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COMPARATIVE STATICS AND LOCAL INDETERMINACY IN OLG ECONOMIES: AN APPLICATION OF THE MULTIPLICATIVE ERGODIC THEOREM

by

John Geanakoplos and Donald J. Brown

This study is an effort to give a simple measure of the local size of the equilibrium set of OLG economies in which there may be more than one good and more than one consumer per period, and in which the generations may differ across time.

We are especially concerned with the meaning and significance of comparative statics, that is the analysis of the changes in the equilibrium set caused by small perturbations of the underlying economy at some given point in time, perhaps occasioned by government intervention in the marketplace. To this end we sharply distinguish between the thought experiment in which the perturbation is anticipated from the beginning of time before any agent has acted, and the perturbation that occurs as a surprise, say at date $t = 1$, after trade and consumption have been carried out for time $t \leqq 0$.

Our major findings can be summarized as follows. First we show that a regular, nondegenerate equilibrium $p = (p_t; -\infty < t < \infty)$ for an economy E is locally unique; it has the property that if the behavior or characteristics of the agents at time $t = 1$ is perturbed, by some sufficiently small policy change, for example, forming the economy $\hat{E}$, then there exists a unique equilibrium $\hat{p}$ for $\hat{E}$ that is near p in the sense that $\frac{|\hat{p}_t - p_t|}{|p_t|}$ declines geometrically to 0 as t approaches ∞ and as t approaches $-\infty$. The comparative statics for perfectly anticipated perturbations of a regular, nondegenerate OLG equilibrium is thus analogous to the comparative statics of a regular Arrow-Debreu equilibrium.

The situation is very different for unanticipated (but publicly announced) shocks or policy changes. Suppose again that p is a regular equilibrium for E and that the perturbation is unanticipated. Let $E^+ = (E|p)$ be the OLG economy, in which $1 \leqq t < \infty$, that is derived from E by fixing all behavior for $t \leqq 0$ at the market clearing prices p. Thus the old at time $t = 1$ maximize their utility, given that their consumption at time $t = 0$ was chosen as in (E,p), on the (erroneous) assumption that they would face prices p_1 at $t = 1$. The set of equilibria $p^+ = (p_1^+, p_2^+, \ldots)$ of E^+ that converge geometrically to p may now display a vast multiplicity, or it may be empty. Our second proposition asserts, however, that if it is nonempty, then it must be a manifold of some dimension no greater than the number of commodities (less one if no individual held any money savings from time 0).

The parametrizable indeterminacy that publicly announced but unanticipated policy can give rise to may be variously interpreted. It has been used in [13], in the context of a specific example, to suggest that Keynesian policy predictions are not inconsistent with utility maximization and market clearing, at least in the short run. For example, if it is generally believed that publicly announced monetary policy will affect quantities, without disturbing certain prices (such as wages) in the short run, then this believed potency of monetary policy can be exactly borne out in equilibrium.

Our third proposition sharpens the result in Proposition 2 in an (historically interesting) special case. Recall that when Samuelson [24] first introduced the one-commodity consumption-loan model he used it to explain how the "social contrivance" of money could effect Pareto improving trade in cases where otherwise the lack of "double coincidence of wants" prevented any trade. Proposition 3 hypothesizes an equilibrium which is regular and nondegenerate, and also autarkic (each individual consumes his endowment) and (forward) Pareto suboptimal. It concludes that almost any neutral monetary policy must have real effects--the indeterminacy, which Proposition 2 asserts is possible, necessarily obtains in this special case.

Throughout this paper our analysis relies on the regularity, and sometimes the nondegeneracy, of equilibrium. Roughly speaking, an equilibrium is defined to be regular if we can associate with it a finite number of well-defined "Lyapunov exponents," i.e., "long-run rates of growth." It is said to be nondegenerate if none of these is equal to 1. Proposition 4 asserts that almost every equilibrium of almost any economy satisfying our assumptions is regular; in particular, this implies that almost every such economy has at least one regular equilibrium. Nondegeneracy is a condition which can always be obtained by arbitrarily small perturbations. We do not analyze its probability except to note in the corollary to Proposition 3 that in the special case when a regular equilibrium is autarkic and Pareto optimal, it must also be degenerate.

A critical hypothesis which we need to derive our propositions is that the behavior of every generation when it is young is sensitive to the expectations it holds about the prices it will face when it is old. It is the possibility that behavior today may be different, not as a result of different exogenous influences, but because of different expectations, which can be rationally held, that gives OLG economies their distinctive character. By postulating expectations-sensitivity we concentrate on the difficult problem of analyzing the infinite regression of expectations about tomorrow's expectations about the day after tomorrow's expectations and so on.

The major analytical tools we rely on in this paper are (1) the multiplicative ergodic theorem proved by Osledec [20], a far-reaching generalization of the law of large numbers to the product of matrices, and (2) the application of this theorem to the theory of nonlinear dynamical systems, developed by Pesin [21] and Ruelle [22].

The obstacle to a straightforward extension of finite-dimensional comparative statics to large economies is that it would require the invertibility of an infinite dimensional matrix, a condition which is impossible to guarantee in general. For expectations-sensitive OLG economies, however, this infinite dimensional matrix may be replaced by the infinite product of finite dimensional matrices. Around a given equilibrium p, an expectations-sensitive economy E determines a transformation φ_t which maps each small change in prices Δp_t into the change in expected prices Δp_{t+1} necessary to induce the change Δp_t in market clearing prices at time t. The infinite iteration $\ldots \varphi_{t+1} \cdot \varphi_t \cdot \ldots \cdot \varphi_1$ is the formal expression of the intuition that lies at the heart of understanding open ended economies, namely that there may be an essential indeterminacy in how an economy responds to a public but unanticipated perturbation because what people do today depends on what they expect people to do tomorrow, which in turn depends on what people tomorrow expect people to do the day after tomorrow, etc.

To explicitly analyze the infinite iteration of the maps φ_t, one turns naturally to the product of the derivatives: $\ldots D\varphi_{t+1} \cdot D\varphi_t \cdot \ldots \cdot D\varphi_1$. But if the characteristics of each generation are chosen at random, then there is no necessary relation between any map φ_t and its successor, or between any matrix $D\varphi_t$ and its successor, and at first glance there does not seem to be any improvement in replacing one infinite dimensional matrix with the product of an infinite number of unrelated finite dimensional matrices. The essential insight of Pesin and Ruelle, however, is that one can deduce the behavior of the nonlinear dynamical system given by the φ_t from the long-run geometric mean of the derivative matrices $D\varphi_t$, provided this mean exists in the sense that the product $\ldots D\varphi_{t+1} \cdot D\varphi_t \cdot \ldots \cdot D\varphi_1$ determines a finite number of "Lyapunov exponents," or "generalized eigenvalues," or "long-run rates of growth."

The mean of an infinite sequence is something that one can hope is tractable. The multiplicative ergodic theorem asserts, among other things, that the product of matrices, drawn randomly from a known distribution, does have such a geometric mean, with probability one. The technical problem we must cope with is to link a given probability measure on the space of agent characteristics with the matrix product of the $D\varphi_t$ defined above, which is an endogenous function of the equilibrium path.

In Section 2 we present our model and discuss the expectations-sensitivity assumption and some ways of weakening it without affecting our analysis. In Section 3 we review the fundamental theorem on nonlinear dynamical systems by Pesin and Ruelle, and in Section 4 we use this theorem to prove Propositions 1-3. In Section 5 we give an illustrative example, and in Section 6 we introduce the multiplicative ergodic theorem, and use it to prove Proposition 4.

Before moving to Section 2, let us mention that since Samuelson's [24] pioneering article on the consumption-loan model there have been a number of analyses of the size of the equilibrium set of special classes of OLG economies. Gale [9] was the first to explicitly recognize the possibility of a one-sided indeterminacy in a time-homogeneous OLG economy with one consumer and one commodity per period. Grandmont [14] proved that in the same model it is possible to have deterministic cycles of all periods.

Balasko and Shell [4] extended the domain of inquiry to allow for heterogeneity between the generations and many commodities per period. They restricted their attention, however, to the case where each generation consists of a single consumer with Cobb-Douglas utilities, and they found that one-sided indeterminacy is impossible. Their analysis was generalized in Geanakoplos-Polemarchakis [12] to allow the single consumer to have an arbitrary intertemporally separable utility, and also by Kehoe-Levine [17], at least locally around a steady state. None of these one-consumer, separable utility models satisfies our expectations-sensitivity hypothesis.

Kehoe and Levine [16] treated a more robust class of economies, allowing for multiple commodities and many consumers per generation. They ruled out, however, the possibility of heterogeneity between generations, and they restricted attention to steady state equilibria. Their main, important contribution was to give a criterion for measuring the degree of one-sided local indeterminacy around a steady state equilibrium. This was the first paper to apply the technique of linearizing an autonomous dynamical system to the study of OLG economies. Our Proposition 2 may be regarded as a generalization to non steady-state equilibria of economies which are not time homogeneous, i.e., a generalization to non-autonomous dynamical systems. Muller and Woodford [19] showed how to extend the Kehoe-Levine analysis to steady states of time-homogeneous OLG economies with land and production and an infinite-lived agent. They also noted a connection between autarkic suboptimality and monetary indeterminacy of steady states.

Finally, let us mention our own earlier work, Geanakoplos-Brown [11] which attempted to complement the local analysis in this paper with a global treatment of the equilibrium set of an OLG economy. It also treated the connection between autarkic suboptimality and monetary indeterminacy, as well as the case of intertemporal separability.

2. The Model

We shall not be able to choose agents independently, instead we must choose them collectively as generations. We thus place our assumptions directly on the set A of possible generation excess demands. If $\xi \in A$, then $\xi = (\xi_y, \xi_0)$, where ξ_y and ξ_0 are functions from $R^{2\ell}_{++}$ into R^{ℓ} representing demand when young and old, given all the prices that will be faced over a lifetime: $\xi(p_a, p_b) = (\xi_y(p_a, p_b), \xi_0(p_a, p_b))$. We assume that:

A1: Any ξ in A can be generated as the aggregate excess demand of a set of H agents, hence it satisfies Walras Law $p_a\xi_y(p) + p_b\xi_0(p) = 0$, and homogeneity: $\xi(p) = \xi(\lambda p)$ for $\lambda > 0$, where $p = (p_a, p_b)$.

A2: ξ is three times differentiable.

A3: There is a compact set $\hat{\Delta} \subset \Delta^{2\ell-1}_{++}$, the $2\ell-1$ simplex of strictly positive prices, such that if ξ and ξ' are in A, and $\xi_0(p_a, p_b) + \xi'_y(p_b, p_c) = 0$, then $(p_a, p_b)/\|(p_a, p_b)\| \in \hat{\Delta}$ and $(p_b, p_c)/\|(p_b, p_c)\| \in \hat{\Delta}$.

A4: A is compact in the $C^3(\hat{\Delta}, R^{2\ell})$ topology of uniform convergence of functions and their derivatives up to third order.

A5: (Expectations sensitivity): For any $(p_a, p_b) \in \hat{\Delta}$, rank $\partial\xi_y/\partial p_b$ = rank $\partial\xi_0/\partial p_a = \ell$.

Differentiability is clearly needed to develop our regularity conditions and compactness is also essential, for it is imperative that the generations not be unboundedly different from each other across time.

A5 implies that demand today is sensitive to any change in tomorrow's prices. Combined with market clearing, it implies that agents can predict perfectly the price change tomorrow, given any price change today.

As we have said earlier, some kind of expectations sensitivity is central to our discussion of indeterminacy. This is embodied in the very strong expectations-sensitivity hypothesis A5. Although we shall shortly weaken this hypothesis, let us give one example to show that it is not contradictory, even if we restrict ourselves to separable (but many) utilities. Consider H Cobb-Douglas consumers $u^h = \sum_{i=1}^{\ell} \alpha_i^h \log x_{ai} + \sum_{j=1}^{\ell} \beta_j^h \log x_{bj}$ and endowments $(w_{a1}^h, \ldots, w_{a\ell}^h, w_{b1}^h, \ldots, w_{b\ell}^h)$. One easily shows that:

$$\frac{d\xi_y}{dp_b} = \begin{bmatrix} 1/p_{a1} & 0 & \cdots & \\ \vdots & 1/p_{a2} & \ddots & 0 \\ 0 & & 0 & 1/p_{a\ell} \end{bmatrix} \left(\sum_{h \in H} \alpha^h w_b^{h'} \right)$$

and similarly for $d\xi_0/dp_a$. When $H \geqq \ell$, a generic choice of the vectors $\{\alpha^h\}$, $\{\beta^h\}$, $\{w_a^h\}$, $\{w_b^h\}$ gives matrices $d\xi_y/dp_b$ and $d\xi_0/dp_a$ that are invertible for all $(p_a, p_b) >> 0$. This example should be contrasted with Balasko-Shell [3], which assumed one Cobb-Douglas consumer per generation.

An economy $E = (\xi^t)_{t\in\mathbb{Z}}$ is a selection from $\mathcal{E} = \prod_{-\infty}^{\infty} A$. We shall always use superscripts, such as with ξ^t , to refer to generations, and subscripts to refer to time periods. Thus $\xi_t(p_{t-1}, p_t, p_{t+1}) = \xi_0^{t-1}(p_{t-1}, p_t) + \xi_y^t(p_t, p_{t+1})$. An equilibrium p for the economy E is a price sequence $(\ldots p_{-1}, p_0, p_1 \ldots)$ in $\Pi = \prod_{-\infty}^{\infty} \mathbb{R}_{++}^L$ such that for all t , $\xi_0^{t-1}(p_{t-1}, p_t) + \xi_y^t(p_t, p_{t+1}) = 0$. As shown by Wilson [25], and Balasko, Cass, and Shell [2], any economy $E \in \mathcal{E}$ must have at least one competitive equilibrium, if A satisfies A1-A4. Moreover, A1-A4 also imply that the equilibrium graph is a compact set in the product topology, if we normalize $p_{01} = 1$.

There is a more useful way of normalizing prices which we shall use from now on. Let $\mathcal{D} = \prod_{-\infty}^{\infty} \hat{\Delta}$. Then $(q_{ta}, q_{tb})_{t\in\mathbb{Z}} \in \mathcal{D}$ is an equilibrium price sequence for the economy E if and only if

<u>E1</u>: $\xi_0^t(q_{ta}, q_{tb}) + \xi_y^{t+1}(q_{t+1a}, q_{t+1b}) = 0$ for all $t \in \mathbb{Z}$ and

<u>E2</u>: There is some $\underline{g} \leqq \overline{g}$ such that for all $t \in \mathbb{Z}$ there is g_t, $\underline{g} \leqq g_t \leqq \overline{g}$ with $q_{t-1b} = g_t q_{ta}$.

From the homogeneity of the ξ^t one can easily derive an isomorphism between equilibria $\overline{p}$ in Π (with $\overline{p}_{01} = 1$) and equilibria $\overline{q}$ in $\mathcal{D}$: simply take $(q_{ta}, q_{tb}) = (p_t, p_{t+1})/\|(p_t, p_{t+1})\|$, for all $t \in \mathbb{Z}$.[1]

Let T be the <u>shift</u> <u>operator</u> on $\mathcal{E} \times \mathcal{D}$:
$T((\xi^t)_{t\in\mathbb{Z}}, (q_{ta}, q_{tb})_{t\in\mathbb{Z}}) \equiv ((\xi^{t+1})_{t\in\mathbb{Z}}, (q_{t+1a}, q_{t+1b})_{t\in\mathbb{Z}})$. Let Q be the graph of the equilibrium correspondence:
$Q \equiv \{(E,q) \in \mathcal{E} \times \mathcal{D} | q \text{ is an equilibrium for } E\}$. We may call $Q \subset \mathcal{E} \times \mathcal{D}$ a graph on $\mathcal{E}$, since for any $E \in \mathcal{E}$, there is some $(E,q) \in Q$. Q is a closed, and hence compact, subset of $\mathcal{E} \times \mathcal{D}$. Moreover Q is shift

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[1]Conversely, $p_0 = q_{oa}/q_{0a1}$, $p_1 = q_{0b}/q_{0a1}$, $p_n = g_{n-1} \cdots g_1 q_{n-1b}/q_{0a1}$ for $n \geqq 2$ and $p_n = g_{n+1} \cdots g_0 q_{na}/q_{0a1}$ for $n \leqq -1$. Note that, also on account of the homogeneity of excess demands, $\partial\xi_y^t/\partial p_{t+1}|_{\overline{p}_t,\overline{p}_{t+1}}$ has full rank along the equilibrium path $\overline{p}$ in Π if and only if $\partial\xi_y^t/\partial q_b|_{\overline{q}_{ta},\overline{q}_{tb}}$ has full rank along the corresponding path $\overline{q}$ in $\mathcal{D}$. The advantage of the price space $\mathcal{D}$ is that equilibrium price sequences need not be normalized at an arbitrary date, and hence the equilibrium conditions are invariant to the relabelling of time periods.

invariant: $T^{-1}Q = Q$.[2] The shift invariance of Q will play a fundamental role in all of the following analyses.

It is possible to weaken assumption A5, and require that for every equilibrium $(E,\bar{q})$ in Q, each generation t is expectations-sensitive at the equilibrium prices $\bar{q}_t$, though perhaps not elsewhere. In fact, if we are interested in the local indeterminacy of a smaller set of equilibria, say some subgraph $\hat{Q}$ of Q, then assumption A5' can be used in place of A5:

<u>A5'</u>: (Expectations-sensitivity in equilibrium): Let $\hat{Q}$ be a closed invariant subgraph of Q. Suppose for all

$$(E,\bar{q}) = ((\xi^t)_{t\in\mathbb{Z}}, (\bar{q}_{ta}, \bar{q}_{tb})_{t\in\mathbb{Z}}) \in \hat{Q} ,$$

$$\text{rank} \left.\frac{\partial \xi_y^1}{\partial p_b}\right|_{\bar{q}_{1a},\bar{q}_{1b}} = \text{rank} \left.\frac{\partial \xi_0^1}{\partial p_a}\right|_{\bar{q}_{1a},\bar{q}_{1b}} = \ell .$$

The expectations-sensitivity of every generation t, at any equilibrium, follows from the expectations sensitivity of generation 1, at any equilibrium, on account of the shift invariance of Q (or $\hat{Q}$).

$\hat{Q}$ may, for example, be the set of monetaryless equilibria $(E, \bar{q})$ where $\bar{q}_{ta} \cdot \xi_y^t(\bar{q}_{ta}, \bar{q}_{tb}) = 0$ for all t. Alternatively, if A consists of a single generation, then we may take $\hat{Q}$ to be the steady state equilibria

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[2]Let $E = (\xi^t)_{t\in\mathbb{Z}}$, and let $T_1E \equiv (\hat{\xi}^t)_{t\in\mathbb{Z}}$, where $\hat{\xi}^t = \xi^{t+1}$, for all $E \in \mathcal{E}$. Similarly let $T_2((q_t)_{t\in\mathbb{Z}}) = (\hat{q}_t)_{t\in\mathbb{Z}}$, where $\hat{q}_t = q_{t+1}$, for all $q \in \mathcal{D}$. Then $T(E,q) = (T_1E, T_2q)$. Evidently T_2q is an equilibrium for T_2E if and only if q is an equilibrium for E. The same could not be said for T_2p, for $p \in \Pi$, since $(T_2p)_{01} = p_{11}$ may not equal 1.

$(E,\overline{q})$, $\overline{q}_t = q_1$ for all t . This is the model studied in Kehoe-Levine [16]. In general, the smaller is $\hat{Q}$, the more likely is assumption A5' to be satisfied. For the steady-state case, nearly any choice of A will satisfy A5', if we take $\hat{Q}$ to be the steady-state equilibria.

The most important consequence of assumption A5' is that, together with the implicit function theorem, it implies that for any $(E,\overline{q}) \in \hat{Q}$ and for all $t \in \mathbb{Z}$, there is an open ball A_t around $(\overline{q}_{ta}, \overline{q}_{tb}) \in \hat{\Delta}$ and a twice differentiable function $F_t : A_t \subset \Delta^{2\ell-1} \to \Delta^{2\ell-1}$ such that if $F_t(q_a, q_b) = (q_a', q_b')$, then there is $\lambda > 0$ with $q_a' = \lambda q_b$ and $\xi_0^t(q_a, q_b) + \xi_y^{t+1}(q_a', q_b') = 0$ and $F_t(\overline{q}_{ta}, \overline{q}_{tb}) = (\overline{q}_{t+1a}, \overline{q}_{t+1b})$. Similarly, from the hypothesis that along any equilibrium path, $\left.\frac{\partial \xi_0^t}{\partial p_a}\right|_{\overline{q}_{ta},\overline{q}_{tb}}$ is invertible, we can show the existence of a twice differentiable function $G_t : A_{t+1}^- \subset \Delta^{2\ell-1} \to \Delta^{2\ell-1}$ satisfying $\xi_0^t(G_t(q_a, q_b)) + \xi_y^{t+1}(q_a, q_b) = 0$ and $G_{tb}(q_a, q_b) = \lambda q_a$ for some $\lambda > 0$, and $G_t(\overline{q}_{t+1a}, \overline{q}_{t+1b}) = (\overline{q}_{ta}, \overline{q}_{tb})$. Clearly for (q_a, q_b) near $(\overline{q}_{ta}, \overline{q}_{tb})$, $G_t \cdot F_t(q_a, q_b) = (q_a, q_b)$.

For any economy E and equilibrium $\overline{q} \in \mathcal{D}$, the equilibria near $\overline{q}$ are given by sequences of the form $\{\ldots G_{-1} \cdot G_0(q), G_0(q), q, F_1(q), F_2 \cdot F_1(q) \ldots\}$ where $q = (q_a, q_b)$ is near $(\overline{q}_{0a}, \overline{q}_{0b})$. It is thus the iterated maps F_t and G_t that we must study.

Notice that $F_t(q_a, q_b)$ gives the prices at time $t+2$ which, if expected at time $t+1$, will allow prices q_a at time t and q_b at time $t+1$ to clear the period $t+1$ markets. Since in a perfect foresight equilibrium these expectations must be realized, we call the $(F_t)_{t \in \mathbb{Z}}$ the perfect foresight (expectations) functions. Similarly we call $(G_t)_{t \in \mathbb{Z}}$ the perfect hindsight functions.

Observe finally that since F_t has a local inverse G_t , each of the $(2\ell-1) \times (2\ell-1)$ matrices $(L_t)_{t \in K} = (DF_t|_{\overline{q}_{ta},\overline{q}_{tb}})_{t \in K}$ is invertible.

3. Dynamical Systems: Regularity and Nondegeneracy

Consider now an abstract dynamical system. Let A be an open ball in $\mathbb{R}^k$ around the origin $0 \in \mathbb{R}^k$, and let $(F_t)_{-\infty<t<\infty}$ be a sequence of differentiable maps (with differentiable inverses) from A into $\mathbb{R}^k$ that map 0 into itself. One may think of $F_t(q)$ as the prices that must be expected to occur at time $t+1$ in order for agents to act in such a way that the prices q clear the market at time t. (When prices are expected to be "normal" at time $t+1$, $q_{t+1} = 0$, they will be "normal" at time t, $q_t = 0$.) What can be said about the set of solutions
$S = \{(q_t)_{t\in\mathbb{Z}} | q_t \in A,\ q_{t+1} = F_t(q_t) \text{ for all } t\}$ or
$S^F = \{(q_t)_{t\geqq 1} | q_t \in A,\ q_{t+1} = F_t(q_t) \text{ for all } t \geqq 1\}$? The difficulty, of course, is that for $q \in A$, there is no guarantee that, say, $F_2(q) \in A$, since the range of F_2 is not confined to A. No matter how close q is to 0, there may be some large enough t such that $F_t \circ \ldots \circ F_1(q) \notin A$.

One hopes that the derivatives $(DF_t|_0)_{t\in\mathbb{Z}}$ can provide some information. For example, if $\|DF_t\|_0 \leq \gamma < 1$ for all $t \geqq 1$ (and if D^2F_t is uniformly bounded in t), then one can prove that for any q near enough to 0, $q_t = F_t \circ \ldots \circ F_1(q) \in A$; in fact, q_t converges geometrically to 0 as $t \to \infty$. In this "contracting case," locally S^F has dimension k. Unfortunately it is very rare that every DF_t is completely contracting: some DF_t may be contracting in some directions, and expanding in others. These DF_t may fit together in strange ways. In particular, the matrices may not commute, and what DF_t contracts, DF_{t+1} may then expand.

The central conclusion of Pesin and Ruelle is that the behavior of the dynamical system depends on a geometric average of the derivatives $L_t = DF_t|_0$, provided it exists.

Definition: Let $(L_t)_{t\in\mathbb{Z}}$ be a sequence of invertible $k \times k$ matrices. Write $L^t \equiv L_t \cdot L_{t-1} \cdot \ldots \cdot L_1$ for $t \geqq 1$, and L^{t*} for L^t transpose. We say the sequence is forward Lyapunov regular if (a) $\lim_{n\to\infty}(L^{n*} \cdot L^n)^{1/2n} = \Lambda$ exists and if (b) there are $s \leqq k$ numbers $0 < \gamma_1 < \gamma_2 < \ldots < \gamma_s$ and subspaces $\{0\} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_s = \mathbb{R}^k$ such that for any $z \in V_i \setminus V_{i-1}$, $\gamma(z) = \lim_{n\to\infty} \sqrt[n]{|L_n \cdot L_{n-1} \cdot \ldots \cdot L_1 z|} = \gamma_i$, for $i = 1, \ldots, s$. The numbers γ_i are the absolute values of the eigenvalues of Λ. They (or their logarithms) are usually called the Lyapunov exponents of $(L_t)_{t\in\mathbb{Z}}$, and $(V_i;\ i=1, \ldots, s)$ is called the corresponding foliation of $\mathbb{R}^k$.

Note that on account of the invertibility of the matrices, the Lyapunov exponents γ_i and their multiplicities (the dimensions of the V_i less V_{i-1}) are independent of the starting point. If, for example, we took $\lim_{n\to\infty} \sqrt[n]{|L_{n+1} \cdot L_n \cdot \ldots \cdot L_2 z|}$, we would get the same γ_i, and a corresponding foliation $V_i' = L_i V_i$, $i = 1, \ldots, s$. Forward Lyapunov regularity only depends on the right tail of the $(L_t)_{t\in\mathbb{Z}}$ sequence. It is thus applicable to a one-sided sequence $(L_t)_{t\geqq 1}$.

In case $k = 1$, the sequence of numbers is Lyapunov regular if and only if the successive geometric means converge. Consider also the special case where $L_i = L$ for all i, and L has k distinct real eigenvalues, which are ordered $0 \leqq |\lambda_1| < \ldots < |\lambda_k|$, corresponding to the linearly independent eigenvectors $e_1, \ldots, e_k$. Let V_1 be the span of e_1, $V_1 = \{\lambda e_1 | \lambda \in \mathbb{R}\}$; let $V_i = \{\lambda_1 e_1 + \ldots + \lambda_i e_i | \lambda_1, \ldots, \lambda_i \in \mathbb{R}\}$. Then it is easy to see that if $z \in V_i \setminus V_{i-1}$, then $z = \lambda_1 e_1 + \ldots + \lambda_i e_i$ where $\lambda_i \neq 0$. Clearly, $L^n z = \lambda_1^n e_1 + \ldots + \lambda_i^n e_i$, and as n grows large, $\sqrt[n]{|L^n z|} \to \sqrt[n]{|\lambda_i^n e_i|} \to |\lambda_i|$. Thus we see that Lyapunov regularity demands that it is possible to decompose $\mathbb{R}^k$ into generalized eigenspaces for the product $\ldots L_n \cdots L_1$. The numbers $\gamma_1, \ldots, \gamma_s$ (or their logarithms) are called the Lyapunov exponents of the product. When the matrices L_i are different, it seems almost incredible that Lyapunov regularity should ever hold. That is the question which is discussed in Section 6.

For our purposes the most striking consequence of Lyapunov regularity is the following theorem on nonlinear dynamical systems. Its proof, in various different forms, can be extracted from Pesin [21] and Ruelle [23]. It has also been discussed in Katok [15].

<u>Theorem A.</u> Let A be an open ball around 0 in $\mathbb{R}^k$. Let $(F_t)_{t\geq 1}$ be a sequence of twice differentiable maps from A into $\mathbb{R}^k$, taking 0 into itself, that has uniformly bounded second derivatives, $\|D^2F_t|_q\| \leq J$ for all $t \geq 1$, $q \in A$. Suppose furthermore that the sequence of derivatives $(DF_t|_0)_{t\geq 1}$ is forward Lyapunov regular, with foliation $V_1, \ldots, V_s$. Let d be the dimension of V_i, the subspace corresponding to the largest Lyapunov exponent γ_i less than one, and let γ satisfy $\gamma_i < \gamma < 1$. Then there are numbers $\Delta > \delta > 0$ such that $S_\gamma^F = \{q \in A \mid |q| < \delta$ and $|F_t \cdot F_{t-1} \cdot \ldots \cdot F_1(q)| < \Delta\gamma^t$ for all $t \geq 1\}$ is a manifold of dimension d, tangent at 0 to V_i.

The theorem states that if the linear approximation $(DF_t|_0)_{t\geq 1}$ of the dynamical system has a (linear) "stable manifold" V_i of dimension d around 0, then so does the nonlinear dynamical system $(F_t)_{t\geq 1}$. The theorem is a generalization of the fundamental stable manifold theorem for dynamical systems for the case when $F_t = F$ for all $t \geq 1$. In the time-homogeneous case it is possible to derive a stronger result when $DF|_0$ has no eigenvalues of modulus 1. Then we can replace S_γ^F by

$S^F = \{q \in A \mid |q-\bar{q}| < \delta$ and $|F^t q - \bar{q}| < \delta\}$. The condition that no Lyapunov exponent be 1 is sufficiently important to be called hyperbolicity or nondegeneracy.

Definition. Let $(L_t)_{t \geq 1}$ be a sequence of $k \times k$ matrices. We say the sequence is nondegenerate (or hyperbolic) if there is no vector $z \in \mathbf{R}^k$ such that $\lim_{n \to \infty} \sqrt[n]{|L_n \cdot L_{n-1} \cdot \ldots \cdot L_1 z|} = 1$.

In addition to forward Lyapunov regularity, one can also define backward Lyapunov regularity and their combination, regularity.

Definition. Let $(L_t)_{t \in \mathbf{Z}}$ be a sequence of invertible $k \times k$ matrices. We say the sequence is backward Lyapunov regular if the sequence $(M_t)_{t \geq 1} \equiv (L_{-t}^{-1})_{t \geq 1}$ is forward Lyapunov regular. If $(L_t)_{t \in \mathbf{Z}}$ is both forward Lyapunov regular, with Lyapunov exponent function $\gamma(z)$, and backward Lyapunov regular, with exponent function $\beta(z)$, and if there are k linearly independent vectors $z_1, \ldots, z_k$ such that $\gamma(z_j) = 1/\beta(z_j)$, $j = 1, \ldots, k$, then it is regular.

4. Local Indeterminacy and Comparative Statics

Let A and $\hat{Q}$ satisfy assumptions A1-A4 and A5'. Let $(E,\bar{q})$ be an equilibrium in $\hat{Q}$. We know that associated to each such equilibrium there is a uniquely given sequence of $(2\ell-1)\times(2\ell-1)$, invertible matrices $(L_t)_{t\in\mathbb{Z}}$. We can thus give:

Definition. The equilibrium $(E,\bar{q})$ is regular iff (1) the associated sequence of $2\ell-1\times 2\ell-1$ matrices $(L_t)_{t\in\mathbb{Z}}$ is regular in the sense of Lyapunov and (2) letting $g_t \equiv |\bar{q}_{t-1b}|/|\bar{q}_{ta}|$, $\lim_{n\to\infty} \sqrt[n]{g_n \cdots g_1}$ exists and equals $\lim_{n\to\infty}(\sqrt[n]{g_{-n} \cdots g_{-1}})^{-1}$. Furthermore, we say that $(E,\bar{q})$ is a nondegenerate equilibrium iff no Lyapunov exponent of the associated $(L_t)_{t\in\mathbb{Z}}$ is equal to one.

Condition (1) refers to the regularity discussed in Section 3, and condition (2) posits that there is a well-defined long run average rate of growth of the prices. If we had chosen the other price normalization $(E,\bar{p})$, $\bar{p}_{01} = 1$, $\bar{p} \in \Pi$ (for the same equilibrium), then condition (2) implies that $\lim_{t\to\infty} \sqrt[t]{\|\bar{p}_t\|} = \lim_{t\to\infty}(\sqrt[t]{\|\bar{p}_{-t}\|})^{-1}$, and is equal to the limit in condition (2).

It can be shown without much trouble (see Balasko-Shell [4]) that if $g < 1$, then the regular equilibrium $(E,\bar{q})$ is forward Pareto optimal: there is no reallocation of the commodities from period 1 onwards that improves one individual's utility without harming another's. On the other hand, if $g > 1$, then the equilibrium is not forward Pareto optimal. In Propositions 1 and 2 we concentrate exclusively on the question of local uniqueness vs. local indeterminacy. In Proposition 3 we turn to the connection between indeterminacy and forward Pareto suboptimality in a special context.

Given an economy $E = (\xi^t)_{t\in Z}$, we define a perturbation as an economy $\hat{E} = (\hat{\xi}^t)_{t\in\mathbb{Z}}$ in which $\hat{\xi}^t = \xi^t$ for all but a finite number of t. The size of the perturbation is given by $\underset{t\in\mathbb{Z}}{\mathrm{Max}}\|\hat{\xi}^t - \xi^t\|$, where we use the C^3 norm. A perturbation $\hat{E}$ of E is of size zero iff $\hat{E} = E$. It is sometimes useful to think of a perturbation not only as an experiment in comparative statics, but possibly as a perfectly anticipated policy change. A rational expectations equilibrium $(q_t)_{t\in Z}$ for $\hat{E}$ may differ at all t from any equilibrium path $(\overline{q}_t)_{t\in\mathbb{Z}}$ of E, even though $\xi^t = \hat{\xi}^t$ until $t = 1$.

Proposition 1. Let $(E,\overline{q}) \in \hat{Q}$ be a regular, nondegenerate equilibrium, where $\hat{Q}$ satisfies A1-A5'. Then every sufficiently small perturbation $\hat{E}$ of E, including E itself, has a locally unique equilibrium near $\overline{q}$. More precisely, for any such perturbation $\hat{E}$ and any $\gamma < 1$ sufficiently big, there are numbers $\delta > 0$, $\Delta > \delta$, $\Delta' > \delta$ such that the set $S_{\delta,\Delta,\Delta'} = \{(q_t)_{t\in\mathbb{Z}} \in \mathcal{D} \mid |q_1 - \overline{q}_1| < \delta$ and $|q_t - \overline{q}_t| < \Delta\gamma^t$ for all $t \geqq 2$ and $|q_t - \overline{q}_t| < \Delta'\gamma^{-t}$ for all $t \leqq 0\}$ contains exactly one equilibrium for $\hat{E}$.

Proof. From the discussion in Section 2, we know that for each $t \geqq 1$ there is an open ball A_t in $\Delta^{2\ell-1}$ centered around $\overline{q}_t = (\overline{q}_{ta}, \overline{q}_{tb})$ on which the perfect foresight function F_t is defined, and similarly for any $t \leqq 0$ there is an open ball A^-_{t+1} around $\overline{q}_{t+1}$ on which the perfect hindsight function G_t is defined. From the compactness and shift invariance of $\hat{Q}$ it follows that the balls A_t and A^-_{t+1} may be taken to be of the same radius $r > 0$, independent of t (and indeed of the equilibrium $(E,\overline{q})$). Furthermore, it must also be true that on each such A_t (A^-_{t+1}), D^2F_t (D^2G_t) is uniformly bounded, again independent of t (or indeed the equilibrium $(E,\overline{q})$).

We have now established that the hypotheses of Theorem A apply to $(F_t)_{t \geqq 1}$ and to $(G_t)_{t \geqq 0}$. The first has a local stable manifold through $\bar{q}_1$ in $\Delta^{2\ell-1}$ of dimension d, and the latter of dimension s. The regularity of $(E,\bar{q})$ implies that the tangent spaces to these stable manifolds are independent: the first is spanned by $z_1, \ldots, z_d$, on each of which the Lyapunov exponent of $(F_t)_{t \geqq 1}$ is less than one and the Lyapunov exponent of $(G_t)_{t \leqq 0}$ is its reciprocal, hence greater than 1 (see Section 3 for the definition of regularity). From the nondegeneracy hypothesis we deduce that $d+s = 2\ell-1$, and hence that the tangent spaces to the stable manifolds span $\mathbb{R}^{2\ell-1}$. In other words, the stable manifolds intersect transversally, hence locally at a single point (namely $\bar{q}_1$). Small perturbations will perturb the manifolds, but they will still intersect at a unique point, in a neighborhood of $\bar{q}_1$. Q.E.D.

The situation is quite different for unanticipated shocks or policy changes. Let $(E,\bar{q})$ be an equilibrium, in which generation 0 consumes and saves when young with certain expectations in mind, namely that it will face prices $\bar{q}_{0b}$ when old. Suppose instead that at the beginning of period 1 its members discover that the world is different from what they thought; they, and all succeeding generations, maximize according to their new expectations. What can be said about the size of the set of one-sided equilibria?

Let $\xi_s^{0h}(q, I^h)$ represent the net purchases individual $h \in H$ of generation 0 would make in his old age if he is surprised to find that the prices turn out to be q, instead (perhaps) of the $\bar{q}_{0b}$ he expected when he purchased $\bar{z}_y^h = \xi_y^{0h}(\bar{q}_{0a}, \bar{q}_{0b})$ in his youth. Of course if he is not surprised, $q = \bar{q}_{0b}$ and $I^h = \bar{M}^h \equiv -\bar{q}_{0a} \cdot \bar{z}_y^h$, then he will act according to plan: $\xi_s^{0h}(\bar{q}_{0b}, \bar{M}^h) = \xi_0^{0h}(\bar{q}_{0a}, \bar{q}_{0b})$. We assume, of course, that the aggregate excess demands $\xi_s^0(q, I^1, \ldots, I^H) = \sum_{h=1}^{H} \xi_s^{0h}(q, I^h)$ and $\xi^0(q_a, q_b)$ can be consistently derived from the maximizing behavior of the same H agents.

As before, we will identify policy changes with perturbations E of the economy $\hat{E}$ (and later of I^h). The idea that the shocks are unanticipated is captured by the definition of one-sided equilibrium that we will give. In order to simplify the statement of the next two propositions, we shall restrict our attention to neutral perturbations with respect to $(E,\overline{q})$, i.e. we will assume that excess demands in $\hat{E}$ remain the same as in E at the prices $\overline{q}$. The set of possible perturbations of E still allows for many changes in $\frac{d\xi_s^0}{dq}$ and $\frac{d\xi^1}{d(q_a, q_b)}$ consistent with rationality and neutrality.[3] Propositions 2 and 3 are proved for "almost all" neutral perturbations.[4]

We call (μ, q^F) , $\mu \in \mathbb{R}$, $q^F = (q_{ta,tb})_{t \geq 1} \in \mathcal{D}^F \equiv \prod_{t=1}^{\infty} \hat{\Delta}$ a one-sided equilibrium of $(\hat{E}, \hat{M}^1, \ldots, \hat{M}^H)$ with respect to $(E,\overline{q})$ iff

$$\text{(E1)} \qquad \hat{\xi}_0^t(q_{ta}, q_{tb}) + \hat{\xi}_y^{t+1}(q_{t+1a}, q_{t+1b}) = 0 \quad \text{for all} \quad t \geq 1$$

$$\text{(E2)} \qquad q_{tb} = \lambda_t q_{t+1a} \text{ for some } \lambda_t > 0 \quad \text{for all} \quad t \geq 1$$

$$\text{(E3)} \qquad \sum_{h=1}^{H} \hat{\xi}_s^{0h}(q_{1a}, \mu\hat{M}^h) + \hat{\xi}_y^1(q_{1a}, q_{1b}) = 0 \ .$$

[3] Let $\overline{z}_0^h \equiv \xi_0^{0h}(\overline{q}_{0a}, \overline{q}_{0b})$. Then $d\hat{\xi}_s^0/dq\big|_{\overline{q}_{0b}, \overline{M}^h, h\in H}$ must satisfy a restriction for Walras Law, homogeneity, and be symmetric and negative definite on $[z_0^1, \ldots, z_0^H]^{\perp}$. Otherwise it can be arbitrary.

[4] Consider a finite dimensional submanifold of K of neutral perturbations $\hat{E}$ which allows for all changes in $d\xi_s^0/dq$ and $d\xi^1/d(q_a, q_b)$ evaluated at the old equilibrium $\overline{q}_{0a}$, $\overline{q}_{0b}$, $\overline{M}^h$, etc. If we can show that for an open and dense subset K' of all sufficiently small perturbations in K , a property holds, and if this is true for all such submanifolds K of the space of all neutral perturbations, then we shall say that the property holds for almost all neutral perturbations.

(E1) and (E2) are as before. At time 1, the old maximize their utility given the prices q_{1a} and their real income $\mu\hat{M}^h$, $h \in H$. We suppose that the savings of the old is in the form of money $\hat{M}^h$. We incorporate the possibility of government monetary policy by allowing that $\hat{M}^h \neq \bar{M}^h \equiv -\bar{q}_{0a}\xi_y^{0h}(\bar{q}_{0a}, \bar{q}_{0b})$. We say that the monetary perturbation $(\hat{M}^1, \ldots, \hat{M}^H)$ is <u>neutral</u> if there is some scalar $\tilde{\mu}$ such that $\tilde{\mu}\hat{M}^h = \bar{M}^h$ for all $h \in H$. We need the scalar μ in (E3) because we renormalize prices every period, and the real value of individual savings depends on the absolute level of prices. Note that if the real and monetary perturbations are both neutral, then $(q_{ta}, q_{tb})_{t \geq 1} = (\bar{q}_{ta}, \bar{q}_{tb})_{t \geq 1}$ and $\mu = \dfrac{|\bar{q}_{1a}||\bar{M}^h|}{|\bar{q}_{1b}||\hat{M}^h|}$ is a one-sided equilibrium for $(E, \hat{M}^1, \ldots, \hat{M}^H)$ with respect to $(E,\bar{q})$. We define the size of the perturbation to be $\text{Max}(\|\hat{E}-E\|, (\hat{M}^h - \bar{M}^h); h \in H)$.

Finally, let us note that by not requiring any connection between $\bar{q}_{0b}$ and q_{1a}, we are considering unanticipated shocks or policy changes at time 1.

Let us begin with the simplest case, in which $(E,\bar{q})$ has no individual savings at time 0, and there is no monetary intervention: $M^h = \bar{M}^h = 0$ for all $h \in H$.

<u>Proposition 2</u>. Let $(E,\bar{q})$ be a (forward) regular equilibrium, where $(A,\hat{Q})$ satisfy A1-A5', in which there is no individual saving at time 0. Then there is an integer r, $0 \leq r \leq \ell-1$, such that for almost all sufficiently small, neutral perturbations $\hat{E}$ with respect to $(E,\bar{q})$, the set of all one-sided locally stable equilibria of $\hat{E}$ with respect to $(E,\bar{q})$ is a manifold of dimension r. More precisely, for almost any such neutral perturbation $\hat{E}$, and any $\gamma < 1$ sufficiently large, there are numbers $\Delta > \delta > 0$ such that $S^F_{\delta,\Delta} = \{(q_t)_{t \geq 1} \,|\, |q_1 - \bar{q}_1| < \delta$ and $|q_t - \bar{q}_t| < \Delta\gamma^t$ for all $t \geq 2\}$ contains exactly an r dimensional manifold of one-sided equilibria q^F.

Proof. As in Proposition 1, consider the forward local stable manifold S_1^F of $(F_t)_{t \geq 1}$ in $\Delta^{2\ell-1}$ through $\overline{q}_1$, where the $(F_t)_{t \geq 1}$ are the perfect foresight connecting functions for E at $\overline{q}$. S_1^F has dimension d, $0 \leq d \leq 2\ell-1$.

Consider the set $\hat{S}_1$ of $(q_{1a}, q_{1b}) \in \Delta^{2\ell-1}$ near $\overline{q}_1$ that solve (E3). For almost all perturbations $\hat{E}$ this is a manifold in $\Delta^{2\ell-1}$ through $\overline{q}_1$ of dimension $\ell-1$. Furthermore, for almost all perturbations $\hat{E}$, $\hat{S}_1$ and $\hat{S}_1^F$ are transverse: they intersect in a manifold of dimension $\mathrm{Max}(0, (\ell-1) + d - (2\ell-1)) = \mathrm{Max}(0, d-\ell)$. If $d-\ell \leq 0$, the intersection is at the single point $\overline{q}_1$, and so $S_{\delta,\Delta}^F$ contains a unique equilibrium $\overline{q}$.

Q.E.D.

In the more general case, neutral unanticipated policy may give rise to a still larger indeterminacy.

Proposition 2'. Let $(E,\overline{q}) \in \hat{Q}$ be a (forward) regular equilibrium, where $(A,\hat{Q})$ satisfy A1-A5'. Then there is an integer r, $0 \leq r \leq \ell$ such that for almost all sufficiently small, neutral (monetary) perturbations $(\hat{E}, \hat{M}^1, \ldots, \hat{M}^H)$ with respect to $(E,\overline{q})$, the set of one-sided locally stable equilibria of $(\hat{E}, \hat{M}^1, \ldots, \hat{M}^H)$ with respect to $(E,\overline{q})$ is a manifold of dimension r.

The proof is similar to Proposition 2; we do not give it. One simply takes into account the extra parameter μ to get the possibility of an extra dimension; $r = \mathrm{Max}[0, d+1-\ell]$, where d is the dimension of the forward stable manifold of $(E,\overline{q})$.

If we had allowed for nonneutral perturbations, the set of one-sided equilibria might have been empty (when $d+1 < \ell$).[5] Propositions 2 and 2' are generalizations of results derived by Kehoe-Levine [16] for the case when all generations are identical for $t \geqq 1$.

The striking difference between Proposition 2 (2'), and Proposition 1, or the analogous finite horizon case, is the possibility of indeterminacy, perhaps of high dimension, when policy is unanticipated. We shall now present one (historically, at least) important class of economies for which the dimension r in Proposition 2' can be taken to be at least one.

Recall that when Samuelson [24] first proposed the consumption loan model, he suggested that it explained how an economy might be in a situation where there was no double coincidence of wants, hence no possibility for trade, without the introduction of the "social contrivance of money."

<u>Proposition 3</u>. Let $(E,\overline{p}) \leftrightarrow (E,\overline{q}) \in \hat{Q}$ be a regular equilibrium, where $(A,\hat{Q})$ satisfy A1-A5'. Suppose in addition that $(E,\overline{q})$ is nondegenerate, and autarkic, and that the unnormalized prices $\overline{p}$ satisfy $\lim \sqrt[n]{|p_n|} > 1$. Then the dimension r of the locally stable solutions to almost any neutral (monetary) perturbation can be taken to be at least 1.

<u>Proof</u>. Recall from Proposition 2' that $r = \text{Max}(0, d+1-\ell)$, so it suffices to show that $d \geqq \ell$, where d is the dimension of the forward stable manifold. Note that since $(E,\overline{q})$ is nondegenerate and regular, there is a backward stable manifold S^B_{-1} at time -1 of dimension $s = (2\ell-1) - d$. It suffices to show that $s \leqq \ell-1$.

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[5] The analysis would also have been more complicated. It would have been necessary to extend the definition of regularity of $(E,\overline{q})$ to include the fact that S^F_1 and S_1 intersect transversally.

Let us apply Proposition 2' in the backward direction. In other words, let E^{-1} be the economy where time runs in the reverse direction, and $\bar{q}^{-1}$ the corresponding equilibrium. We know from Proposition 2' that for almost any neutral monetary perturbation of $(E^{-1}, \bar{q}^{-1})$, there is a locally stable manifold of one-sided equilibria of dimension $\mathrm{Max}(0, s+1-\ell)$. Suppose $s \geqq \ell$. Then this manifold has dimension at least one. In particular it contains one-sided equilibria distinct from $\bar{q}^{-1}$. But since $(E^{-1}, \bar{q}^{-1})$ is autarkic, any such different equilibrium must Pareto dominate $(E^{-1}, \bar{q}^{-1})$ in the forward direction. But in the forward direction of $(E^{-1}, \bar{q}^{-1})$, the unnormalized prices grow at a rate $\lim_{n\to\infty} \sqrt[n]{|p_{-n}|}$ less than one. Hence in this direction $(E,\bar{q})$ is Pareto optimal. Thus we must have $s \leqq \ell-1$, and the theorem is proved. Q.E.D.

Corollary 1. If an equilibrium $(E,\bar{q}) \in \hat{Q}$ is regular and both forward and backward Pareto optimal, then it is degenerate.

Proof. If not, either $d \geqq \ell$ or $s \geqq \ell$. Q.E.D.

Other versions of Proposition 3 have appeared in Geanakoplos-Brown [11], and Muller-Woodford [19].

Let us finally note, following an observation by Kehoe, Levine, and Mas-Colell,[6] that if at the equilibrium $\bar{q}$ of E all the excess demands satisfy the gross-substitutes conditions $\left.\frac{\partial \xi_i^t}{\partial q_{tj}}\right|_{\bar{q}_t} > 0$ if $1 \leqq i \neq j \leqq 2\ell$, then the dimension r of local indeterminacy determined in Proposition 2 must be zero. For if $\hat{q}$ is another equilibrium of a neutral perturbation

[6] Private correspondence.

$\hat{E}$ of E that is sufficiently near t, then it follows immediately from gross substitutes that $\max_{1 \leq i \leq 2\ell} \frac{\hat{q}_{ti}}{\bar{q}_{ti}} \leq \max_{1 \leq i \leq 2\ell} \frac{\hat{q}_{t+1i}}{\bar{q}_{t+1i}}$, hence it is impossible that $\hat{q}_t - \bar{q}_t$ converges geometrically to zero.

5. An Illustrative Example

We can illustrate Propositions 1-3 in a concrete example in which every generation t from $-\infty$ to ∞ consists of a single consumer with endowment $(w_t, 1-w_t)$ for the single good in each period of his two period life, and utility

$$u^t(x_t, x_{t+1}) = \alpha_t \log x_t + (1-\alpha_t)\log x_{t+1} .$$

As long as $0 < w_t < 1$ and $0 < \alpha_t < 1$, the excess demand for commodities by generation (α_t, w_t) satisfies the sensitivity conditions $\partial\xi_t/\partial p_{t+1} \neq 0 \neq \partial\xi_{t+1}/\partial p_t$ at any $(p_t, p_{t+1}) >> 0$. If we randomly choose each generation (α_t, w_t) from the rectangle $[a,b] \times [c,d]$, with $0 < a, b, c, d < 1$, then there typically will be no intergenerational homogeneity, no steady states, and no deterministic cycles. It is precisely this kind of setting to which our analysis is designed to apply.

Consider first, however, the simple steady-state case where the rectangle $[a,b] \times [c,d]$ reduces to a single point and for all t, $\alpha_t = \alpha < w = w_t$. Let us normalize prices, as we have throughout this analysis, by setting $r_t \equiv p_{t+1}/p_t$. Let $\hat{Q}$ be the set of steady state equilibria; there are two of these, one with $r_t = 1$ for all t, and in the other, autarkic steady state equilibrium, $r_t = \bar{r} = \frac{w}{\alpha}\frac{1-\alpha}{1-w} < 1$ for all t. Each of these steady-state equilibria is locally unique: for any perfectly anticipated change in the behavior of the generations alive at time 1, there is a unique path r_t such that r_t geometrically approaches $r = \bar{r}$ (or $r = 1$) as t approaches ∞ or $-\infty$. Thus Proposition 1 is confirmed for $(E,q) \in \hat{Q}$. Both these equilibria are regular (and nondegenerate) in the sense that around each equilibrium the $(DF_t|_{\bar{r}=r \text{ or } 1})_{t\in\mathbb{Z}}$ has a forward and a (reciprocal) backward Lyapunov exponent (different from one; in both equilibrium they are $1/\bar{r}$ and $\bar{r}$, respectively).

Proposition 2 asserts that the maximal dimension of one-sided indeterminacy around a regular equilibrium with no personal saving, such as the autarkic equilibrium $\bar{r}$, is $\ell-1 = 0$. Allowing for monetary policy, or more generally a regular equilibrium with personal saving, expands the maximal one-sided indeterminacy to $\ell = 1$ dimension, according to Proposition 2'. And indeed it can be confirmed (just as Proposition 3 predicts) that around the autarkic equilibrium $\bar{r}$, a "neutral" gift of money from the government to the old at time 0 will produce a continuum of possible locally stable one-sided equilibria.

In addition to the two steady equilibria, there are a continuum of other equilibria which satisfy the property that $r_t \to \bar{r}$ as $t \to \infty$ and $r_t \to 1$ as $t \to -\infty$. All of these equilibria are forward Lyapunov regular (and backward Lyapunov regular), so Proposition 2' applies to them: there is a one-dimensional one-sided indeterminacy in unanticipated policy. On the other hand, none of these equilibria is regular, since the forward and backward Lyapunov exponents are not reciprocals. As a result, none of them is locally unique. This illustrates the important point that we cannot in general expect all equilibria of nice economies to be regular. (In this example there is a forward and a backward "sink" which assures us that every shift-invariant measure λ on Q is concentrated on the steady state equilibria; in Section 6 it will become clear why this in turn might limit the class of regular equilibria for this example to steady states.)

Let us return now to the more general example where the rectangle $[a,b] \times [c,d]$ has positive area, $a < b$, $c < d$. The domain of Q is now not a point but an infinite dimensional space. Still Propositions 1-3 apply to regular equilibria. But how can be we sure there are any? Since there is only one consumer and one commodity per period, for any choice $(\alpha_t, w_t)_{t\in\mathbb{Z}}$, there must again be an autarkic equilibrium $(\bar{r}_t)_{t\in\mathbb{Z}}$. One can calculate the derivative of the perfect foresight connecting functions $(F_t)_{t\in\mathbb{Z}}$, $\left.\frac{dF_t}{dr_t}\right|_{\bar{r}_t} = \frac{\alpha_t^2(1-w_t)^2}{(1-\alpha_t)w_t\alpha_{t+1}(1-w_{t+1})}$. From the pointwise ergodic theorem, one can show that for $\bar{\mu}$-almost all choices of economies $(\alpha_t, w_t)_{t\in\mathbb{Z}}$, the autarkic equilibrium $(\bar{r}_t)_{t\in\mathbb{Z}}$ is regular. When there are several commodities per period, direct calculation will usually not demonstrate the existence of regular equilibria. Nevertheless, in Section 6 we will derive their existence from the multiplicative ergodic theorem.

The above calculation allows us to illustrate one more important point: it is evident that for some measures μ on $[a,b] \times [c,d]$, the geometric mean of $\left(\left.\frac{dF_t}{dr_t}\right|_{\bar{r}_t}\right)_{t\geq 1}$ is 1. To eliminate nondegeneracy, one may have to perturb the parameter space A or the measure. The classic case of degeneracy we have identified in Corollary 1: a Pareto optimal, autarkic equilibrium. Thus in the special steady state example when $\alpha_t = a = b = c = d = w_t$ for all t, Q contains a unique equilibrium, which is Pareto optimal and autarkic, and degenerate. The slightest perturbation of a, b, c or d removes this problem.

6. Regularity with Probability One

Needless to say, the utility of Propositions 1-3 depends crucially on how "likely" it is that Lyapunov regularity holds at a particular equilibrium. If it is unlikely, then Propositions 1-3 do not really permit us to analyze any economies except the steady state economies, where every generation is the same for all t. In fact, although regularity seems improbable, once we give the proper interpretation to likely, we shall show it almost always obtains.

It would be nice to show (as Debreu did in 1970 for regular finite economies) that for almost any economy, all of its equilibria are regular. But as the example in Section 5 shows, this is impossible. Instead we shall demonstrate that for almost all economies, almost all of their equilibria satisfy the Lyapunov regularity property. This will be deduced from the famous multiplicative ergodic theorem.

Thus we shall assign our probability measure λ directly to the equilibrium graph Q (or $\hat{Q}$, if we are interested only in a subgraph). Recall that $Q \subset E \times \mathcal{D}$, where $E = \prod_{-\infty}^{\infty} A$ is the space of economies. Perhaps there is a natural Borel measure $\bar{\mu}$ on E. For example, if μ is a Borel measure on the compact set A, then we might take $\bar{\mu} = \prod_{-\infty}^{\infty} \mu$. In this case our notion of a random economy corresponds to an independent drawing of the generations characteristics, governed by the measure μ. If there is a natural measure $\bar{\mu}$ on E, then we shall require that the marginal distribution of λ on E be equal to $\bar{\mu}$. In keeping with our point of view that the labelling of calendar time is not in itself significant, we suppose that $\bar{\mu}$ is stationary, or shift invariant. In other words, if $T_1 : E \to E$ is the one shift operator on E defined by

$T_1((\xi^t)_{t\in Z}) = (\hat{\xi}^t)_{t\in Z}$ with $\hat{\xi}^t = \xi^{t+1}$, and if F is any measurable subset of E, then we must have $\bar{\mu}(T_1^{-1}(F)) = \bar{\mu}(F)$. Of course if $\bar{\mu}$ is the product measure $\bar{\mu} = \prod_{-\infty}^{\infty} \mu$, then it is indeed shift invariant. We shall also concentrate on measures λ on Q (or $\hat{Q}$) which are shift invariant: for any measurable $G \subset Q$, $\lambda(T^{-1}(G)) = \lambda(G)$, where T is the shift operator on $E \times \mathcal{D}$ introduced in Section 2.

We are now in a position to invoke the multiplicative ergodic theorem to demonstrate that with λ probability one, all equilibria are regular. The multiplicative ergodic theorem is an extraordinary generalization of the pointwise ergodic theorem, itself a great leap up from the strong law of large numbers. The theorem was first proved by Osledecs, and can be derived from Kingman's subadditive ergodic theorem. Let us state it with the notation most suggestive for our purposes:

<u>Multiplicative Ergodic Theorem</u>. Let (Q,λ) be a probability space and let $\tau : Q \to Q$ be a transformation which preserves the measure λ: $\lambda(\tau^{-1}(G)) = \lambda(G)$ for all measurable $G \subset Q$. Let $L : Q \to M_{k\times k}$ map each x in Q into an invertible $k \times k$ matrix and suppose $\int_Q \text{Max}[0, \log(\|L^{-1}(x)\|]d\lambda < \infty$. Then $(L_t)_{t\in Z}$, $L_t = L(\tau^t(x))$ is regular in the sense of Lyapunov.

We now use this theorem to prove that equilibria are regular with probability one.

<u>Proposition 4</u>. Let $(A,\hat{Q})$ satisfy assumptions A1-A5'. Let λ be any shift-invariant probability measure on the equilibrium subgraph $\hat{Q}$. Then with λ-probability one each equilibrium is regular.

Proof. We must show first that for λ-almost all equilibria $(E,\bar{q})$ in $\hat{Q}$, the matrices $(L_t)_{t\in Z}$ are regular, where $L_t = DF_t|_{\bar{q}_{ta},\bar{q}_{tb}}$ and F_t is the perfect foresight function associated with the equilibrium $(E,\bar{q})$ at time t, as explained in Section 2.

Let $L : Q \to M_{2\ell-1\times 2\ell-1}$ be defined by $L(E,\bar{q}) = DF_1|_{\bar{q}_{1a},\bar{q}_{1b}}$. Then it is easy to see that $L_1 = L(E,\bar{q})$, and $L_2 = L(T(E,\bar{q}))$, and similarly for all t, $L_t = L(T^t(E,\bar{q}))$. By hypothesis, λ is shift-invariant, i.e. T preserves the measure λ. Moreover, since $DF_1|_{\bar{q}_1}$ is invertible, and since L is a continuous function on the compact set $\hat{Q}$, it follows that $\int_Q \mathrm{Max}[0, \log||L^{-1}(E,\bar{q})||]d\lambda < \infty$. Applying the multiplicative ergodic theorem we conclude that $(L_t)_{t\in Z}$ is a regular sequence of matrices.

Secondly, we must show that $\lim_{n\to\infty} \sqrt[n]{|\bar{q}_{1b}|/|\bar{q}_{2a}| \cdots |\bar{q}_{nb}|/|\bar{q}_{n+1a}|}$ exists and is equal to $\lim_{n\to\infty}(\sqrt[n]{|\bar{q}_{0a}|/|\bar{q}_{-1b}| \cdots |\bar{q}_{-n+1a}|/|\bar{q}_{-nb}|})^{-1}$, for λ-almost all equilibria $(E,\bar{q})$ in $\hat{Q}$. To this end let $\hat{L} : \hat{Q} \to M_{1\times 1}$ be defined by $L(E,\bar{q}) = |\bar{q}_{1b}|/|\bar{q}_{2a}|$. Another application of the multiplicative ergodic theorem concludes the proof. Q.E.D.

Corollary 2. Let $(A,\hat{Q})$ satisfy assumptions A1-A5', and let $\bar{\mu}$ be a stationary Borel probability measure on the space of economies $\mathcal{E}$. Then there is a probability measure λ on $Q \subset \mathcal{E} \times \mathcal{D}$, with marginal distribution $\bar{\mu}$ on $\mathcal{E}$, such that for $\bar{\mu}$-almost all $E \in \mathcal{E}$, the conditional measure λ_E on the equilibria of E is well-defined. Moreover, for $\bar{\mu}$-almost all economies E in $\mathcal{E}$, λ_E-almost all of their equilibria are regular. In particular, $\bar{\mu}$-almost all economies E in $\mathcal{E}$ have at least one regular equilibrium $(E,\bar{q}) \in \hat{Q}$.

<u>Proof</u>. Any Borel measure λ on the compact graph $\hat{Q}$ can be "disintegrated" into a marginal measure on E, and conditional measures λ_E, see [7]. Thus in view of Proposition 4, it suffices to show that there is a shift-invariant measure λ on $\hat{Q}$, with marginal distribution $\bar{\mu}$ on E.

We know, since $\hat{Q}$ is a closed graph, that there is some Borel measurable selection f from $\hat{Q}$. Let Γ be its graph: $\Gamma = \{(E, f(E))\}$ where E takes on all values in E. We can put a probability measure λ_0 on $\hat{Q}$ by setting, for any Borel measurable $G \subset \hat{Q}$,
$\lambda_0(G) \equiv \bar{\mu}\{E|(E, f(E)) \in G\}$. This clearly has marginal distribution $\bar{\mu}$ on E. Now, given the probability measure λ_0 on $\hat{Q}$, we can define probability measures λ_t on $\hat{Q}$ by $\lambda_1(G) \equiv \lambda_0(T^{-1}(G))$ and similarly $\lambda_t(G) \equiv \lambda_{t-1}(T^{-1}(G))$ for all $t \geqq 1$, and measurable $G \subset \hat{Q}$. Notice that if $\bar{\mu}$ is stationary, then all these measures λ_t on $\hat{Q}$ retain the same marginal distribution $\bar{\mu}$ on E. Finally, letting $\bar{\lambda}_t = \frac{1}{t+1} \sum_{s=0}^{t} \lambda_s$ we get a countable collection of measures $\bar{\lambda}_t$ on the compact set $\hat{Q}$, which by Prokhorov's theorem must have a limit measure λ (for some subsequence). It is easy to see that λ must be shift invariant and have marginal distribution $\bar{\mu}$ on E. Q.E.D.

Observe that Corollary 2 cannot be strengthened in any obvious way, for if $\hat{\hat{Q}}$ is a shift invariant closed subgraph of $\hat{Q}$ then it is possible that every shift invariant probability λ on $\hat{Q}$ with marginal distribution $\bar{\mu}$ on E is concentrated on $\hat{\hat{Q}}$. Our example in Section 6 makes this clear.

One can also use the multiplicative ergodic theorem to show that in the class of autarkic equilibria, regularity still occurs with probability one.[6]

[6] Let A^* be a collection of generation utility functions. Let B be a collection of generation endowments. Let $Q^* \subset \prod_{-\infty}^{\infty} A^* \times \prod_{-\infty}^{\infty} B$ be the set

Having shown that regularity occurs with λ-probability one, it might be conjectured that the same is true for nondegeneracy. Indeed, given any degenerate equilibrium, it can be perturbed away. If A is rich enough, then nondegeneracy is an open and dense condition in Q (with respect to the ℓ_∞ topology). However, as the example in Section 6 makes clear, such a conjecture is false. Nondegeneracy requires a long run average to be different from 1. The real numbers distributed between 0 and 2 form a rich set, but with the uniform probability their average is 1. To prove nondegeneracy is a probability one event, one would need to choose the measure $\bar{\mu}$, instead of taking it as arbitrary, as we did for regularity. We have not pursued this problem in this paper.

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of all utility and endowment assignments which are also (autarkic) competitive equilibrium. Then Q^* is a compact, shift-invariant, graph on $E^* = \prod_{-\infty}^{\infty} A^*$, given assumptions on A^* analogous to A1-A5. One can prove, in exactly analogous fashion to Proposition 4, that a typical autarkic equilibrium in Q^* is regular.

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