

Why is too much leverage bad for the economy?

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Abstract

In this paper we show that competitive equilibrium prices and margin requirements naturally lead to too much leverage relative to the constrained optimum. We describe two mechanisms through which equilibrium forces lead agents to borrow too much and to hold too little collateral.

To illustrate the first mechanism we present a very simple example without collateral and default where restricting borrowing leads to a Pareto-improvement over the competitive equilibrium allocation because financial markets are incomplete. Limiting borrowing naturally leads to a change in spot-prices that makes all agents better off. We then introduce collateral, default, endogenous margin requirements and production and we illustrate the second mechanism by showing that the endogenous margin requirements are suboptimal because they result in too much default. Finally we show how the two effects interact - forcing agents to leverage less leads to a Pareto-improvement because it reduces default and because it reduces borrowing.

1 Introduction

In this paper we examine how too much borrowing and too much leverage naturally leads to constrained suboptimal equilibria in a general equilibrium model where margin requirements are endogenously determined. We present examples where restricting borrowing or restricting leverage via increasing margin requirements leads to a Pareto-improvement over the competitive equilibrium allocation. The examples reveal two separate mechanisms through which the competitive equilibrium can be constrained suboptimal. The first mechanism is well known from the literature on general equilibrium with incomplete asset markets (see e.g. Geanakoplos and Polemarchakis (1986) or Carvajal and Polemarchakis (2011)) – when markets are incomplete a reallocation of assets in the first period will perturb equilibrium spot prices in the second period and may lead to better risk-sharing. In particular, it seems natural to conjecture that in a model with a durable good (housing) and a perishable good (food), the agent who likes housing is a natural borrower in the first period. When this agent is poor in a given state in the second period, the price of housing will be low and both borrower and lender would like to insure against this price-uncertainty. When markets are incomplete this might be impossible, but restricting borrowing might stabilize the price in that state, making both borrower and lender better off. We argue that this mechanism, which has nothing to do with collateral, leverage or default, is prevalent source of constrained suboptimality whenever financial markets are incomplete. The second mechanism builds on the obvious empirical and well known theoretical fact that debt-overhang can lead to inefficiencies (going back to Myers (1977)). In the context of housing (but also of corporate default) agents often forgo efficient production possibilities if they understand that their debt is so large that they will not benefit from undertaking production. Concretely, we assume that houses are traded with 'gardens' (i.e. a commodity different to the basic house that consists of land and structure that determines the quality of the house). The gardens depreciate but the owner of the house can rebuilt them in the second period at small costs. If the owner understands that the market price of the house is so low that he will not be able to repay the loan even if he builds the maximum amount of gardens he will default and build no gardens whatsoever (there is overwhelming empirical evidence that this effect is quantitatively important, see e.g. Melzer (2012) or Campbell et al. (2011)). This inefficiency is known to borrower and lender and priced into the loan contracts. However, agents do not understand that decreasing leverage decreases the fractions of households who do not build gardens and thereby increases the price of all houses. This second mechanism is relevant even in a model without aggregate uncertainty but also reinforces the first mechanism in a general model.

The idea that 'too much borrowing' is bad for economic growth and leads financial crises has a long history in economic thought. It has also been realized for some time that borrowing against collateral might have detrimental effects for welfare and increases volatility in asset prices. Geanakoplos (2009) lists several reasons why too much leverage is bad for the economy. There are now many interesting contributions that highlight the possibly negative effects of collateral and too much borrowing (e.g., among many others, Geanakoplos (2003), Krishnamurthy (2003), Fostel and Geanakoplos (2008), Bianchi (2011), Brunnermaier and Sannikov (2013)). While these are all inefficiencies caused by too much leverage it is far from clear that regulating leverage can lead to a Pareto-improvement. For example, the fact that it is costly to seize collateral is certainly reflected in the price of those contracts which are secured by little collateral and on which agents default frequently. In itself this is not an externality but needs to be viewed as part of the contracting

technology. There is now also several papers on constrained suboptimal outcomes in the presence of collateral and borrowing constraints (see e.g. Lorenzoni (2008) or Korinek (2012)). However, the mechanism examined in those papers is substantial different from ours. The main contribution of our paper is to give two extremely simple examples that illustrate the roles of two different important mechanisms.

In the first example, we consider a simple model with incomplete markets, but without collateral constraints, default or production. When markets are incomplete, Geanakoplos and Polemarchakis (1986) showed that generically there is an intervention at time 0 alone that can lead to a Pareto improvement by changing the asset ownership structure. When markets clear in period 1, the new distribution of assets leads to a change in prices that itself redistributes wealth across states in a way that was not spanned by the asset payoffs. This is the key idea to the example. Borrowers buy a durable good because they have a high marginal propensity to consume this good. They borrow to finance consumption in the first period. Unfortunately, they face endowment uncertainty in the second period and in the state where they are poor and want to sell housing while in the state where they are rich (and the lenders are relatively poor) they want to buy housing. Reducing borrowing increases the price of housing and helps the seller of housing in both states. Since the borrower is the seller for states where he has high marginal utility while the lender is the seller in those states where his marginal utility is high, both borrower and lender gain from these price-changes.

It is often argued that reducing leverage by regulating margin requirements on loans can have beneficial effects for the economy (see e.g. Geanakoplos (2003) or Ashcroft et al. (2010)). In order to formalize this idea we construct a model with endogenous leverage, production and costly default. As in Geanakoplos (1997), we endogenize the margin requirements by assuming that agents have access to a large set of contracts which distinguish themselves only by their margin requirements. Equilibrium prices for all the contracts are determined in a competitive equilibrium, but since collateral is scarce only few of them are actively traded. In fact, in our setup below we can show that only one contract is actively traded and hence that the margin requirement is uniquely determined in equilibrium. In order to model a negative externality associated with default, we assume that owners of the durable good have a profitable production opportunity. However, they only undertake the production in the second period if the value of their house is not too low and they do not default. Hence, default becomes socially costly. In a model with production reducing leverage can then lead to a Pareto-improvement through the following mechanism. If agents face idiosyncratic shocks to their productivity equilibrium leverage might be set at a level where a large fractions of agents (the ones with a relatively bad productivity shock) defaults. Decreasing leverage makes borrowers with a good productivity shock wealthier and increases production since it reduces the number of agents that default. Hence it increases the price of housing in the second period. Even though at fixed equilibrium prices, forcing agents to leverage less makes everybody worse off the fact that housing prices in the second period go up leads to a Pareto-improvement because it leads to more production.

We end the paper by presenting an example with both aggregate uncertainty and idiosyncratic shocks. We show that in this environment both effects play an important role and they reinforce themselves. In a bad aggregate shock a large number of agents defaults. Decreasing leverage leads to a Pareto-improvement through less default in the bad state and through the fact that prices of housing increase in both states, making the lender and the borrower better off.

The paper is organised as follows. In Section 2 we consider simple incomplete markets example

without collateral or default. In Section 3 we introduce default, collateral and production. We explain how idiosyncratic shocks alone can lead to a constrained suboptimal equilibrium. In Section 4 we present a model with both idiosyncratic and aggregate shocks and illustrate how our two effects interact. Section 5 concludes.

2 Too much borrowing in economies with incomplete markets

We first show how *over-borrowing* occurs naturally in models with incomplete markets and several commodities. We shall tell a simple story that makes clear the intuition, and then we shall give a recipe for finding examples that fit the story. Finally, we focus on one concrete example in which our effect holds even more generally than in the story.

2.1 The Intuition

2.1.1 The Setting

The setting of our story involves a pure exchange economy with two periods and two states in the second period, so we write $s = 0, U, D$. There are two goods: housing, a durable good, and food, which is perishable. The only financial market is a bond, which promises the delivery of one unit of food in both states U and D . There are two agents, who have von Neumann Morgenstern utilities for consumption of food and housing in states U and D and constant marginal utility of food in state 0 , which makes calculations easier.

Without loss of generality, we may suppose the marginal utility for food at time 0 is 1 for both agents. If there is an interior equilibrium, then any adjustment of food endowments at time 0 will give rise to the same adjustment of food consumption in state 0 , without any other changes in prices or consumption. It follows that if a government intervention leads to a new equilibrium that increases the sum of utilities compared to the old *laissez faire* equilibrium, then there must be a transfer of food between the agents at time 0 such that the combined effect of the intervention and the transfer makes both agents better off. (Nonetheless, we shall give a concrete example in which the Pareto improving effect of limiting borrowing can be obtained without any transfers at time 0).

Consider the intervention of restricting the amount agents are allowed to borrow. This may be implemented either by forcing the agents to borrow and lend a smaller amount, at a price fixed by the government, or more simply by passing a law limiting borrowing. In the former case, markets in $0, U$, and D would be allowed to find prices at which supply equals demand for food and housing, assuming deliveries are made on the assigned bond transactions. In the latter case, markets would be allowed to find prices that clear supply and demand for all consumption; for bonds, market clearing will need a slightly modified definition in which lenders imagine they can purchase as much as they want of the bonds at the going interest rate, but the borrowers think they can sell as much of the bonds as they want up to (but not beyond) the government decreed limit. As long as the bond transactions were the same in the two implementations, the future consumptions would be the same, as would be the consumption of housing at time 0 . The only difference would be the consumption of food at time 0 . With constant marginal utility of food, this difference can be erased with the proper adjustment in the food transfer.

2.1.2 The Story

Imagine that agent B has higher marginal propensity to consume housing in every state $0, U, D$ than A does, and suppose that B needs to borrow in state 0. Finally, we suppose that B has a higher endowment of food in state U than in state D, but otherwise endowments (and utilities) are the same across the two states. Thus we think of U as the good state (for B and for the economy) and D as the down state. We shall check that in equilibrium, Agent B has a lower marginal utility of income in state U than agent A, but a higher marginal utility of income in state D than agent A. Finally, suppose that B is a net buyer of housing in state U, and a net seller of housing in state D.

A government intervention to restrict borrowing in state 0 transfers wealth from A to B in states U and D, since then B has a smaller debt to pay. From the marginal propensity to consume hypothesis this raises net demand for housing in both states, and by Walrasian tatonnement stability, this raises the price of housing in both states. From the envelope theorem, the change in utility for agents A and B in each state U,D is their marginal utility of income (where income is denoted in units of food) multiplied by the change in value of their old net supply for housing. Thus the rise in housing prices helps A (and hurts B) in state U, where A was the seller of housing. Since A cares about an extra dollar of income in state U more than B does, this increases the sum of their utilities in state U. Similarly, the rise in housing prices helps B (and hurts A) in state D, where B was the seller of housing. Since B cares about a dollar of income in state D more than A does, this increases the sum of their utilities in state D as well, giving us our desired conclusion (assuming that housing consumption at date 0 is unchanged across the two equilibria).

These hypotheses, if we could verify them in a concrete example, necessarily imply that a small restriction in the amount B can borrow from A must lead to an increase in the sum of utilities after equilibrium is restored in states U and D. An additional transfer of food between A and B at 0 can then be found which makes both agents better off than they were in the original equilibrium. We now turn to constructing a class of examples which embody all the aforementioned hypotheses.

In passing we shall cook up an example in which the increase in the price of housing in state U is exactly the same as the increase in state D. This added feature enables us to deduce the same conclusion, that restricting borrowing leads to a Pareto improvement, no matter who is the buyer and who is the seller in states U and D, provided that B buys more houses net in U than he does in D.

2.2 The Model

Utility functions are given by

$$\begin{aligned} u^A(x_{0F}, x_{0H}, x_{UF}, x_{UH}, x_{DF}, x_{DH}) &= x_{0F} + 0x_{0H} + \delta^A[\pi_U v^A(x_{UF}, x_{UH}) + (1 - \pi_U)v^A(x_{DF}, x_{DH})] \\ u^B(x_{0F}, x_{0H}, x_{UF}, x_{UH}, x_{DF}, x_{DH}) &= x_{0F} + 1x_{0H} + \delta^B[\pi_U v^B(x_{UF}, x_{UH}) + (1 - \pi_U)v^B(x_{DF}, x_{DH})] \end{aligned}$$

Let the endowments be

$$\begin{aligned} e^A &= ((e_{0F}^A, e_{0H}^A), (e_{UF}^A, e_{UH}^A), (e_{DF}^A, e_{DH}^A)) \\ e^B &= ((e_{0F}^B, e_{0H}^B), (e_{UF}^B, e_{UH}^B), (e_{DF}^B, e_{DH}^B)). \end{aligned}$$

We assume that there is a single risk-less asset (bond) available for trade, promising one unit of food in both states U and D – for now we assume that agents cannot default and that promises

do not have to be backed by collateral. Let q denote the price of the bond in period 0 and θ^A, θ^B the bond-holdings of agents A and B . The definition of a GEI equilibrium is standard but it is useful to spell out what it means for a good to be durable. We normalize the price of food to be 1 at each state and define $p_s = p_{sH}$ for all states $0, U, D$. A GEI equilibrium consists of prices q, p_s , $s = 0, U, D$ and choices $(\theta^h, x^h)_{h=A,B}$ such that

$$\begin{aligned} (\theta^h, x^h) &\in \arg \max u^h(x) \text{ s.t.} \\ p_0(x_{0H} - e_{0H}^h) + (x_{0F} - e_{0F}^h) + q\theta^h &= 0 \\ p_s(x_{sH} - x_{0H} - e_{sH}^h) + (x_{sF} - e_{sF}^h) - \theta^h &= 0, s = U, D \end{aligned}$$

and

$$\begin{aligned} \sum_{h=A,B} \theta^h &= 0, \quad \sum_{h=A,B} (x_0^h - e_0^h) = 0, \\ \sum_{h=A,B} (x_{sF}^h - e_{sF}^h) &= 0, \quad \sum_{h=A,B} (x_{sH}^h - e_{sH}^h - e_{0H}^h) = 0, \quad s = U, D. \end{aligned}$$

Geanakoplos and Polemarchakis (1986) show that GEI equilibria are generically constrained sub-optimal in the sense that restricting agents' portfolio choices while allowing for transfers at period 0 can Pareto-improve on the equilibrium allocation. We now want to give a general example to illustrate that simply restricting borrowing provides such a Pareto-improving intervention.

Suppose that the von Neumann Morgenstern utilities are additively separable: for $h = A, B$

$$v^h(x_F, x_H) = v_F^h(x_F) + v_H^h(x_H)$$

where each v_c^h is twice differentiable, strictly increasing, and concave.

We say that v^B displays *universally higher marginal propensity to consume H* than v^A iff for all $(x_F^A, x_H^A), (x_F^B, x_H^B) \gg 0$

$$\frac{dv_H^A(x_H^A)/dx}{dv_F^A(x_F^A)/dx} = \frac{dv_H^B(x_H^B)/dx}{dv_F^B(x_F^B)/dx} \Rightarrow \frac{-d^2v_H^A(x_H^A)/dx^2}{-d^2v_F^A(x_F^A)/dx^2} > \frac{-d^2v_H^B(x_H^B)/dx^2}{-d^2v_F^B(x_F^B)/dx^2}$$

This condition is equivalent to requiring that for any prices and incomes, so long as A and B are both demanding strictly positive amounts of food and housing, an equal but small increase in income to each party will increase B's demand for housing by more than it increases A's demand for housing. To see why, observe that at utility maximizing choices, both agents must be setting the ratio of their marginal utility of housing to their marginal utility of food equal to the price ratio of housing to food. When income goes up, holding prices fixed, each agent will increase his consumption of housing and food in such a way as to decrease his marginal utility of housing by the same percentage as he decreases his marginal utility of food. Each additional unit of housing consumption reduces his marginal utility of consumption by a percentage equal to the second derivative of housing utility divided by the first derivative of housing utility. Hence the change in his consumption of housing (and food) due to a small increase in income will be inversely proportional to the ratio of his second derivative of housing utility divided by the first derivative of his housing utility.

2.3 A Specific Utility

Utility functions are given by

$$u^A(x_{0F}, x_{0H}, x_{UF}, x_{UH}, x_{DF}, x_{DH}) = x_{0F} + 0x_{0H} + \pi_U(x_{UF} + \alpha \log x_{UH}) + (1 - \pi_U)(x_{DF} + \alpha \log x_{DH})$$

$$u^B(x_{0F}, x_{0H}, x_{UF}, x_{UH}, x_{DF}, x_{DH}) = x_{0F} + 1x_{0H} + \pi_U(\beta \log x_{UF} + x_{UH}) + (1 - \pi_U)(\beta \log x_{DF} + x_{DH})$$

Notice that agent A has marginal propensity to consume food of one, and so marginal propensity to consume housing of zero. The reverse is true of agent B , so indeed B always has higher marginal propensity to consume housing. For simplicity we assume that agent A derives no utility from housing consumption in 0; thus in equilibrium agent B will consume the entire aggregate housing endowment in period 0.

We focus on equilibria where agent A is the lender, i.e. $\theta^A > 0$. We define $\theta = \theta^A$. Throughout this example we also assume that the only difference in endowments (or utilities) between states U and D is that $e_{UF}^B > e_{DF}^B$. We define $e_H^A = e_{UH}^A = e_{DF}^A$ to be the housing endowment of agent A in the second period. If all housing comes from the first period (no new houses being build in the second period), this will naturally be zero, but this is just one particular example.

Note that since agent A receives no utility from housing in period 0, B will consume all houses in the first period. It simplifies notation to define

$$\tilde{e}_{sH}^B = e_{sH}^B + x_{0H}^B = e_{sH}^B + e_{0H}^A + e_{0H}^B$$

For now, we focus on a pre-equilibrium i.e. assume that non-negativity constraints on consumption do not bind. The first order conditions for consumption in states U and D are

$$\frac{\alpha}{x_{sH}^A} = p_s$$

$$\frac{\beta}{x_{sF}^B} = \frac{1}{p_s}$$

We then obtain from the first order conditions and the budget sets for both states $s = U, D$,

$$x_{sF}^A = (e_{sF}^A - \alpha) + \theta + p_s e_{sH}^A, \quad (1)$$

$$x_{sH}^A = \frac{\alpha}{p_s} \quad (2)$$

$$x_{sF}^B = \beta p_s, \quad (3)$$

$$x_{sH}^B = \frac{e_{sF}^B - \theta}{p_s} + \tilde{e}_{sH}^B - \beta \quad (4)$$

From the market-clearing condition, obtain that

$$p_s = \frac{\alpha - \theta + e_{sF}^B}{\beta + e_{sH}^A}, \quad s = U, D \quad (5)$$

and therefore

$$\frac{dp_s}{d\theta} = \frac{-1}{\beta + e_{sH}^A} < 0, \quad s = U, D. \quad (6)$$

Since we assumed e_{sH}^A is the same for $s = U, D$, it follows that so is $\frac{dp_s}{d\theta}$. Since the only difference in endowments (or utilities) between states U and D is that $e_{UF}^B > e_{DF}^B$, we see from the above equations

that $p_U > p_D$, and hence that $x_{UF}^B > x_{DF}^B$. We also see that $x_{UH}^A < x_{DH}^A$, hence $x_{UH}^B > x_{DH}^B$. It follows from $e_{UH}^B = e_{DH}^B$ that

$$(x_{UH}^B - \tilde{e}_{UH}^B) > (x_{DH}^B - \tilde{e}_{DH}^B) \quad (7)$$

It also follows that the marginal utility of money to agent B is skewed more toward D than it is for agent A .

$$\begin{aligned} \mu_U^A &= \pi_U, \mu_D^A = (1 - \pi_U) \\ \mu_U^B &= \pi_U \frac{\beta}{x_{UF}^B}, \mu_D^B = (1 - \pi_U) \frac{\beta}{x_{DF}^B} \\ \frac{\mu_U^A}{\mu_D^A} &= \frac{\pi_U}{1 - \pi_U} > \frac{\pi_U x_{DF}^B}{(1 - \pi_U) x_{UF}^B} = \frac{\mu_U^B}{\mu_D^B} \end{aligned}$$

Assuming that both agents consume a positive amount of every good, we must have that the price of the bond is equal to A's marginal utility, as well as B's marginal utility, hence

$$q = \mu_U^A + \mu_D^A = 1 = \mu_U^B + \mu_D^B$$

It follows that $\mu_U^A > \mu_U^B$.

Thus we have verified all the conditions we hypothesized at the outset of this section.

2.4 Pareto Improving Interventions with Transfers

We first show that by restricting borrowing, both agents are always made better off if we allow for first period transfers. This result is independent of who sells or buys the durable good in the second period. We also identify conditions that ensure that an improvement is possible without first period transfers – this will depend crucially on the lender's housing endowments in the second period.

In our setup, with identical linear utility in food in state 0, a Pareto-improvement from restricting borrowing is possible (with first period transfers) if and only if the sum of second period utilities increases as borrowing is restricted.

According to the envelope theorem, an infinitesimal increase in θ will cause second period utilities to change by the change in the revenue from selling the same amount of H , times the marginal utility of income, which is

$$\begin{aligned} \frac{du_1^A}{d\theta} &= \mu_U^A \frac{dp_U}{d\theta} (e_{UH}^A - x_{UH}^A) + \mu_D^A \frac{dp_D}{d\theta} (e_{DH}^A - x_{DH}^A) \\ \frac{du_1^B}{d\theta} &= \mu_U^B \frac{dp_U}{d\theta} (\tilde{e}_{UH}^B - x_{UH}^B) + \mu_D^B \frac{dp_D}{d\theta} (\tilde{e}_{DH}^B - x_{DH}^B) \end{aligned}$$

Recalling that for $s = U, D$

$$(e_{sH}^A - x_{sH}^A) + (\tilde{e}_{sH}^B - x_{sH}^B) = 0$$

we can write

$$\begin{aligned} \frac{du_1^A}{d\theta} &= \mu_U^A \frac{dp_U}{d\theta} (x_{UH}^B - \tilde{e}_{UH}^B) + \mu_D^A \frac{dp_D}{d\theta} (x_{DH}^B - \tilde{e}_{DH}^B) \\ \frac{du_1^B}{d\theta} &= \mu_U^B \frac{dp_U}{d\theta} (\tilde{e}_{UH}^B - x_{UH}^B) + \mu_D^B \frac{dp_D}{d\theta} (\tilde{e}_{DH}^B - x_{DH}^B) \end{aligned}$$

Since

$$\frac{dp_U}{d\theta} = \frac{dp_D}{d\theta} < 0,$$

by substituting $\mu_D^A = 1 - \mu_U^A$, and $\mu_D^B = 1 - \mu_U^B$, by adding the above two equalities, and noting that the sum of excess demands is zero in each state, and dividing by $\frac{dp_U}{d\theta}$, we can write

$$\left[\frac{du_1^A}{d\theta} + \frac{du_1^B}{d\theta} \right] / \frac{dp_U}{d\theta} = (\mu_U^A - \mu_U^B) [(x_{UH}^B - \tilde{c}_{UH}^B) - (x_{DH}^B - \tilde{c}_{DH}^B)] > 0. \quad (8)$$

Restricting borrowing is 'potentially' (meaning if one allows for arbitrary period 0 transfers) Pareto improving iff $\frac{du_1^A}{d\theta} + \frac{du_1^B}{d\theta} < 0$. Thus decreasing θ (thereby reducing borrowing) always increases the sum of utilities. With a reallocation of consumption at time 0 this can always make both agents better off. Note that this argument is independent of the borrower being a seller of houses in the down-state or a buyer in the up-state.

REMARK 1 *We present the argument for the case where the single asset available for trade is a risk-free bond. This is the naturally case for our question. However, it is useful to note that the same argument goes through if there is only a single risk-free asset available for trade. As long as Equation (7) holds (which is guaranteed for the case of the risk-free asset but becomes a condition in the case of a risky asset), the analysis carries through independent of the asset's payoffs.*

2.4.1 Computing Equilibrium

Assuming that both agents consume a positive amount of every good, we must have that the price of the bond is equal to A's marginal utility, as well as B's marginal utility, hence

$$\begin{aligned} q &= \mu_U^A + \mu_D^A = 1 \\ q &= \mu_U^B + \mu_D^B = \pi_U \frac{\beta}{x_{UF}^B} + (1 - \pi_U) \frac{\beta}{x_{DF}^B} \end{aligned}$$

Thus in equilibrium we must have

$$\begin{aligned} 1 &= \pi_U \frac{\beta}{x_{UF}^B} + (1 - \pi_U) \frac{\beta}{x_{DF}^B} \\ &= \pi_U \frac{\beta}{\beta p_U} + (1 - \pi_U) \frac{\beta}{\beta p_D} \\ &= \pi_U \frac{\beta + e_{UH}^A}{\alpha - \theta + e_{UF}^B} + (1 - \pi_U) \frac{\beta + e_{DH}^A}{\alpha - \theta + e_{DF}^B} \end{aligned} \quad (9)$$

This results in a quadratic equation that turns out to have a unique economically meaningful solution. Since agent B has linear utility in housing, if he consumes positive amounts of housing at all states, the period zero price of housing is given by $p_0 = 2$.

2.5 Simple numerical example

We conclude the discussion of the simple economy by giving a concrete numerical example. So far, we have allowed for zero period transfers. In this subsection we would like to identify a situations where an improvement is possible without period zero transfers.

For this, we focus on the case where $\pi_U = \pi_D = 1/2$ and $\alpha = \beta$ to simplify the computations and the expressions below. In this case, the quadratic Equation (9) resulting from agent B 's first

order conditions has a unique economically meaningful solution that is given by

$$\theta = \theta^A = \frac{1}{2} \left(\alpha + e_{UF}^B + e_{DF}^B - e_H^A - \sqrt{(\alpha + e_H^A)^2 + (e_{DF}^B - e_{UF}^B)^2} \right) \quad (10)$$

Suppose that $e_{UF}^B = 5, e_{DF}^B = 1$ and $\tilde{e}_{DH}^B = \tilde{e}_{UH}^B = 1$ while $e_{UH}^A = e_{DH}^A = 1.5$ and $e_{UF}^A = e_{DF}^A = 3$. For $\alpha = \beta = 1.5$ we obtain from Equation (10) that in equilibrium $\theta^A = 0.5$.

As we pointed out before first period price of housing is always 2 and the price of the bond is always 1. It can be easily computed that prices of houses in the second period are $p_{UD} = 2/3$ and $p_{DH} = 2$. Second period equilibrium consumptions of the borrower are given by

$$x_{UF}^B = 3, x_{DF}^B = 1, \quad x_{UH}^B = 1.75, x_{DH}^B = 0.25.$$

Consumption of the lender is

$$x_{UF}^A = 3, x_{DF}^A = 2, \quad x_{UH}^A = 0.75, x_{DH}^A = 2.25.$$

Clearly this leads to bad risk-sharing – when the borrower is poor his houses fetch a low price. As in our introductory story, the borrower sells the house if he is poor and he buys housing in state U when he is rich.

If borrowing is restricted to be at most 0.45 and $\theta = 0.45$ instead of the equilibrium value of 0.5 we obtain that prices stabilize and increase in both states by $1/60$. In the D-state the borrower benefits since he sells the house there, in the U-state the lender benefits.

Consumption of the borrower in the second period is given by

$$x_{UF}^B = 3.025, x_{DF}^B = 1.025, \quad x_{UH}^B = \frac{425}{242} \sim 1.756, x_{DH}^B = \frac{25}{82} \sim 0.305.$$

One observes that housing consumption of the borrower in the D state increases substantially while it only increase slightly in the U-state although there also he has more money available since he borrowed less. However, the price effect benefits the lender who is a seller in that state. When moving from the GEI equilibrium to the situation of restricted borrowing, second period utility of the lender increases by 0.0929 utils and utility of the borrower increases by 0.0053 utils. Limiting borrowing leads to a Pareto-improvement without the need to redistribute endowments.

3 Borrowing, default and production

We now extend the basic GEI model from the last example to incorporate collateral and default as well as production and idiosyncratic shocks. We extend the model along three dimensions. First, we assume that agents can only borrow if they pledge housing as collateral. All loans are non-recourse and in the last period the agents can default on their debt-obligations and hand over the collateral associated with the loan to the lender. (In a two period model, such as ours, the borrower will default if and only if the debt is greater than the value of the collateral). The need for collateral limits borrowing, and thus attenuates the beneficial effects of government interventions to constrain borrowing. Indeed, if the collateral constraint is binding, then the marginal utility of income in the future to the borrower will be lower than to the lender, meaning that if future prices remain unaltered as borrowing changes, there is too little borrowing from a social point of view. Constraining

borrowing still further, by government intervention, seems counterproductive. It is even worse if the borrowers are net sellers of housing next period. Restricting borrowing will have the added effect of raising the price of housing next period, again lowering total utility because it will hurt the lenders who are buying housing more than it will help the borrowers who are selling the houses.

To argue the wisdom of government intervention to limit borrowing, we introduce another feature of collateral. We assume that the owner of the durable good in period 0 has a production opportunity at the beginning of period 1. (During the crisis, this might have been repairs to the house). Using a linear technology, one unit of the house and one unit of labor input can be used to produce one unit of the house and δ units of food. Think of a garden with fruit and vegetable plants, which is a substitute for the other good we called food. For simplicity, we assume no utility for leisure so that the production can be viewed as costless. We want to interpret this as the limiting case of small *positive* costs, i.e. the owner of the house only undertakes production if he gets something positive in return. The food produced from one unit of housing is part of the house. The key point is that if the collateral is seized, it will include the garden. An owner who owes more than the value of the house plus the garden he *could* build will default and thus not build the garden at all, even though it is socially useful and almost costless to construct. The lender would then receive only the value of the house with no additional food.

The possibility that debt overhang destroys the incentive to produce does not necessarily mean the original debt contract is suboptimal. The lender can perfectly well foresee that the borrower will not build his gardens if the debt is too high in some state. The lender thus has himself a reason not to give bigger debt contracts on the same collateral; in short, the lender will constrain leverage. Why then should the government intervene to constrain it more? The reason is that the lenders collectively do not realize that if they all lend less, the borrowers who were repaying in full will need to pay less, and since they have higher marginal propensity to consume housing, this will raise the price of houses. Then fewer buyers will be under water, and so more gardens will be built (and the lenders will collectively collect more money)..

Third, and finally, we assume that agents face idiosyncratic shocks to δ . While all individuals within one type are ex ante identical they differ ex post by the realisation of the productivity shock, δ . We assume that δ realizes in $[0, \Delta]$ with some distribution function G_s in each state $s = 1, \dots, S$.

We first focus on the case of only one state, i.e. no aggregate shock and then consider a case with 2 aggregate states, similar to Example 1 above. With one shock we find that there are two socially negative effects of government intervention: the borrowers value money more in the present vs the future than the lenders, and curtailing their borrowing further hurts both borrowers and lenders, assuming all prices remain the same. Second, curtailing lending will raise the price of housing in the future, which can lower social utility as we saw above. We show in the example that the beneficial effects of the price externality on garden production can be bigger than the two negative effects of curtailing borrowing. (We also show that things could go in the other direction). When we add uncertainty, we shall find that two of the three effects argue for curtailing borrowing.

3.1 The Model

We now imagine that each of the 2 types of agents represent a continuum of identical individuals. Thus we think of the set of agents as lying on the continuum $(0, 2]$, where all the agents $t \in (h - 1, h]$ are identical in endowments and utility to agent A if $h = 1$ and to agent B if $h = 2$. The maximum productivity of the agents for gardens depends on an idiosyncratic shock δ_s^t that is realized

simultaneously with the aggregate state $s \in S$. Hence we can write

$$\begin{aligned} e^t &= e^A = ((e_{0F}^A, e_{0H}^A), (e_{sF}^A, e_{sH}^A)_{s=1, \dots, S}) & \text{for } t \in (0, 1] \\ e^t &= e^B = ((e_{0F}^B, e_{0H}^B), (e_{sF}^B, e_{sH}^B)_{s=1, \dots, S}) & \text{for } t \in (1, 2] \end{aligned}$$

but the final consumptions may depend on the idiosyncratic shocks. We invoke the law of large numbers and restrict attention to consumptions in which each agent's consumption depends on the state and on his own idiosyncratic shock, but not on the shocks of others. Utility functions are given by

$$\begin{aligned} U^A(x_{0F}, x_{0H}, (x_{sF}(\delta_s), x_{sH}(\delta_s))_{\delta_s \in [0, \Delta], s=1, \dots, S}) &= x_{0F} + 0x_{0H} + \sum_{s=1}^S \pi_s \int_0^\Delta (x_{sF}(\delta_s) + \alpha \log x_{sH}(\delta_s)) dG_s(\delta_s) \\ U^B(x_{0F}, x_{0H}, (x_{sF}(\delta_s), x_{sH}(\delta_s))_{\delta_s \in [0, \Delta], s=1, \dots, S}) &= x_{0F} + 1x_{0H} + \sum_{s=1}^S \pi_s \int_0^\Delta (\beta \log x_{sF}(\delta_s) + x_{sH}(\delta_s)) dG_s(\delta_s) \end{aligned}$$

As explained above, this utility function has the advantage that in any state $s = 1, \dots, S$, A has marginal propensity to consume F of 1, and so marginal propensity to consume H of 0. The reverse is true of B, so indeed B always has higher marginal propensity to consume H. So as before, the price of houses increases as wealth is transferred from A to B.

As in Geanakoplos (1997), we endogenize the margin requirements by assuming that a large set of contracts can be traded; equilibrium prices for all the contracts must be determined, and then one can check which contracts are positively traded. Often there will be just one. A contract is an ordered pair (collateral, promise). We assume that all contracts require one house as collateral, and that any non-contingent promise of food can be made on that collateral. We index the contracts traded by their promise and write $j \in \mathcal{J} \subset \mathbb{R}_+$; contract j promises j units of food in each state s , backed by one house. A borrower can default on his promise in which case the housing-collateral associated with the contract is handed over to the lender. We allow \mathcal{J} to be infinite, but assume that an agent is restricted to trade in finitely many of the contracts.

Given the set of available contracts, \mathcal{J} , agent $h \in \{A, B\}$ faces the following maximization problem

$$\begin{aligned} &\max_{\phi \in \mathbb{R}^{\mathcal{J}}, x(\delta) \in \mathbb{R}_+^{1+S} \times [0, \Delta]} U^h(x) \text{ s.t.} \\ &x_{F0}^h - e_{F0}^h + p_0(x_{H0}^h - e_{H0}^h) = \sum_j q_j \phi_j \\ &x_{Fs}^h(\delta_s) - e_{Fs}^h + p_s(x_{Hs}^h(\delta_s) - e_{Hs}^h) = (x_{H0}^h - \sum_{j \in \mathcal{J}: \phi_j > 0} \phi_j)(p_s + \delta_s) + \\ &\quad \sum_{j: \phi_j > 0, (p_s + \delta) \geq j} \phi_j ((p_s + \delta_s) - j) - \sum_{j \in \mathcal{J}: \phi_j < 0} \phi_j d_{sj} \quad s = U, D, \delta_s \in [0, \Delta] \\ &(x_{H0}^h - \sum_{j \in \mathcal{J}: \phi_j > 0} \phi_j) \geq 0 \end{aligned}$$

Here, we use the fact that by the law of large numbers the distribution of δ_s across the agents in any state s will be deterministic, and so we can assume the prices depend only on the aggregate state. The first equation in the budget set says that net money spent on food and housing must come out of the revenue obtained by selling contracts j , where $\phi_j > 0$ represents the number of

contracts of type j sold, and $-\phi_j > 0$ represents the number of contracts of type j purchased. The second equation says that in every state s , an agent who receives idiosyncratic shock δ_s must finance his net purchase of food and his holdings of houses beyond his new endowment from sales of his unencumbered housing, which with the gardens he will build will each fetch $p_s + \delta_s$, plus the sales of encumbered houses for which it is worthwhile to pay off the entire loan and pocket the revenue $(p_s + \delta_s) - j$ that remains after the loan is paid off, plus the revenue d_{sj} obtained from each loan that he purchased.

Given a promise j it is useful to define the cut-off value $\delta_s^*(j, p_s)$ as

$$\delta_s^*(j, p_s) = \begin{cases} 0 & \text{if } j - p_s \leq 0 \\ j - p_s & \text{if } 0 \leq j - p_s \leq \Delta \\ \Delta & \text{if } j - p_s \geq \Delta \end{cases} \quad (11)$$

Any agent whose production shock realizes below or equal to δ_s^* will fully default on the contract and produce nothing, while any agent whose production shock is greater than δ_s^* will produce and pay back in full. The lender is thus left either with a seized house and no garden, or with the payout in full of j . The payout of a contract with promise j in state s is then

$$d_{sj} = G_s(\delta_s^*(j, p_s))p_s + (1 - G_s(\delta_s^*(j, p_s)))j.$$

A GEI equilibrium with endogenous margin requirements is defined as a price-function $q : \mathcal{J} \rightarrow \mathbb{R}_+$, spot-prices $p_0, (p_s)_{s=1}^S$ and choices of all agents such that markets clear and agents optimize. The finitely many contracts that are traded determine the margin requirements straightforwardly. We have that for each contract j , its loan-to-value ratio (LTV) can be written as $LTV(j) = \frac{q_j}{p_0}$.

3.2 One State Example

We first analyze the model under the assumption that $S = 1$. We want to determine equilibrium borrowing, margin requirements and prices. As before we assume that there is one house at $s = 0$, and that agent A does not derive utility from the house, so it is purchased by agent B.

Throughout this section, we assume that $\alpha = \beta$. This simplifies the computation considerably and allows us to derive closed-form conditions for a Pareto-improving decrease in leverage.

We make the following assumption on endowments.

$$e_{1F}^B > 1 + e_{1H}^A. \quad (12)$$

We also assume that the idiosyncratic δ -shock is atomless with cdf G and support $[0, 1]$ and normalize its average to be $1/2$, i.e. $\int \delta dG(\delta) = \frac{1}{2}$. Later we will consider the concrete case where the shock is uniform.

We will first compute equilibrium, assuming that only a single arbitrary contract j is traded. Next we will find the equilibrium contract j^* , and confirm that indeed no agent would want to trade any other contract. We will then compute the sum of expected utilities for agents of type A and B, as a function of j . Lastly we show that reducing leverage will generally Pareto-improve in the sense that it increases the sum of second period utilities (i.e. a Pareto-improvement possibly involves transfers in the first period).

3.2.1 Characterizing equilibrium for a fixed contract j

We assume for now that in equilibrium only one contract j is traded. We then show how j is chosen endogenously, assuming that only one contract is traded in equilibrium (this is the problem of endogenous leverage). We then derive sufficient conditions on G so that indeed nobody would want to trade any other j in addition to the endogenously chosen one. We confine our attention to interior equilibria, in which both agent types consume positive amounts of food and housing in period 1, and positive amounts of food in period 0.

The type B borrowers will have different wealth levels in the last period, depending on their idiosyncratic productivity shocks. But in interior equilibria, they will all set the ratio of the marginal utility of food to housing equal to the ratio of the price of food to houses, giving

$$\begin{aligned}\frac{\beta}{x_{F1}^B(\delta)} &= \frac{1}{p_1} \\ x_{F1}^B(\delta) &= \beta p_1\end{aligned}$$

Hence total demand for food by type B agents is the same as it was in the last section, despite all the idiosyncratic shocks. Since the rest of their money is spent on housing, total demand for housing (as a function of price and their total income) by type B agents is also the same as before.

The same logic applies to the type A agents. Despite the idiosyncratic shocks, each type A agent will choose housing consumption so that

$$\begin{aligned}\frac{\alpha}{x_{H1}^A(\delta)} &= \frac{p_1}{1} \\ x_{H1}^A(\delta) &= \frac{\alpha}{p_1}\end{aligned}$$

just as in the last section. Hence again total demand by type A agents for housing and food (as a function of p_1 and their total income) must be the same as in the previous section.

In collateral equilibrium, no agent is ever required to deliver anything out of his endowment; only the collateral can be confiscated. Hence agents of type A can acquire at most one extra house put up as collateral from period 0, and some of the produced gardens, while agents of type B will retain at least all of their food endowment in state 1. Thus from the formula for p_1 developed in Section 1, and the assumptions $\alpha = \beta$ and $e_{1F}^B > 1 + e_{1H}^A$, we know that

$$p_1 \geq \frac{\alpha + e_F^B}{\beta + e_H^A + 1} = \frac{\alpha + e_F^B}{\alpha + e_H^A + 1} > 1.$$

Thus at an interior equilibrium, the marginal utility of the numeraire food is 1 for agents of type A in state 0 and in state 1, and for agents of type B it is also 1 in state 0 but $1/p_1 < 1$ in state 1. B will therefore always want to borrow more from A if he can guarantee full delivery of his promises. It follows that at equilibrium, all of the housing (which B must own) is used as collateral, and $j \geq p_1$.¹

We now compute equilibrium for any fixed loan j between p_1 and $(\Delta + p_1)$. We already said it can never be part of equilibrium to trade exclusively in a contract which promises less than p_1 , and any promise j above $\Delta + p_1$ will be equivalent to one promising exactly $\Delta + p_1$.

¹We shall show in a moment that because the distribution function G is continuous, we must have $j > p_1$, giving actual default.

The crucial idea is that the agents with shocks $\delta \leq j - p_1$ will not produce gardens at all. So let us define total garden production by

$$P(j, p_1) = \int_{x > j - p_1} x dG(x) = \int_{x > \delta(j, p_1)} x dG(x)$$

It is evident, and of crucial importance, that the output of gardens is decreasing in j and increasing in p_1 .

Since $j \geq p_1$, we might as well assume that the house is transferred as payment to the lender. The only question is how much extra food he gets. The total extra transfer of food $T(j, p_1)$ is easily seen to be the product of the additional payment $(j - p_1)$ made by each agent who produces any garden at all, by the number of people $(1 - G(j - p_1))$ who choose to produce

$$T(j, p_1) = (j - p_1)(1 - G(j - p_1)).$$

Therefore, the 'average' borrower has total food endowments in the second period of $e_F^B + \int_{x > j - p_1} (x - (j - p_1)) dG(x) = e_F^B + P(j, p_1) - T(j, p_1)$. Using the equation for the spot price derived in Example 1, we obtain that the price of housing in the second period is given by

$$\tilde{p}_1(j, p_1) = \frac{\alpha + e_F^B + \int_{x > j - p_1} (x - (j - p_1)) dG(x)}{1 + \beta + e_H^A} = \frac{\alpha + e_F^B + P(j, p_1) - T(j, p_1)}{1 + \beta + e_H^A}.$$

Note that, independently of G , we must have that $\frac{\partial \tilde{p}_1(j, p_1)}{\partial j} < 0$ since we always have that $\int_{x > \delta} (x - \delta) dG(x)$ is decreasing in δ . Of course in equilibrium we must have that

$$\tilde{p}_1(j, p_1) = p_1$$

All that remains in order to characterize equilibrium when borrowing is restricted to the single bond j is to find the price of the bond q_j and the price of housing p_{0H} in period 0, together with period 0 consumption. In an interior equilibrium, the lender is not restricted in how much he lends, so the bond must trade at a price that reflects his marginal utility. Since agents of type A have marginal utility of 1 in both states 0 and 1, it follows that the price of the bond is determined by its delivery, measured in units of food. Hence

$$q(j, p_1) = d_1(j, p_1) = j(1 - G(j - p_1)) + G(j - p_1)p_1 = p_1 + T(j, p_1)$$

The marginal utility to agents of type B of food in period 0 is 1, and in period 1 is $1/p_1$. The type B agents will all buy one unit of housing in period 0 with leverage, that is, by simultaneously selling contract j . As described in Geanakoplos (1997), the marginal disutility of the downpayment on the house must then be equal to the marginal utility of owning the house at time 0, net of making the deliveries on j . Recalling that the agents must all give up the house and possibly some gardens, we obtain

$$p_{0H} - q_j = 1 + \frac{1}{p_1}(P(j, p_1) - T(j, p_1)) \quad (13)$$

so

$$p_{0H} = p_1 + T(j, p_1) + 1 + \frac{1}{p_1}(P(j, p_1) - T(j, p_1)).$$

It is interesting to observe that the price of the house is higher than its marginal utility to the B

agents who buy it. The gap was called the collateral value by Fostel-Geanakoplos (2008), and exists because the house, by virtue of serving as collateral, enables its owners to take out the loan j . We can compute the collateral value κ

$$\begin{aligned}\kappa &= p_1 + T(j, p_1) + 1 + \frac{1}{p_1}(P(j, p_1) - T(j, p_1)) - [1 + 1 + \frac{1}{p_1}P(0, p_1)] \\ &= (1 - \frac{1}{p_1})(p_1 + T(j, p_1)) - \frac{1}{p_1}(P(0, p_1) - P(j, p_1))\end{aligned}$$

where the first term in the last line above is the benefit that B agents get because they can borrow at a better rate than they would be willing to pay for a loan, and the second term represents the loss in productivity that results from going into so much debt on each house used as collateral.

3.2.2 Endogenous Leverage, Endogenous j

We conjecture that in equilibrium, there will only be one loan j traded. The question is, which j ? The method of dealing with the endogeneity of leverage was introduced in Geanakoplos (1997). In equilibrium, every agent will correctly anticipate the price p_1 of houses that will prevail in state 1. Every borrower will observe the prices q_j of *all* the loans, and decide which one he wants to take out. Note that each collateral can back only one loan. Of course a borrower could take out ε loans of type j backed by ε houses, and ε' loans of type j' backed by ε' different houses. But we conjecture he (and everyone else) will choose the same loan type j^* . If after exclusively taking out loans of type j^* , every buyer recognizes that he would do no better by substituting an infinitesimal amount of any loan j for the same amount of loan j^* , then indeed we can say that every borrower does prefer the same loan j^* to every other loan j .

How does the borrower evaluate which loan to take out? No matter how or what kinds of loans he takes out, the last infinitesimal loan should bring him more benefits than any other loan that could have been taken out on the same collateral. Geanakoplos-Zame (2013) defined the liquidity value of a loan to the borrower in collateral equilibrium as the price of (an infinitesimal amount of) the loan minus the marginal disutility of making the actual deliveries (on the infinitesimal amount of the loan) divided by the marginal utility of a dollar at time 0, assuming all prices stay fixed at their equilibrium values.

In the context of the present model, define the price of each loan j by the formula in the last section for $q(j, p_1)$. Then the liquidity value of loan j backed by collateral of one house is

$$\Lambda(j, p_1) = (1 - \frac{1}{p_1})(p_1 + T(j, p_1)) - \frac{1}{p_1}(P(0, p_1) - P(j, p_1))$$

No matter what loan j is taken, the second term above represents the expected loss in gardens that occurs when the house becomes collateral that is under water, discounted by the marginal utility $1/p_1$ of the numeraire in state 1. The first term denotes the gain in utility that occurs when the expected delivery $p_1 + T(j, p_1)$, which brings disutility of only $\frac{1}{p_1}(p_1 + T(j, p_1))$, can be used to borrow $q(j, p_1) = (p_1 + T(j, p_1))$ because the A agents are willing to lend at 0 interest. In general, the borrowing agents will choose loan j^* only if

$$j^* \in \arg \max_{j \in \mathcal{J}} \Lambda(j, p_1)$$

So we complete our definition by adding the above expression plus

$$j = j^*$$

to our set of equilibrium equations from the last section. The equations in the last section plus this condition determine equilibrium.

Under the assumption of differentiability and interiority we obtain that a necessary condition for loan j^* to be chosen is that

$$\frac{\partial \Lambda(j, p_1)}{\partial j} = \left(1 - \frac{1}{p_1}\right) \frac{\partial T(j, p_1)}{\partial j} + \frac{1}{p_1} \frac{\partial P(j, p_1)}{\partial j} = 0 \quad (14)$$

The expression for the liquidity value $\Lambda(j, p_1)$ appears to be exactly the same expression as the collateral value of the house. The only difference is that the price p_1 in the formula for $\Lambda(j, p_1)$ is the same for all j , namely the equilibrium price, corresponding to market clearing when the single equilibrium loan j^* is traded. As Geanakoplos-Zame (2013) pointed out, the collateral value is equal to the liquidity value of the loan(s) that is (are) chosen to be backed by the collateral: $\kappa = \Lambda(j^*, p_1)$. For other loans $j \neq j^*$, we must have $\Lambda(j, p_1) \leq \Lambda(j^*, p_1) = \kappa$. We shall present an example of computing equilibrium in the next section.

For now, observe that

$$\begin{aligned} \frac{\partial T(j, p_1)}{\partial j} \Big|_{(j, p_1) = (p_1, p_1)} &= 1, & \frac{\partial T(j, p_1)}{\partial j} \Big|_{(j, p_1) = (p_1 + \Delta - \varepsilon, p_1)} &< 0 \\ \frac{\partial P(j, p_1)}{\partial j} \Big|_{(j, p_1) = (p_1, p_1)} &= 0, & \frac{\partial P(j, p_1)}{\partial j} \Big|_{(j, p_1) = (p_1 + \Delta - \varepsilon, p_1)} &< 0 \end{aligned}$$

where the last inequality holds assuming Δ is in the support of G . Raising j above zero increases the payment, which helps lenders more than it hurts borrowers, and has a negligible effect on the production of gardens. Eventually, raising j decreases transfers and production of gardens. It follows that an optimal j^* must satisfy $p_1 < j^* < p_1 + \Delta$. In particular, there will be default in equilibrium.

3.2.3 Uniform shocks

To illustrate the method of computing equilibrium, we consider the concrete case where the δ -shock is uniformly distributed on $[0, 1]$. We then have that $P(j, p_1) = \frac{1}{2}(1 - (j - p_1)^2)$ and $T(j, p_1) = (j - p_1)(1 - (j - p_1))$. Letting $\delta = j - p_1$, we get that promise $j = p + \delta_*$ is traded in equilibrium only if

$$\delta_* = \arg \max_{\delta} \left(\left(1 - \frac{1}{p_1}\right)[p_1 + \delta(1 - \delta)] - \frac{1}{p_1} \left[\frac{1}{2} - \frac{1}{2}(1 - \delta^2)\right] \right)$$

Note that the maximand is a strictly concave function of δ , hence for any anticipated price p_1 , there is always a unique level of δ , and hence promise j , that maximises the liquidity value. Our conjecture that only one contract will be traded in equilibrium is verified.

Setting the derivative of the above maximand equal to 0,

$$\begin{aligned} \left(1 - \frac{1}{p_1}\right)(1 - 2\delta_*) - \frac{1}{p_1}\delta_* &= 0 \\ \left(1 - \frac{1}{p_1}\right) &= \left(1 - \frac{1}{p_1}\right)2\delta_* + \frac{1}{p_1}\delta_* = \delta_*(2 - \frac{1}{p_1}) \\ \delta_* &= (p_1 - 1)/(2p_1 - 1) \end{aligned}$$

Now we return to our equilibrium equations from the last section to compute equilibrium. Observe first that

$$P(j, p_1) - T(j, p_1) = \frac{1}{2}(1 - \delta_*^2) - \delta_*(1 - \delta_*) = \frac{1}{2} - \delta_* + \frac{1}{2}\delta_*^2.$$

Substituting this into the expression for $\tilde{p}_1(j, p_1) = p_1$, we get

$$p_1 = \frac{\frac{1}{2} + \alpha - \delta_* + \frac{1}{2}\delta_*^2 + e_F^B}{1 + \beta + e_H^A}. \quad (15)$$

We then obtain

$$\delta_* = \frac{p_1 - 1}{2p_1 - 1} = \frac{\frac{1}{2} + \alpha - \delta_* + \frac{1}{2}\delta_*^2 + e_F^B - 1 - \beta - e_H^A}{2(\frac{1}{2} + \alpha - \delta_* + \frac{1}{2}\delta_*^2 + e_F^B) - 1 - \beta - e_H^A}$$

Multiplying out and collecting terms this implies that the equilibrium promise is determined by the following equation.

$$2\delta_*^3 + 5\delta_*^2 - 2(1 + \alpha - e_H^A + 2e_F^B)\delta_* - 2e_H^A + 2e_F^B - 1 = 0 \quad (16)$$

By Descartes' rule of signs this polynomial of degree 3 has at most 2 real positive solutions (the rule bounds the number of real positive solutions by the number of sign-changes of the coefficients, see e.g. Kubler and Schmedders (2010)).

It is easy to check that under our assumptions the expression is positive at $\delta = 0$ and negative at $\delta = 1$ – therefore the number of solutions in $[0, 1]$ must be odd and hence the solution must always be unique. Thus we have shown how to compute the unique equilibrium for this class of economies.

The following numerical example illustrates some key properties of equilibrium.

EXAMPLE 1 *Suppose that $\alpha = \beta = 1$, that endowments at $t = 0$ are given by*

$$e_{0F}^A = 1, e_{0H}^A = 0.8, \quad e_{0F}^B = 1, e_{0H}^B = 0.2$$

and the endowments at $t = 1$ are

$$e_{1F}^A = 1, e_{1H}^A = 0, \quad e_{1F}^B = 2, e_{1H}^B = 0$$

To compute equilibrium, we first solve Equation (16) and obtain $\delta_* \simeq 0.279$. Using Equation (15) we obtain that the equilibrium price for housing in the second period is given by $p_1 \simeq 1.630$. The unique promise traded is therefore given by $j_* \simeq 1.909$. The price of the promise is

$$q = p_1 + \delta_*(1 - \delta_*) \simeq 1.831.$$

Instead of producing 0.5 units of food in the second period, total production is only

$$P(j_*, p_1) = \frac{1}{2}(1 - \delta_*^2) \simeq 0.461.$$

Using Equation (13) we obtain that the price of housing at $t = 0$ is given by

$$p_0 = q + 1 + \frac{1}{p_1}(P(j_*, p_1) - \delta_*(1 - \delta_*)) \simeq 2.990$$

The consumption allocation and utilities can now be computed straightforwardly and are summarized in the following table.

	x_{F0}	x_{H0}	x_{F1}	x_{H1}	U
Agent A	1.561	0	1.831	0.613	2.9026
Agent B	0.439	1	1.630	0.387	2.3143

One can see in equilibrium that although agents could have chosen debt contracts j that would not lead to default, they all chose to trade a contract j^* for which 28% defaulted. The prices q_j represent rational anticipations of the default rate and the reduced garden production, caused by the debt overhang. Despite their rational awareness of the disadvantages to such high levels of debt, the agents all willingly chose to enter into the contract j^* .

The reason is that the borrowers valued money much more in state 0 than in state 1, compared to the lenders. The only way to borrow more is to create more debt, and more debt overhang. Observe that $p_1 = 1.63 > 1$, so type B agents value period 0 numeraire vs state 1 numeraire p_1 times as much as type A agents do. This ratio is called the liquidity wedge in Fostel-Geanakoplos (2008). As a result, getting access to loans is incredibly advantageous to agents of type B. Their marginal benefit or liquidity value to each loan is 0.762. They do not take out any larger or smaller loans, including loans that would not default, because the others give strictly lower liquidity value. They do not take out more loans of the same type j^* because that would involve using more houses as collateral. And houses are a bad deal for type B agents (and even worse for type A agents). The price of houses exceeds the marginal use value of the house to type B agents by a considerable amount; in fact, by exactly the liquidity value of type j^* loans.

3.2.4 Pareto-improving intervention

Agents choose to trade loan contract j^* because it is in their individual self interest. That does not mean it is socially optimal. In this section we calculate the social utility of trading each contract j , to see whether j^* is optimal. We find that typically it is not. The reason is not that the agents don't foresee the inefficiency consequences of default. They do. What they overlook is that if they collectively borrow and lend less, they will affect the future price of housing. And a fall in the future price of houses will cause more defaults and more inefficiency.

As was the case in Example 1, a restriction in borrowing will increase the future price of houses. The restriction by itself, absent a change in future housing prices, is bad for the economy, because the borrowers (who are already constrained from borrowing by the need to post collateral) value the money today more than the lenders. If the borrowers are future sellers of houses, the rise in prices helps them but hurts the lenders more, because they value the same amount of money in the future more. On the other hand, the rise in future houses brings some households up from underwater on their debt. And this increases production. If this production effect is sufficiently strong, it can overwhelm the other two disadvantages to restricting borrowing.

We begin by computing the equilibrium utilities for all the agents, assuming they are restricted to trade the contract j , by plugging in equilibrium consumptions into the utility function. We have

that period 1 utilities are

$$\begin{aligned} U_1^A(j, p_1) &= \alpha \log\left(\frac{\alpha}{p_1}\right) + e_{1F}^A + T(j, p_1) + p_1(e_{1H}^A + 1 - \frac{\alpha}{p_1}) \\ U_1^B(j, p_1) &= \beta \log(\beta p_1) + e_{1H}^B + \frac{1}{p_1}[e_{1F}^B + P(j, p_1) - T(j, p_1) - \beta p_1] \end{aligned}$$

As in Example 1, we now add the utilities. Since we assume $\alpha = \beta$ the non-linear terms in the sum of second period utilities reduce to constants, independent of p_1 , because $\beta \log(x_F^B) + \alpha \log(x_H^A) = \alpha(\log(\alpha) - \log(p_1) + \log(\beta) + \log(p_1)) = 2\alpha \log(\alpha)$. Therefore the sum of second period utilities can be written

$$\begin{aligned} U_1^A(j, p_1) + U_1^B(j, p_1) &= 2\alpha \log(\alpha) + e_{1F}^A + e_{1H}^B - \alpha - \beta + \frac{e_{1F}^B + P(j, p_1) - T(j, p_1)}{p_1} + p_1(1 + e_H^A) + T(j, p_1) \\ &= C + \frac{e_{1F}^B + P(j, p_1) - T(j, p_1)}{p_1} + p_1(1 + e_H^A) + T(j, p_1) \\ &= C + 1 + \frac{e_{1F}^B + P(0, p_1)}{p_1} + p_1 e_H^A + (1 - \frac{1}{p_1})(p_1 + T(j, p_1)) - \frac{1}{p_1}(P(0, p_1) - P(j, p_1)) \end{aligned}$$

The last two terms are exactly $\Lambda(j, p_1)$, and the first four terms do not depend on j . Hence we see that the market would choose the socially optimal contract j^* if the choice of contract did not affect future prices.

The socially optimal contract must take into account that changing j will affect p_1 . We show how to compute these effects. Observe first that both $P(j, p_1)$ and $T(j, p_1)$ depend on (j, p_1) only through $j - p_1$. Hence the same is true for \tilde{p}_1 . Therefore we can write them all as a function of $\delta = j - p_1$. Letting $\mathcal{U}(\delta) = U_1^A(p_1 + \delta, p_1) + U_1^B(p_1 + \delta, p_1)$ gives

$$\mathcal{U}(\delta) = C + \frac{e_F^B + P(\delta) - T(\delta)}{p_1(\delta)} + p_1(\delta)(1 + e_H^A) + T(\delta)$$

Not taking into account the effect of a change in δ on the price, note that the first order conditions for a utility maximizing δ (holding prices fixed) become

$$\frac{1}{p_1}(P'(\delta) - T'(\delta)) + T'(\delta) = 0$$

which is identical to the first order condition for the equilibrium promise, Equation (14) above.

However, the condition for the social utility maximizing level of δ does take into account that prices change as δ changes and we obtain

$$\mathcal{U}'(\delta) = (P'(\delta) - T'(\delta)) \left(\frac{\alpha(1 + e_H^A + \alpha)}{(e_F^B + \alpha - T(\delta) + P(\delta))^2} + \frac{e_H^A + 1}{1 + e_H^A + \alpha} \right) + T'(\delta)$$

We want to show that at equilibrium δ determined by (14) we have that $\mathcal{U}'(\delta) < 0$. A sufficient condition for this is obviously that

$$\mathcal{K}'(\delta) > \mathcal{U}'(\delta) \text{ for all } \delta, \tag{17}$$

where $\mathcal{K}'(\delta) = \frac{1}{p} (P'(\delta) - T'(\delta)) + T'(\delta)$ is the derivative without taking price-changes into account (i.e. in equilibrium $\mathcal{K}'(\delta) = 0$).

Since $P'(\delta) - T'(\delta) < 0$ this holds, whenever

$$\frac{\alpha(1 + e_H^A + \alpha)}{(e_F^B + \alpha - T(\delta) + P(\delta))^2} + \frac{e_H^A + 1}{1 + e_H^A + \alpha} > \frac{1}{p}$$

Multiplying out and substituting for p we obtain that this is equivalent to

$$\frac{\alpha}{e_F^B + \alpha + P(\delta) - T(\delta)} + \frac{(e_F^B + \alpha + P(\delta) - T(\delta))(e_H^A + 1)}{(1 + \alpha + e_H^A)^2} > 1$$

Multiplying out, this turns out to be equivalent to

$$(e_F^B + P(\delta) - T(\delta) - 1 - e_H^A)(-\alpha^2 + (1 + e_H^A)(e_F^B + P(\delta) - T(\delta))) > 0$$

A sufficient condition for this is that Assumption (12) holds and, in addition

$$\alpha^2 < e_F^B(1 + e_H^A). \quad (18)$$

If second period endowments are 'sufficiently' large, it will always be the case that agent B borrows too recklessly which leads to too much default. We will see below that if second period endowments are sufficiently small and the condition does not hold, it might very well be the case that decreasing margin requirements leads to a Pareto improvement.

To shed some light on the economic effect, it is useful to decompose the effect of a price change on welfare. $\mathcal{U}(\delta)$ can be rewritten as

$$\mathcal{U}(\delta) = \frac{e_F^B + P(\delta)}{p} - \frac{T(\delta)}{p} + p(e_H^A + G(\delta)) + (p + \delta)(1 - G(\delta))$$

We can decompose \mathcal{U}' into three terms

$$\begin{aligned} \mathcal{U}'(\delta) &= T'(\delta)\left(1 - \frac{1}{p}\right) + p'(\delta)(1 - G(\delta))\left(1 - \frac{1}{p}\right) + \\ & p'(\delta) \left(-\frac{e_F^B + P(\delta) - T(\delta)}{p^2} + e_H^A + G(\delta) + \frac{(1 - G(\delta))}{p} \right) + \\ & \frac{P'(\delta)}{p} \end{aligned}$$

The first term, $E1 = T'(\delta)\left(1 - \frac{1}{p}\right) + p'(\delta)(1 - G(\delta))\left(1 - \frac{1}{p}\right)$, is the direct transfer effect, since $T'(\delta)$ is the effect of an increase in δ on delivery of the bond and the second term in the sum is the price effect - a reduction in δ leads to a higher price and hence more delivery on the house. Note that, since $p > 1$, through this effect a reduction in δ decreases utility if $T'(\delta) > 0$ (i.e. in the relevant region). Note that here it plays a role that we express everything in terms of δ instead of the promise j .

To interpret the second effect, we first rewrite it as follows.

$$E2 = p'(\delta) \left(\frac{1}{p} \left(-\frac{e_F^B + P(\delta) - T(\delta) - \beta p}{p} - (1 + G(\delta)) \right) + e_H^A + G(\delta) - \frac{\beta}{p} \right)$$

Observe that in equilibrium, by market clearing, we have

$$e_F^B + P(\delta) - x_F^B = e_F^B + P(\delta) - \beta p = x_F^A - e_F^A.$$

Using also agent A 's second period budget constraint, we obtain

$$\frac{e_F^B + P(\delta) - \beta p}{p} - \frac{T(\delta)}{p} - (1 - G(\delta)) = \frac{x_F^A - e_F^A - T(\delta) - p(1 - G(\delta))}{p} = e_H^A + G(\delta) - \frac{\alpha}{p}.$$

Since $\alpha = \beta$ we can write the second term in the sum as

$$E2 = p'(\delta) \left(-\frac{e_H^A + G(\delta)}{p} + \frac{\alpha}{p^2} - \frac{\beta}{p} + e_H^A + G(\delta) \right) = p'(\delta) \left(e_H^A + G(\delta) - \frac{\alpha}{p} \right) \left(1 - \frac{1}{p} \right)$$

This is the effect of the change of price on utility in equilibrium, by the envelope theorem. Note that if agent A is a buyer of the house, i.e. $e_H^A + G(\delta) - \frac{\alpha}{p} < 0$, a reduction in δ decreases utility through this effect as well.

The third effect, $E3 = \frac{P'(\delta)}{p}$ is the effect on production, less agents produce if δ increases. This is evaluated at the marginal utility of the borrower. An decrease in δ always increases utility through this effect. As we have argued above, this effect is typically stronger than the other 2.

Note that we can rewrite the decomposition of the three effects as in terms of j as follows. The pure price effect and the effect on production stay as before except for the fact that one needs to replace $p'(\delta)$ by $\frac{dp}{dj}$ in E2 and that E3 now reads as $\frac{d\delta}{dj} \frac{P'(\delta)}{p}$. The first term is more interesting and becomes

$$(1 - G - G'\delta \frac{d\delta}{dj}) \left(1 - \frac{1}{p} \right),$$

where the term $1 - G - G'\delta$ is simply the derivative of the delivery with respect to j , taking into account that $p'G$ already is accounted for in the income effect of the lender in the term $E2$.

For the uniform case, we can verify that the first order conditions for welfare maximizing δ are as follows.

$$-\delta + \alpha \frac{1 - \delta}{1 + \alpha + e_H^A} + \frac{4\alpha(\delta - 1)(1 + \alpha + e_H^A)}{(2\alpha + (\delta - 1)^2 + 2e_F^B)^2} = 0 \quad (19)$$

This first order condition is necessary and sufficient since it can be verified that \mathcal{U} is globally concave. Simply taking derivatives, we obtain

$$\mathcal{U}''(\delta) = -1 - \frac{\alpha}{1 + \alpha + e_H^A} - \frac{16\alpha(\delta - 1)^2(1 + \alpha + e_H^A)}{(2\alpha + (\delta - 1)^2 + 2e_F^B)^3} + \frac{4\alpha(1 + \alpha + e_H^A)}{(2\alpha + (\delta - 1)^2 + 2e_F^B)^2} < 0,$$

where the inequality follows from the fact that the third negative term of the sum is always greater or equal in absolute values to the last term of the sum.

To illustrate what is going on it is useful to return to the numerical Example above. Recall that in Example 1, we had $\alpha = \beta = 1$, $e_H^A = 0$ and $e_F^B = 2$. In equilibrium we have $\delta^* \sim 0.279$ and $p \sim 1.630$. Clearly both the first and the second effect go in the wrong direction, the lender is a buyer of the house and a higher house price hurts him more than it helps the borrower. In fact, putting in numbers we obtain that $E1 = 0.0705$, $E2 = 0.0467$. An increase of δ would make both agents better off according to these effects. However, we obtain that $E3 = \frac{\delta}{p} = -0.171$ – the third effect is much stronger than the first two combined and reducing δ increases the sum of the utilities.

It is useful to illustrate the mechanism by comparing the optimal promise with the equilibrium

promise. Recall that in the unregulated equilibrium we had $p_1 \simeq 1.630$ and the unique promise traded was given by $j_* \simeq 1.909$. In the example, solving Equation (19) we obtain that the optimal level of δ is given by $\delta_o \simeq 0.239$. The associated second period housing price is $p_1 \simeq 1.645$. Hence if agents are required not to promise more than $\bar{j} = 1.8840$ the new regulated equilibrium must Pareto-dominate the unregulated one. In the new regulated equilibrium the price of the promise is

$$q = p_1 + \delta_o(1 - \delta_o) \simeq 1.827.$$

Although the promise itself decreases substantially, the price it fetches in period zero only decreases slightly. Instead of producing 0.461 units of food in the second period, as they do in equilibrium, total production is now

$$P(\bar{j}, p_1) = \frac{1}{2}(1 - \delta_o^2) \simeq 0.471.$$

Using Equation (13) we obtain that the price of housing at $t = 0$ increases to

$$p_0 = q + 1 + \frac{1}{p_1}(P(j_*, p_1) - \delta_o(1 - \delta_o)) \simeq 3.003$$

The consumption allocation and utilities can now be computed straightforwardly and are summarized in the following table.

	x_{F0}	x_{H0}	x_{F1}	x_{H1}	U
Agent A	1.5754	0	1.826	0.608	2.9038
Agent B	0.4246	1	1.645	0.392	2.3144

Compared to the utility levels in Section 3.2.3. one observes that both agents' utilities increase. This is of course an artifact of the choice of housing endowments in the first period. Since housing prices increase due to the regulation, the borrower might very well be worse off in the new equilibrium because he has to spend more resources for housing (this would be the case, for example, if $e_H^B < 0.195$). What we prove is that the sum of utilities increases, i.e. there exist transfers in the first period that make both better off.

Finally it is useful to point out that our results do not hold for all economies where the borrower is fully leveraged. The above condition 18, $\alpha^2 < e_F^B(1 + e_H^A)$ is in fact needed in order to ensure that the housing-price effect in the second period which we have shown to be decremental to welfare, does not swamp out the production effect. In fact, we can find examples where things can go the other way, i.e. it might be optimal to put upper bounds on margin requirements. So illustrate this, consider the following

EXAMPLE 2 Consider Example 1, but suppose that $\alpha = 2$ and $e_{F1}^B = 1$ while $e_{H1}^A = 0$.

Note that this example does satisfy the condition for the borrower to be fully leveraged in equilibrium but also note that it does not satisfy condition 18. It turns out that the utility maximizing level of production is $\delta_o = 0.1247$. However, the equilibrium is $\delta_* = 0.1053$. In this economy the borrower is fully leveraged but 'too timid' - he insists on a relatively high margin requirement. A little more default but substantially more borrowing would make the lender so much better off that his utility increase would compensate for the loss of the borrower and the social loss due to less production. The decrease in the price of housing would help the lender more than the decrease in production hurts borrower and lender combined.

4 Example 3: Aggregate and idiosyncratic uncertainty

In our previous case, we saw that a regulation of margin requirements results in three welfare effects. Only one of these effects unambiguously increased welfare. In particular, if the borrower is a seller of the house in the second period, an increase in the price of the house (without changing production) is always bad for welfare. In our first example, with two aggregate states, we argued that the opposite is true. If the borrower is a seller of the house in the state where he is poor and a buyer in the state where he is rich an increase of housing prices in both states always increases the sum of utilities.

We now combine the previous two examples and illustrate that the two effects we discover enforce themselves in a model with aggregate uncertainty and idiosyncratic production shocks. As in the first example, we assume that there are two (aggregate) states, $s = U, D$ and that endowments of agent A as well as housing endowments of agent B are identical across the two states. The states differ in that $e_{FU}^B > e_{FD}^B$. As in Example 2, we assume that $\alpha = \beta$.

To show that the two effects from Examples 1 and 2 reinforce each other, we proceed as in Example 2. We assume that only one contract is traded in equilibrium and that the borrower is fully leveraged. We then characterize the equilibrium promise made and show that indeed under our assumptions on endowments, the borrower is fully leveraged. We also argue that only one contract is traded. We then proceed to argue that the equilibrium margin requirements are too low and that both the effect from Example 1 and the effect from Example 2 lead to this inefficiency.

4.1 Equilibrium promise

As before we start off by assuming that only promise is traded and that the borrower is fully leveraged – as before we give sufficient conditions on the fundamentals for this under the assumption of uniform shocks below.

Similarly to Section 3, define

$$\delta(j, p) = \begin{cases} 0 & \text{if } j \leq p \\ j - p & \text{if } p \leq j \leq p + \Delta \\ 1 & \text{if } j \geq p + \Delta \end{cases}$$

and define $P(j, p) = \int_{x > \delta(j, p)} x dG(x)$ and $T(j, p) = \delta(j, p)(1 - G(\delta(j, p)))$. Our assumptions on endowments guarantee that $p_U > p_D$. Since the borrower is fully leveraged, we can take $j \geq p_D$.

Given all trade takes place in promise j (and agent B uses the entire unit of the house in the first period as collateral, i.e. is fully leveraged) the price of housing in the D-state is given by

$$p_D = \frac{\alpha + e_{FD}^B + P(j, p_D) - T(j, p_D)}{1 + \beta + e_{HD}^A}.$$

In the U -state, the borrower pays $\max[p_U, j] + T(j, p_U)$ to the lender and we have

$$p_U = \frac{\alpha + e_{FU}^B + P(j, p_U) - \min[p_U, j] - T(j, p_U)}{\beta + e_{HU}^A}.$$

As in Section 3.2 above, in order to verify that a given contract j^* is the only one that is traded

in equilibrium, we define the liquidity value of a loan j as

$$\begin{aligned}\Lambda(j, p_U, p_D) &= \pi_D \left(\left(1 - \frac{1}{P_D}\right)(p_D + T(j, p_D)) - \frac{1}{p_D}(P(0, p_D) - P(j, p_D)) \right) + \\ &\quad \pi_U \left(\left(1 - \frac{1}{P_U}\right)(\min[p_U, j] + T(j, p_U)) - \frac{1}{p_U}(P(0, p_U) - P(j, p_U)) \right)\end{aligned}$$

As above, an agent will choose to trade loan j^* only if

$$j^* \in \arg \max_{j \in \mathcal{J}} \Lambda(j, p_U, p_D)$$

We want to focus on the case where the equilibrium promise satisfies $j^* < p_D + \Delta$ and for simplicity we want to restrict ourselves to economies where $j^* < p_U$. If there is default both in the U - and in the D - state the calculations become more complicated without making the point more interesting. What is important is that there is not full default in the D -state, i.e. a local change in the promise has an effect on prices in both states.

Assuming that $p_U \geq j$, promise $j = p_D + \delta$ is traded in equilibrium only if $\frac{d\Lambda}{d\delta} = 0$ which can be written as follows

$$\pi_D \left(T'(\delta) - \frac{1}{p_D}(T'(\delta) - P'(\delta)) \right) + \pi_U \left(1 - \frac{1}{p_U} \right) = 0. \quad (20)$$

Obviously, compared to Example 2, the fact that the borrower is rich in the U -state will generally lead to higher leverage and a higher δ in equilibrium.

4.1.1 Uniform shocks

We now assume again that δ is distributed uniformly on $[0, 1]$ and that $\pi_U = \pi_D = \frac{1}{2}$. We want to focus on the case where there is no default in the U -state whenever $j \leq p_D + \delta$. A simple sufficient condition is obviously the following.

$$e_{FD}^B + 1 \leq e_{FU}^B \quad (21)$$

In order to guarantee that the borrower is fully leveraged, we assume in addition

$$\begin{aligned}2e_{FD}^B + \alpha &> e_H^A \text{ and} \\ e_{FU}^B - e_{FD}^B &> \frac{(1 + 2\alpha(1 + e_H^A - e_{FD}^B) + 2e_H^A(1 + e_H^A - e_{FD}^B))(1 + 2\alpha + 2e_{FD}^B)}{2(1 + \alpha + e_H^A)(\alpha - e_H^A + 2e_{FD}^B)}\end{aligned} \quad (22)$$

To see that the condition is sufficient, note that if the collateral constraint for the borrower does not bind, he will always choose $j \leq p_D$. Therefore, using the fact that the borrower's marginal utility is increasing in the promise, a sufficient condition for the borrower to be fully leveraged is that even if he promises p_D his marginal utility for second period consumption will be lower than that of agent A , i.e. at $j = p_D$, we have

$$\pi_D \left(1 - \frac{1}{p_D} \right) + (1 - \pi_D) \left(1 - \frac{1}{p_U} \right) > 0. \quad (23)$$

Using the fact that without default total production in each state is 0.5 the prices are given by

$$p_D = \frac{\alpha + e_{FD}^B + \frac{1}{2} - p_D}{\beta + e_H^A} = \frac{\alpha + e_{FD}^B + \frac{1}{2}}{1 + \beta + e_H^A} \text{ and } p_U = \frac{\alpha + e_{FU}^B + \frac{1}{2} - p_D}{\beta + e_H^A}$$

and defining $\gamma = e_{FU}^B - e_{FD}^B$ we obtain that (23) is equivalent to

$$\begin{aligned} & (-1 - 2\alpha - 2e_{FD}^B)(2 + 3\alpha + 3e_H^A + 2\alpha e_H^A + 2(e_H^A)^2 + 2e_{FD}^B - 2(1 + \alpha + e_H^A)(e_{FD}^B + \gamma)) - \\ & (1 + \alpha + e_H^A)(-1 - 2\alpha - 2e_{FD}^B + 2(1 + \alpha + e_H^A)\gamma) > 0 \end{aligned}$$

If $2e_{FD}^B > e_H^A - \alpha$ this is equivalent to

$$\gamma > \frac{(1 + 2\alpha(1 + e_H^A - e_{FD}^B) + 2e_H^A(1 + e_H^A - e_{FD}^B))(1 + 2\alpha + 2e_{FD}^B)}{2(1 + \alpha + e_H^A)(\alpha - e_H^A + 2e_{FD}^B)}$$

Under Conditions (21) and (22) it must therefore always be true that the borrower is fully leveraged and there there is no default in the U -state whenever $j < p_D + 1$. We therefore now need to search for conditions that ensure existence of an equilibrium where a unique contract is traded which promises less than $p_D + 1$.

In our case of uniform shocks and under the assumption that there is only one contract traded with $j = p_D + \delta$, we obtain

$$p_D = \frac{\alpha + \frac{1}{2} - \delta + \frac{1}{2}\delta^2 + e_{FD}^B}{1 + \alpha + e_H^A}.$$

We also obtain

$$p_U = \frac{\alpha + 2\alpha^2 - 2\alpha\delta - \delta^2 + e_H^A + 2\alpha e_H^A - 2\delta e_H^A - 2e_{FD}^B + 2(1 + \alpha + e_H^A)e_{FU}^B}{2(\alpha + e_H^A)(1 + \alpha + e_H^A)}$$

It is easy to see that, both p_D and p_U are decreasing in δ - i.e. if agents are forced to promise less than in equilibrium, prices in both states will increase.

Using the above analysis we obtain that promise $j = p + \delta_*$ is traded in equilibrium only if

$$\delta_* = \arg \max_{\delta} \left(\pi_D \left((1 - \frac{1}{p_1}) [p_1 + \delta(1 - \delta)] - \frac{1}{p_1} [\frac{1}{2} - \frac{1}{2}(1 - \delta^2)] \right) + \pi_U (p_U + \delta) \left(1 - \frac{1}{p_U} \right) \right)$$

Note that as in the case of no aggregate uncertainty the maximand is a strictly concave function of δ , hence for any anticipated price (p_D, p_U) there is at most one level of δ , and hence promise j , that maximises the liquidity value and satisfies $\delta < 1$. Setting the derivative of the above maximand equal to 0,

$$\begin{aligned} \left(1 - \frac{1}{p_D}\right)(1 - 2\delta_*) - \frac{1}{p_1}\delta_* + \left(1 - \frac{1}{p_U}\right) &= 0 \\ \left(1 - \frac{1}{p_D}\right) + \left(1 - \frac{1}{p_U}\right) &= \delta_* \left(2 - \frac{1}{p_D}\right) \\ \delta_* &= (p_D - 1 + p_D - \frac{p_D}{p_U}) / (2p_D - 1) \end{aligned}$$

A simple sufficient condition for $\delta_* \leq 1$ is obviously $2p_D - 1 > 0$ which always holds if

$$2e_{FD}^B + \alpha > e_H^A + 1. \quad (24)$$

We have therefore verified that under Conditions (21), (22) and (??) there always exists an equilibrium where a single j^* is traded, where the borrower is fully leveraged and where there is no default in the U -state. Note that (??) always implies the first part of (22).

Using the expressions for p_D and p_U it is easy to see that the equilibrium level of δ_* solves the

following equation.

$$\frac{2(1 - \delta_*)(\alpha + (-2 + \delta_*)\delta_* - e_H^A + 2e_{FD}^B)}{2\alpha + (1 - \delta_*)^2 + 2e_{FD}^B} = \frac{2(\alpha + e_H^A)(1 + \alpha + e_H^A)}{\alpha + 2\alpha^2 - 2\alpha\delta_* - \delta_*^2 + e_H^A + 2\alpha e_H^A - 2\delta e_H^A - 2e_{FD}^B + 2(1 + \alpha + e_H^A)e_{FU}^B}$$

Unfortunately, this is a polynomial of degree 5 and can therefore only be solved numerically. However, since it is a univariate polynomial all positive solutions can be found with arbitrary accuracy. In the numerical examples reported here, the solution is always unique.

To illustrate the model, consider the following simple numerical example.

EXAMPLE 3 *Suppose $\alpha = \beta = 1$ and endowments are given by*

$$e_{H0}^A = 0.5, e_{F0}^A = 1, \quad e_{H0}^B = 0.5, e_{F0}^B = 1$$

$$e_{HU}^A = e_{HD}^H = 1, \quad e_{FU}^A = e_{FD}^A = 1, \quad e_{HU}^B = e_{HD}^B = 0, \quad e_{FD}^B = 1, e_{FU}^B = 3.$$

Solving for the equilibrium promise, we obtain that $\delta_* = 0.199$. Equilibrium prices are given by

$$p_U = 1.76381 \quad p_D = 0.77368$$

The equilibrium promise is $j^* = 0.97267$. The price of the promise is determined by the lender and is given by

$$q = \frac{1}{2}(p_D + \delta_*(1 - \delta_*)) + \frac{1}{2}j^* \simeq 0.95287$$

Instead of producing 0.5 units of food in the D -state in the second period, total production is only

$$P(j^*, p_D) = \frac{1}{2}(1 - \delta_*^2) \simeq 0.4802.$$

As in the case of no aggregate uncertainty, the marginal disutility of the downpayment on the house must then be equal to the marginal utility of owning the house at time 0, net of making the deliveries on j . We obtain that the price of housing at $t = 0$ is given by

$$p_0 = 1 + q + \frac{\pi_D}{p_D}(P(j^*, p_D) - T(j^*, p_D)) + \frac{\pi_U}{p_U}(p_U - j^*) \quad (25)$$

For our example, we obtain

$$p_0 \simeq 3.3845$$

The consumption allocation and utilities can now be computed straightforwardly and are summarized in the following table.

	x_{F0}	x_{H0}	x_{FD}	x_{HD}	x_{FU}	x_{HU}	U
Agent A	1.7394	0	1.2263	1.2925	2.2672	0.5670	2.9026
Agent B	0.2606	1	0.7737	0.7075	1.7638	1.4330	2.3143

Note that the lender is a buyer of housing in the D -state and a seller in the U -state, as in our leading example in Section 2.

4.2 Optimal promise – possible Pareto improvements

We proceed as in Section 3 above to show that generally a reduction of the promise relative to its equilibrium value makes everybody better off. As before, it is useful to derive an expression for the sum of to agents' second period utility as a function of δ . We obtain

$$\begin{aligned} \mathcal{U}(\delta) &= \pi_D \left(\frac{e_{FD}^B + \int_{x \geq \delta} (x - \delta) dG(x)}{p_D} + p_D(e_H^A + 1) + \int_{x \geq \delta} \delta dG(x) \right) + \\ &\quad \pi_U \left(\frac{e_{FU}^B - \delta - p_D}{p_U} + p_U e_{HU}^A + p_D + \delta \right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{U}'(\delta) &= \pi_D \left((P'(\delta) - T'(\delta)) \left(\frac{\alpha(1 + e_H^A + \alpha)}{(e_F^B + \alpha - T(\delta) + P(\delta))^2} + \frac{e_H^A + 1}{1 + e_H^A + \alpha} \right) + T'(\delta) \right) + \\ &\quad \pi_U \left(1 - \frac{1}{p_U} + E(\delta) \right), \end{aligned}$$

where

$$E(\delta) = \frac{\delta p'_U}{p_U^2} + \frac{-p'_D p_U - p'_U (e_{FU}^B - p_D)}{p_U^2} + p'_U e_{HU}^A + p'_D.$$

Note that using

$$p_U = \frac{\alpha + e_{FU}^B - p_D - \delta}{\beta + e_{HU}^A},$$

and the fact that $\alpha = \beta$ we can write

$$\begin{aligned} E(\delta) &= \left(1 - \frac{1}{p_U}\right) p'_D + \frac{-p'_U e_{FU}^B - p_D - \delta}{p_U^2} + p'_U \alpha + p'_U e_{HU}^A \\ &= p'_D \left(1 - \frac{1}{p_U}\right) + p'_U e_{HU}^A \left(1 - \frac{1}{p_U}\right) - \frac{\alpha}{p_U} p'_U \left(1 - \frac{1}{p_U}\right) \end{aligned}$$

which is always negative if $e_{HU}^A > \alpha$.

The equilibrium level of δ , determined by Equation (20) is now inefficient through 2 effects. First, similar to the effect in Example 2 we have that if

$$\left(T'(\delta) - \frac{1}{p_D} (T'(\delta) - P'(\delta)) \right) = -\frac{\pi_U}{\pi_D} \left(1 - \frac{1}{p_U} \right)$$

then $\mathcal{U}'(\delta) \leq 0$ because as we have seen we always have $\mathcal{U}'(\delta) \leq \mathcal{K}'(\delta)$.

Second, we have that even if

$$\pi_D \left((P'(\delta) - T'(\delta)) \left(\frac{\alpha(1 + e_H^A + \alpha)}{(e_F^B + \alpha - T(\delta) + P(\delta))^2} + \frac{e_H^A + 1}{1 + e_H^A + \alpha} \right) + T'(\delta) \right) + \pi_U \left(1 - \frac{1}{p_U} \right) = 0$$

it is still true that $\mathcal{U}'(\delta) < 0$ because $E(\delta) < 0$. This second effect is comparable to Example 1: In the U-state, the lender is relatively poor and wants to save. A decrease in borrowing/saving leads to an increase of the price of housing and agent 1, being a net seller of the house in the U-state, benefits.

4.2.1 Decomposing the effect

As above, it is useful to decompose the welfare effect in three effects: The effect of the transfer, the price effect and the effect on production. For this, we can treat the D -state separately and we can decompose the effect there exactly as in Section 3.2.4 above.

We can write

$$\begin{aligned} \mathcal{U}'(\delta) = & \pi_D \left(1 - \frac{1}{p_D}\right) (T'(\delta, p_D) + (1 - G(\delta))p'_D) + \pi_U \left(1 - \frac{1}{p_U}\right) (1 + p'_U(\delta)) + \\ & \pi_D p'_D \left(1 - \frac{1}{p_D}\right) \left(e_H^A + G(\delta) - \frac{\alpha}{p_D}\right) + \pi_U \left(1 - \frac{1}{p_U}\right) p'_U \left(e_H^A - \frac{\alpha}{p_U}\right) + \\ & \pi_D P'(\delta, p_D) \end{aligned}$$

As in the case without aggregate uncertainty, the first effect is unambiguously negative - a reduction in borrowing leads to a decrease in total welfare because the borrower is constrained and has on average lower marginal utility for consumption than the lender. The crucial difference between this case and the case without aggregate uncertainty is now that naturally the second effect is positive. In the D -state, where the borrower is poor, he is likely to be a seller of housing and $p_D < 1$ so his marginal utility is above the lender's. In the U state, the lender is likely to be a seller of the house and with $p_U > 1$ an increase of the price increases total utility in both states.

4.2.2 The uniform case

Finally, to illustrate how regulation in Example 3 increases utility we consider the uniform case. It is easy to see that for the uniform case the function $\mathcal{U}'(\delta)$ is given by

$$\begin{aligned} \mathcal{U}'(\delta) = & 4 - 4\delta + \frac{2(-1 + \delta)}{2 + e_H^A} + \frac{2(-1 + \delta)(1 + e_H^A)}{2 + e_H^A} - \frac{2e_H^A(1 + \delta + e_H^A)}{(1 + e_H^A)(2 + e_H^A)} - \frac{4(-1 + \delta)(2 + e_H^A)(1 - 2\delta + \delta^2 + 2e_{FD}^B)}{(3 - 2\delta + \delta^2 + 2e_{FD}^B)^2} + \\ & \frac{4(-1 + \delta)(2 + e_H^A)}{3 - 2\delta + \delta^2 + 2e_{FD}^B} - \frac{4(1 + e_H^A)(1 + \delta + e_H^A)(2\delta(2 + e_H^A - e_{FU}^B) + \delta^2 e_{FU}^B + (3 + 2e_{FD}^B)e_{FU}^B)}{(-3 + \delta^2 + 2\delta(1 + e_H^A) + 2e_{FD}^B - 4e_{FU}^B - e_H^A(3 + 2e_{FU}^B))^2} + \\ & \frac{4(1 + e_H^A)(2 + e_H^A + (-1 + \delta)e_{FU}^B)}{-3 + \delta^2 + 2\delta(1 + e_H^A) + 2e_{FD}^B - 4e_{FU}^B - e_H^A(3 + 2e_{FU}^B)} \end{aligned}$$

The equation $\mathcal{U}'(\delta) = 0$ can be rewritten as a polynomial equation of degree 9 and also has to be solved numerically.

Returning to Example 3 above, we compute the optimal level of δ and note that it is in fact zero. In this case it is socially optimal to have no default at all, while in equilibrium there is substantial default in the D -state. To understand why the difference is so big, we note that both the effect from Example 1 and the effect from Example 2 play a role. The effect from Example 2 is clear: Reducing borrowing increases the price of housing and increases production. The effect from Example 1 plays an important role as well, however. Obviously, since $x_{Hs}^A = \frac{1}{p_s}$, the lender is a buyer of houses in the D -state and a seller in the U -state. The marginal utility of the borrower is higher in the U state and lower in the D -state. As we have seen in Example 1 an increase of the price makes both better off. At $\delta = 0$ we obtain

$$p_U = 1.8333 \quad p_D = 0.83333.$$

Both prices increase substantially, making both agents better off, even in absence of the production-effects.

5 Conclusion

In this paper we present two classes of simple examples where competitive equilibria are constrained inefficient and a reduction in borrowing or in leverage makes all agents better off. While the examples are very simple and stylised they reveal a few economically important mechanisms. The first one is standard in the GEI literature but its connection to over-borrowing has not been widely explored. The second mechanism appears to be new. Most notably, we show that the two mechanisms interact in an important way.

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