_g = 1$ V and $E_B = 6$ meV, one gets

$$\frac{a_{N-1}}{\lambda} \approx \frac{4}{\Delta\nu + 1} \ . \tag{38}$$

We see that in the range of interest ($N - 0.3 < \nu_0 < N + 0.3$), ratio a_{N-1}/λ is quite large. It is well known that, under these conditions, even a small amount of disorder affects significantly the tunneling rate causing its increase.[15–17] The dependence $\ln L_{N-1,N} \sim (a_{N-1}/\lambda)^2$ that is valid in the "clean case" is altered by disorder and changes the quadratic function in the above estimate to almost linear dependence: $\ln L_{N-1,N} \sim (2a_{N-1}/\lambda)[\ln A]^{1/2}$, where $A \gg 1$ at small disorder and decreases with increasing disorder. For the data of Ref. 4, the latter estimate seems to be more appropriate.

It is interesting to mention that the number of completely filled edge states changes by one when $\Delta\nu$ changes sign ($M = N - 1$ for $\Delta\nu < 0$, $M = N$ for $\Delta\nu > 0$). It means

that, in principle, one more equilibration length $L_{N,N+1}$ related to the width a_N may become relevant. But from our point of view, under the conditions of Ref. 4, the corresponding ratio

$$\frac{a_N}{\lambda}=\left[\frac{eV_g}{\pi^2 E_B}\right]^{1/2}\frac{1}{\Delta\nu}\approx\frac{4}{\Delta\nu} \tag{39}$$

for $0<\Delta\nu<0.3$ is too large to make equilibration observable, and the bulk of the sample is completely decoupled from the Nth channel.

Now we would like to give a more detailed interpretation of the experimental observations made in Ref. 4. We start with the magnetic field corresponding to ν_0 greater than integer number N ($\nu_0\approx N+0.4$). According to our picture, there are N incompressible liquid strips dividing the 2DEG edge into N edge channels and the bulk region occupied by compressible liquid. Equilibration among N edge channels occurs easily due to small distances between them. However, the bulk region is separated from the Nth edge channel by a dipolar strip that is wide enough to prevent their equilibration. At this magnetic-field resistance measurement with nonideal contacts injecting and detecting current in the outermost Landau level only will yield the result $R=h/e^2N$. Now we increase the magnetic field, thereby decreasing ν_0. This leads to the growth of the widths of dipolar strips. The first dipolar strip to become wide enough to quench equilibration (besides the Nth one, which is already very wide) is $(N-1)$. This leads to a gradual decoupling of the Nth edge channel. When the value of ν_0 crosses N, the Nth dipolar strip becomes infinitely large and disappears, creating a new bulk region out of the Nth edge channel to take the place of the old one. However, this should not affect the described measurement, as the $N+1$ incompressible region was already uncoupled. As we keep decreasing ν_0, the $(N-1)$ st dipolar strip grows wider and wider, making the equilibration into the Nth channel harder and harder to occur. This makes R closer to its value for $N-1$ channels $R=h/e^2(N-1)$. Finally, at some value $\nu_0\approx N-0.3$, the $(N-1)$st dipolar strip becomes so wide that no measurable equilibration through it occurs. Then we find the quantized value of $R=h/e^2(N-1)$. Further increase of magnetic field does not affect R (forming a plateau on the R versus H plot) until the width of the $(N-2)$ dipolar strip becomes large enough to quench equilibration into the $(N-1)$ edge channel. And then the whole cycle repeats itself.

Thus our theory of edge states suggests a quite satisfactory explanation of experimental observations of the anomalous QHE with so-called nonideal contacts, which probe only some edge channels.[4] In the same experiment,[4] the usual "bulk" QHE was observed while using standard probes. There is a significant difference in the physics of "bulk" and anomalous QHE's. Quantization of the Hall resistance in the former case is due to the localization of the bulk electron states. Quantization observed with nonideal probes occurs at different values of magnetic field and is due to the lack of equilibration. This effect is not a macroscopic one (it should vanish in sufficiently long samples) and usually the quantization is not as good as in the "bulk" QHE. While disorder is crucial for the observation of the bulk effect, it may destroy the anomalous QHE. We have already discussed disorder-assisted tunneling between the edge states. Now we consider the effect of long-range disorder on the edge states geometry using the approach due to Efros.[12,13,18]

The spatial scale of the random potential created by the random distribution of donors is of the order of the spacer layer thickness s. Let us use w to designate the amplitude of the random potential. If $w<\hbar\omega_c$ and $s<a_k$, then disorder does not change the general structure of alternating compressible and incompressible liquid strips. Changes occur only at the edges of compressible strips where the density of electrons (in almost-empty Landau level) or holes (in almost-filled Landau level) is less than the charge density needed to compensate for the random potential. The strips of localized compressible liquid appear at the edges of compressible strips (see Fig. 5). Equilibration between delocalized states of compressible liquid involves hopping through localized strips as well as tunneling through the incompressible strip. If the temperature is not too low, a typical hopping length is of the order of s. It means that, even in the presence of disorder with $s<a_k$, the equilibration process is dominated by the tunneling through distance a_k. Thus our conclusion about periodical dependence of $L_{N-1,N}$ on ν_0 remains valid when the random potential satisfies conditions $w<\hbar\omega_c$ and $s<a_k$. If disorder is strong ($w>\hbar\omega_c$), then continuous incompressible strips of the width a_k do not exist. Many islands of compressible liquid are formed inside each strip. The only relevant hopping length in this case is s, and we cannot arrive at the periodical dependence of $L_{N-1,N}$ on ν_0.

Now we turn our attention to the FQHE regime. Let us make an estimate of the positions and widths of the incompressible liquid strips under the conditions of experiment performed by Kouwenhoven *et al.*[5] This experiment (similar to the one discussed above) was performed at the bulk filling factor $\nu_0=1$, and the existence of the decoupled fractional edge channel was demonstrated on the lengths exceeding 2 μm. According to our theory, two dipolar strips are formed at filling factors $\frac{1}{3}$ and $\frac{2}{3}$. Their positions are [Eq. (35)] $x_{1/3}=\frac{5}{4}l$ and $x_{2/3}=\frac{13}{5}l$. Substituting $V_g=3$ V, $n_0=1.8\times10^{11}$ cm^{-2} in Eq. (9), we get $l=3600$ Å. From Eq. (36), using $\lambda=90$ Å ($B=7.8$ T)

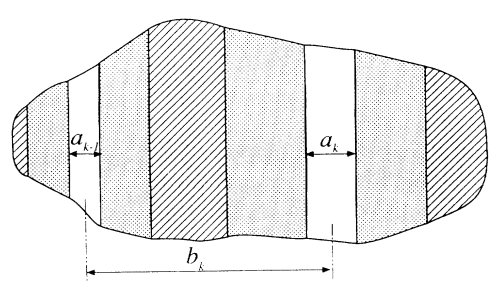

FIG. 5. Edge channels in the presence of disorder. Shaded areas represent delocalized compressible liquid. Dotted regions are occupied by localized compressible liquid. The rest is incompressible liquid.

and $c_{1/3}=c_{2/3}=0.03$,[19] we find

$$a_{1/3}=200\ \text{Å}=2.3\lambda\ , \tag{40}$$

$$a_{2/3}=460\ \text{Å}=5.1\lambda\ . \tag{41}$$

This gives an idea of why only the innermost channel was decoupled in the experiment.[5] The same measurements were done[5] on another sample with higher electron concentration $n_0=2.3\times10^{11}\ \text{cm}^{-2}$ and consequently smaller l at given voltage. At $V_g=3$ V no decoupling was observed ($a_{2/3}/\lambda=4.5$), but at $V_g=4.5$ V ($a_{2/3}/\lambda=5.6$) they saw the decoupling of the innermost channel. These observations are in qualitative agreement with our theory.

VI. CONCLUSION

In this paper we have studied the distribution of the electron density in 2DEG near the gate-induced edge. This is an electrostatic problem that can be solved analytically by exploiting the smallness of the 2DEG screening length in comparison with the depletion width $2l$. In the absence of magnetic field, l is the only relevant length scale for the electron density distribution. The magnetic field does not change this distribution on a rough scale. The exceptions are only the narrow strips near the lines where an integer number of Landau levels is fully occupied. A small portion of charge is redistributed, forming dipolar strips in the vicinity of those lines. The dipolar strip produces a steep drop in the electrostatic potential, which brings the next Landau level to the Fermi level; see Fig. 1(e). A complete analytical description of dipolar strips is obtained. The width of such a strip of incompressible liquid is much smaller than the width of an adjacent strip of compressible liquid. Moreover, these widths obey the universal relation [see Eq. (27)], which does not depend on magnetic field or their distance from the 2DEG boundary. We associate the equilibration between two neighboring edge states with the tunneling through the dipolar strip that separates them. This should give a better estimate of the equilibration rate than the one-electron model, because the dipolar strips are relatively narrow. Formulas for the widths of these strips obtained in this paper allows us to analyze the dependence of the equilibration length on magnetic field and gate voltage. In particular, we explain the experimentally observed periodic dependence of the rate of equilibration between the two innermost edge channels on the filling factor in the IQHE regime. The knowledge of incompressible strip widths was also used to discuss the difference in equilibration rates for $\frac{1}{3}$ and $\frac{2}{3}$ FQHE edge states.

Note added in proof. After submission of this paper we learned from Kane that he had performed a similar calculation of the dipolar strip.[20]

ACKNOWLEDGMENTS

We are indebted to P. L. McEuen, A. D. Stone, and R. G. Wheeler, who called our attention to the importance of edge state electrostatics. The authors wish to thank K. A. Matveev for a careful reading of the manuscript and for making a series of helpful comments. We are grateful to E. B. Foxman, P. A. Lee, P. L. McEuen, and I. M. Ruzin for helpful discussions. We would like to thank A. L. Efros and the authors of Ref. 4 for sending us results of their work prior to publication. Two of us (L.G. and B.S.) acknowledge support by NSF Grant No. DMR-91-17341. One of us (L.G.) was also supported in part by Research Funds of the Graduate School of the University of Minnesota; another (D.C.) was supported by NSF Grant No. DMR-89-13624.

APPENDIX

We have to solve the Laplace equation $\Delta\phi(\mathbf{r})=0$ in the half-space $z<0$ with the following boundary conditions:

$$\left.\frac{d\phi(x,z)}{dz}\right|_{z\to-0}=\tau(x),\quad |x|<l\ ,$$
$$\phi(x,z=0)=0,\quad |x|>l\ . \tag{A1}$$

As boundary conditions are independent of y, the problem becomes two dimensional. Following Ref. 6, we solve by means of the analytic functions theory. Let us represent $\phi(x,z)$ as an imaginary part of the analytic function $F(\zeta)$, where $\zeta=x+iz$. $F(\zeta)$ should satisfy boundary conditions

$$\text{Re}\left[\frac{dF}{d\zeta}\right]=\tau(x),\quad |x|<l\ ,$$
$$\text{Im}\left[\frac{dF}{d\zeta}\right]=0,\quad |x|>l\ . \tag{A2}$$

Now we introduce the analytic function

$$f(\zeta)=i\frac{dF}{d\zeta}(l^2-\zeta^2)^{1/2}\ , \tag{A3}$$

for which we know the imaginary part everywhere on the real axis:

$$\text{Im}[f(x)]=\tau(x)(l^2-x^2)^{1/2},\quad |x|<l\ ,$$
$$\text{Im}[f(x)]=0,\quad |x|>l\ . \tag{A4}$$

With this information we regenerate $f(\zeta)$ in the lower half-plane using the Schwartz integral:

$$f(\zeta)=\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\text{Im}[f(x)]}{x-\zeta}dx+c$$
$$=\frac{1}{\pi i}\int_{-l}^{l}\frac{\text{Im}[f(x)]}{x-\zeta}dx+c\ , \tag{A5}$$

where c is a constant of integration. We set $c=0$. For $\tau(x)=(4\pi e n_0/\epsilon)$, we obtain

$$\phi(x,z)=\frac{4\pi e n_0}{\epsilon}\text{Im}[\zeta+i(l^2-\zeta^2)^{1/2}]\ . \tag{A6}$$

For $\tau(x)=(4\pi e x dn/dx|_{x=x_k}/\epsilon)$, Eq. (46) yields

$$\phi(x,z)=\frac{4\pi e dn/dx|_{x=x_k}}{\epsilon}\text{Im}\ \{\tfrac{1}{2}\zeta[\zeta+i(l^2-\zeta^2)^{1/2}]\}\ . \tag{A7}$$

[1]C. W. J. Beenakker and H. van Houten, in *Solid State Physics,* edited by H. Ehrenreich and D. Turnbull (Academic, New York, 1991), Vol. 44; M. Büttiker, in *Semiconductors and Semimetals,* edited by M. Reed (Academic, New York, 1992), Vol. 35.

[2]M. Büttiker, Phys. Rev. B **38**, 9375 (1988).

[3]C. W. J. Beenakker, Phys. Rev. Lett. **64**, 216 (1990); A. M. Chang, Solid State Commun. **74**, 871 (1990).

[4]B. W. Alphenaar, P. L. McEuen, R. G. Wheeler, and R. N. Sacks, Phys. Rev. Lett. **64**, 677 (1990); B. W. Alphenaar, Ph.D. thesis, Yale University, 1991; B. W. Alphenaar, P. L. McEuen, R. G. Wheeler, and R. N. Sacks (unpublished).

[5]L. P. Kouwenhoven, B. J. van Wees, N. C. van der Vaart, C. J. P. M. Harmans, C. E. Timmering, and C. T. Foxon, Phys. Rev. Lett. **64**, 685 (1990).

[6]L. I. Glazman and I. A. Larkin, Semicond. Sci. Technol. **6**, 32 (1991).

[7]J. C. Maxwell, *A Treatise on Electricity and Magnetism* (Clarendon, Oxford, 1892).

[8]K. K. Choi, D. C. Tsui, and K. Alavi, Appl. Phys. Lett. **50**, 110 (1987).

[9]P. L. McEuen, E. B. Foxman, J. Kinaret, U. Meirav, M. A. Kastner, N. S. Wingreen, and S. J. Wind, Phys. Rev. B **45**, 11 419 (1992).

[10]S. Luryi, in *High Magnetic Fields in Semiconductor Physics,* edited by G. Landwehr (Springer, New York, 1987).

[11]U. Wulf, V. Gudmundson, and R. R. Gerhardts, Phys. Rev. B **38**, 4218 (1988).

[12]A. L. Efros, Solid State Commun. **67**, 1019 (1988).

[13]A. L. Efros, Phys. Rev. B **45**, 11 354 (1992).

[14]B. J. van Wees, E. M. M. Willems, C. J. P. M. Harmans, C. W. J. Beenakker, H. van Houten, J. G. Williamson, C. T. Foxon, and J. J. Harris, Phys. Rev. Lett. **62**, 1181 (1989).

[15]B. I. Shklovskii, Pis'ma Zh. Eksp. Teor. Fiz. **36**, 43 (1982) [JETP Lett. **36**, 53 (1982)]; B. I. Shklovskii and A. L. Efros, Zh. Eksp. Teor. Fiz. **84**, 811 (1983) [Sov. Phys. JETP **57**, 470 (1983)].

[16]D. G. Polyakov and M. E. Raikh, Solid State Commun. **74**, 1209 (1990).

[17]T. Martin and S. Feng, Phys. Rev. B **44**, 9084 (1991).

[18]A. L. Efros, Solid State Commun. **65**, 1281 (1988).

[19]A. M. Chang, in *The Quantum Hall Effect,* edited by R. E. Prange and S. M. Girvin (Springer-Verlag, New York, 1987).

[20]B. E. Kane, Ph.D. thesis, Princeton University, 1988.

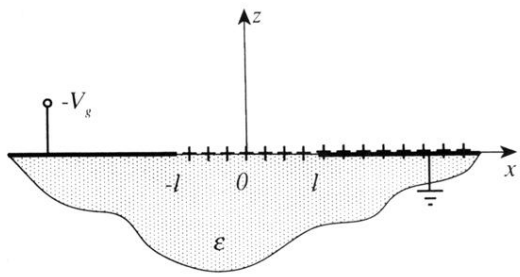

FIG. 2. Two-dimensional capacitor formed at the 2DEG edge. Thick lines represent two conductors: the gate at potential $-V_g$ on the left, and the grounded 2DEG on the right. Plusses represent a uniform positive background due to donors. The dotted area is occupied by a semiconductor with high dielectric constant ϵ.

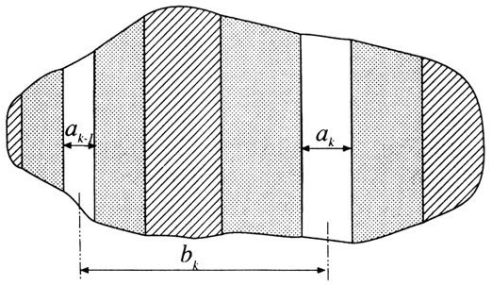

FIG. 5. Edge channels in the presence of disorder. Shaded areas represent delocalized compressible liquid. Dotted regions are occupied by localized compressible liquid. The rest is incompressible liquid.