Cowles Foundation Paper 577

Journal of Development Economics 13 (1983) 223–236. North-Holland

A GEOMETRIC EXPLANATION OF THE TRANSFER PARADOX IN A STABLE ECONOMY

John GEANAKOPLOS and Geoffrey HEAL*

Yale University, New Haven, CT 06520, USA

Columbia University, New York, NY 10027, USA

Received November 1982, final version received March 1983

It has recently been shown [Chichilnisky, *Journal of Development Economics* (1980)] that in a three-agent two-good economy, the transfer paradox may occur at a Walrasian stable equilibrium. Our paper gives a geometric demonstration of the result, making the role of the third agent clear. It also generalises the earlier result in certain respects, showing inter alia that what is important is that the number of goods should exceed the number of agents.

1. Introduction

It is possible that the receipt of a gift could make the recipient worse off? Leonticf (1936) showed that this is possible, but Samuelson (1952) pointed out that the Leontief example rcquires the initial equilibrium to bc unstable. Since then, the so-called transfer paradox has been associated with instability; Balasko (1978) proved formally that this paradox is incompatible with Walrasian stability in a two-agent economy. However, Chichilnisky (1980) has recently shown that with three agents the transfer paradox may occur, even at a Walrasian stable competitive equilibrium.

In this paper we are able to give a geometric proof of Chichilnisky's result, replacing her assumptions about non-singularity of a Jacobean with a simple boundary condition and a picture, which also makes clear the significance of a third agent. Moreover our diagram allows us to prove that it does not matter which good or combination of goods is transferred: under the specified conditions the paradox will occur (for small enough gifts) whenever the transfer raises the recipient's income at the initial transfer prices. This holds cven if some of the goods are transferred in negative amounts, that is, even if we think more gcnerally of exchanges rather than transfers. What is crucial is the income effect of the transfer or exchange.

It is clear that in a 2-agent world, if a transfer makes the recipient worse off, then it must make the donor better off. However, this need not be true with more than two agents. In fact, in Chichilnisky's example both the donor

*We wish to acknowledge helpful conversations with Graciela Chichilnisky and Heraklis Polemarchakis. This paper was written while Heal was Visiting Professor at the Cowles Foundation, Yale University.

and recipient lose, while the third party gains, as she points out in her discussion of coalitions. Accordingly, in our last section, as in Chichilnisky's section 4, we consider a stronger transfer paradox: in a suitable economy E all sufficiently small gifts of commodities from N to S will make the recipient S worse off and the donor N better off, despite the fact that both E and the new economy E' are globally Walrasian stable. We can thus distinguish two paradoxes: what we have called the strong transfer paradox, which involves the recipient losing and the donor gaining, and the advantageous reallocation paradox [Guesnerie and Laffont (1978)] which involves both donor and recipient losing, or gaining if the transfer is reversed.

We conclude the paper by giving a still briefer abstract proof of both transfer paradoxes.

2. The model

The model and notation are identical with those in Chichilnisky (1980), though we shall refer to agents rather than income groups. There are three agents, denoted H, L, and S, and two goods, denoted A and B. Preferences are of the Leontief fixed-proportions type,[1] with the agent H consuming proportionately more of A than the agent L who in turn consumes proportionately more of A than the agent S. The utility functions are (see fig. 1)

$$U_H = \min(A, aB), \quad a > 1, \qquad U_L = \min(A, B), \qquad U_S = \min(cA, B), \quad c > 1.$$

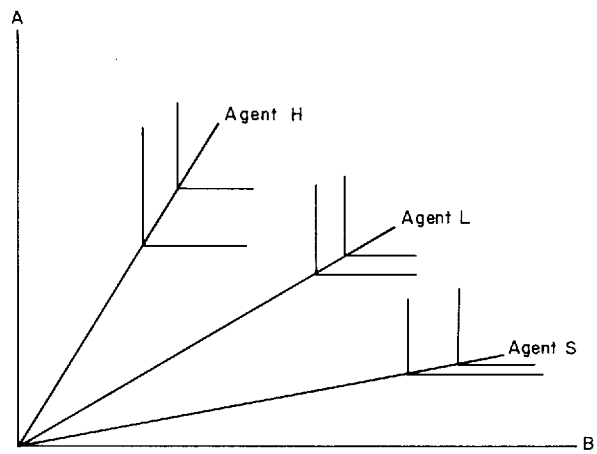

Fig. 1

[1]The use of fixed proportions preferences makes it possible to relate the occurrence of 'paradoxical' outcomes to the underlying data of the economy, namely preferences and endowments. Indeed, Chichilnisky's paper in this journal gives necessary and sufficient conditions on these data. While the same outcomes can of course be shown to occur with smooth preferences, it seems impossible in this case to give conditions for their occurrence in terms of the data of the economy.

In addition to her preferences, each agent is characterized by a strictly positive endowment vector, and these are denoted (H_A, H_B), (L_A, L_B), and (S_A, S_B) in the obvious notation. P_A and P_B are of course the prices of A and B.

Let us consider the demand functions of each of the agents. If $P_A>0$ and $P_B>0$, then a Leontief agent with utility $U=\min(A,\lambda B)$ ($\lambda=a$, 1, or $1/c$) and endowment (E_A, E_B) would choose C_A and C_B so that

$$P_A C_A + P_B C_B = P_A E_A + P_B E_B \quad \text{or} \quad P_A \lambda C_B + P_B C_B = P_A E_A + P_B E_B,$$

or

$$C_B = \frac{(P_A/P_B)E_A + E_B}{1+(P_A/P_B)\lambda}. \quad \text{Similarly} \quad C_A = \frac{E_A + (P_B/P_A)E_B}{1+(P_B/P_A)(1/\lambda)}.$$

In case one of the prices P_A or P_B is zero, the agent will be able to afford an entire line segment of consumption bundles which make her indifferent. We assume she chooses the unique point on the consumption ray $\{(A,B): A=\lambda B\}$. With this convention we note that the demands at $P_A=1$, $P_B=0$ are $C_B=(1/\lambda)E_A$ and $C_A=E_A$ while for $P_A=0$, $P_B=1$ we have

$$C_B = E_B \quad \text{and} \quad C_A = \lambda E_B.$$

Our demand functions are continuous on the whole price simplex, $\Delta=\{(P_A,P_B)\in\mathbb{R}^2_+ \mid P_A+P_B=1\}$. In what follows it will prove convenient to work with the normalized price space $\{p\in\mathbb{R}\mid p>0\}$, where $p=P_A/P_B$. Then $C_B(p)=(pE_A+E_B)/(1+p\lambda)$ and $C_B(0)=\lim_{p\to 0} C_B(p)=E_B$ and $C_B(\infty)\equiv\lim_{p\to\infty} C_B(p)=(1/\lambda)E_A$. Note also that $\lim_{p\to 0} C_A(p)=\lambda E_B$.

Let us now consider the excess demand function for commodity B for the economy as a whole. It is given by:

$$B(p) = \frac{pH_A+H_B}{1+pa} - H_B + \frac{pL_A+L_B}{1+p} - L_B + \frac{pS_A+S_B}{1+(1/c)p} - S_B$$

$$= \frac{p(H_A-aH_B)}{1+pa} + \frac{p(L_A-L_B)}{1+p} + \frac{p(S_A-(1/c)S_B)}{1+(1/c)p}.$$

Clearly $B(0)=0$ and $\lim_{p\to\infty} B(p)=(1/a)H_A-H_B+L_A-L_B+cS_A-S_B$. Similarly we could write the expression for the aggregate excess demand $A(p)$. We shall not need this expression, except to note that

$$A(0) \equiv \lim_{p\to 0} A(p) = aH_B - H_A + L_B - L_A + (1/c)S_B - S_A, \quad \text{and}$$

$$A(\infty) \equiv \lim_{p\to 0} A(p) = 0.$$

Observe that the prices $(P_A, P_B)=(0,1)$, or $p=0$, are defined as a competitive equilibrium if and only if $A(0)<0$, since in that case the B market clears and there is excess supply in the market A which has zero price. Similarly the prices $(P_A, P_B)=(1,0)$, or $p=\infty$, are defined as a competitive equilibrium if and only if $B(\infty)<0$. Of course if $B(p)=0$ for some $0<p<\infty$, then by Walras Law we must also have that $A(p)=0$ and p is then a competitive equilibrium.

Definition. We shall say that the economy described above is *globally Walrasian stable* iff it possessed a unique equilibrium $\bar{p}$, satisfying

(1) $0<\bar{p}<\infty$,

(2) if $0<p<\bar{p}$, then $B(p)<0$,

(3) if $\bar{p}<p<\infty$, then $B(p)>0$.

The reader can convince himself that this is precisely the traditional definition of Walrasian stability: if $B(p)>0$, the only price ratio which clears the market is a $\bar{p}$ for which commodity B is more expensive, $\bar{p}<p$. In fig. 2 we find an example of a function $B(p)$ satisfying the definition. Before proceeding to the assumptions and our theorem we introduce the function $F(p)=B(p)[(1+pa)(1+p)[1+(1/c)p]]/p=B(p)g(p)$. Observe that for $0<p<\infty$ $g(p)>0$. Hence for $0<p<\infty$ $F(p)$ is negative, zero or positive precisely when $B(p)$ is. Furthermore, recall from the formula for $B(p)$ that $F(p)$ is a quadratic function in p (with coefficients depending on H_A, H_B, etc.).

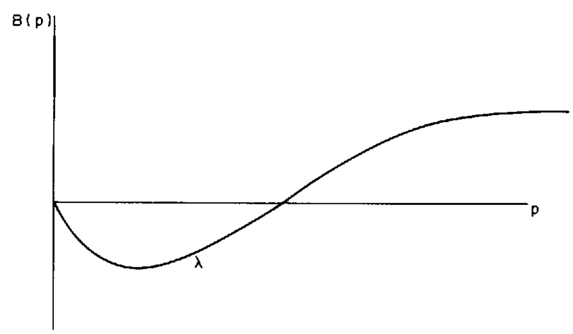

Fig. 2

3. The assumptions

(*A.1*) The endowment vectors are strictly positive and no agent's endowment lies on her own consumption ray.

(*A.2*) When $P_B=0$, there is positive excess demand for good B, that is $B(\infty) \equiv \lim_{p\to\infty} B(p)=(1/a)H_A-H_B+L_A-L_B+cS_A-S_B>0$.

(*A.3*) When $P_A=0$, there is positive excess demand for good A, that is $A(0) \equiv \lim_{p\to 0} A(p)=aH_B-H_A+L_B-L_A+(1/c)S_B-S_A>0$.

(*A.4*) $L_A<L_B$.

These are closely related to the assumptions made by Chichilnisky. In particular (A.2) is her (C.1) and (A.4) is her assumption $\lambda>0$. We have dropped her differentiability assumptions, showing that they follow from (A.3). The reader can easily convince himself that these assumptions are mutually consistent, although they do have consequences about the data of the economy (namely the endowments). For example we shall show that they imply that $cS_A>S_B$.[2] In what follows we shall hold the utilities fixed; accordingly we parameterize any economy by its vector E of endowments, $E=(H_A, H_B, L_A, L_B, S_A, S_B)$. The set $\mathscr{E}$ of economics E which satisfy assumptions (A.1)–(A.4) is clearly open. In particular, if $E\in\mathscr{E}$ then a small transfer of commodities from H to S will produce an economy $E'\in\mathscr{E}$ also satisfying assumptions (A.1)–(A.4).

Lemma 1. Let the economy E satisfy assumptions (A.1)–(A.3). Then E is globally Walrasian stable.

Proof. From assumptions (A.2) and (A.3) we know that there is no equilibrium with $P_A=0$ or with $P_B=0$. Consider the quadratic function $F(p)$. $F(0)\equiv\lim_{p\to 0} F(p)=H_A-aH_B+L_A-L_B+S_A-(1/c)S_B<0$ by assumption (A.3). On the other hand, $\lim_{p\to\infty} F(p)=\infty$. By continuity there must be a $\bar{p}$, $0<\bar{p}<\infty$, such that $F(\bar{p})$, and hence $B(\bar{p})$, equals zero. Furthermore, since F is quadratic $\bar{p}$ must be the only point between 0 and ∞ at which $F(\bar{p})=0$, for if there were a second there would necessarily be a third root, which is impossible for a quadratic function. Finally, by continuity again it follows that if $0<p<\bar{p}$ then $F(\bar{p})<0$ and if $\bar{p}<p<\infty$ then $F(p)>0$. The same must therefore hold true for $B(p)$. Q.E.D.

Fig. 2 is therefore a faithful representation of the excess demand function $B(p)$. The reader interested only in the strong transfer paradox may proceed directly to the simpler proof in section 5. Both theorems are reproved in section 6.

[2]This has created some controversy, for Chichilnisky also included the assumption that S is endowed with 'mostly B goods'. Of course this is possible if c is large and she herself pointed out the consequence $cS_A>S_B$.

4. The transfer paradox and advantageous reallocation

We shall now give a geometric proof of essentially Chichilnisky's (1980) Theorem 1, that under assumptions (A.1)–(A.4) a transfer of commodities from H to S makes S worse off, even in the presence of global Walrasian stability. We point out that in this case H's utility must also decline.[3] In section (5) we replace Chichilnisky's assumption (A.4) with assumptions (B.1)–(B.3) which allow us to derive an even stronger transfer paradox: under these conditions the transfer of commodities from H to L necessarily lowers the utility of L and raises the utility of H, even in the presence of global Walrasian stability.

Lemma 2. Let E satisfy assumptions (A.1)–(A.4). Then $S_A > (1/c)S_B$. *In particular, at the unique equilibrium* $\bar{p}$ *S imports good B and exports good A.*

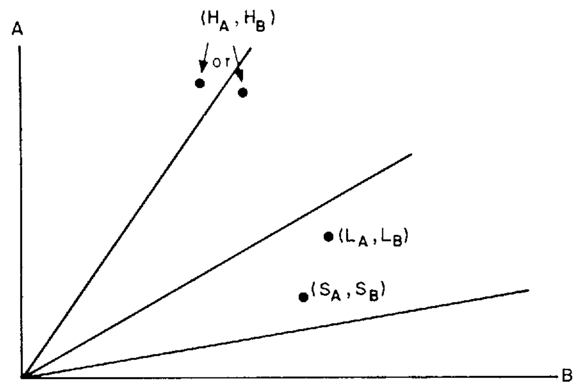

Fig. 3

Proof. From Lemma 1 we know that there is a $\bar{p}$, $0<\bar{p}<\infty$ with $B(\bar{p})=0$. Assume, contrary to fact, that $S_A \leqq (1/c)S_B$. Then since

$$0=B(\bar{p})=\frac{\bar{p}(H_A-aH_B)}{1+\bar{p}a}+\frac{\bar{p}(L_A-L_B)}{1+\bar{p}}+\frac{\bar{p}(S_A-(1/c)S_B)}{1+(1/c)\bar{p}}$$

and $L_A<L_B$, it follows that $H_A>aH_B$. But then

$$0=B(\bar{p})<\frac{p(H_A-aH_B)}{1+\bar{p}}+\frac{\bar{p}(L_A-L_B)}{1+\bar{p}}+\frac{\bar{p}(S_A-(1/c)S_B)}{1+\bar{p}}.$$

[3]This is shown in section 4 of Chichilnisky's paper.

Factoring out $\bar{p}/(1+\bar{p})$ contradicts assumption (A.3). Hence we must have that $S_A > (1/c)S_B$. Q.E.D.

Theorem 1 (Transfer Paradox). Let E satisfy assumptions (A.1)–(A.4), with unique equilibrium $\bar{p}$ and corresponding utility levels (at the equilibrium consumption bundles) of $\bar{U}_H$, $\bar{U}_L$, $\bar{U}_S$. Then the transfer of a sufficiently small bundle of goods from H to S necessarily reduces the utility $\bar{U}'_S$ of S at the new equilibrium $\bar{p}'$, despite the fact that both E and the new economy E′ are globally Walrasian stable. On the other hand, such a transfer also reduces the utility $\bar{U}'_H$ of H, and raises the utility $\bar{U}'_L$ of L.

Proof. Suppose we maintained the old equilibrium prices $\bar{p}$ in the new economy E'. Then of course H would be worse off and S better off. However, at these prices there is necessarily an excess demand in E' for good B, $B'(\bar{p}) > 0$, since wealth has been shifted from a low propensity-to-consume-B agent to a high propensity-to-consume-B agent. Since E' is globally stable, it follows that $\bar{p}' < \bar{p}$. Observe from fig. 4 that since $cS_A > S_B$ it follows that the lower is $\bar{p}$, the worse off is S. Our strategy consists in showing that even at the prices $\hat{p} < \bar{p}$ at which S would just buy his original equilibrium consumption bundle $\bar{S}$ there will still be excess demand for B, $B'(\hat{p}) > 0$. Hence necessarily $\bar{p}' < \hat{p}$ and S is worse off. We must distinguish two cases, according to whether $H_A > aH_B$ or $H_A \leqq aH_B$.

In fig. 4 we have considered the case where $H_A > aH_B$. Observe that by assumption the transfer $H - H^\delta = S^\delta - S$. We shall show that $\hat{L}_B - \bar{L}_B > \bar{H}_B - \hat{H}_B$. Since $\bar{p}$ is the equilibrium for E, it follows that $(\bar{S} - S) + (\bar{H} - H) = -(\bar{L} - L)$. On the other hand since all the angles correspond and at least one side is identical, the two triangles $\Delta \bar{S}SS^\delta$ and ΔOHH^δ are congruent. Hence $\bar{H} - 0 = (H - 0) + (\bar{H} - H) = (\bar{S} - S) + (\bar{H} - H) = -(\bar{L} - L)$. Finally since $\gamma < \beta$, it follows that $\hat{L}_B - \bar{L}_B > \bar{H}_B - \hat{H}_B$. To see this, draw a line parallel to $\bar{L}\hat{L}$ starting from $\bar{H}$ and see that it intersects OH to the left of $\hat{H}_B$. This new triangle is identical to $\Delta L\bar{L}\hat{L}$. Clearly H is also worse off at $\hat{p}$ in E', and he will be still worse off at $\bar{p}'$.

In fig. 5 we consider the case where $H_A \leqq aH_B$. Again we have that $H - H^\delta = S^\delta - S$, and hence that the triangles ΔOHH^δ and $\Delta \bar{S}SS^\delta$ are congruent. Since from the fact that $\bar{p}$ is an equilibrium for E we that $(\bar{L} - L) + (\bar{H} - H) = S - \bar{S}$, it follows that $\bar{H} - 0 = L - \bar{L}$. But now the fact that $\gamma < \beta$ again implies that $\hat{L}_B - \bar{L}_B > \bar{H}_B - \hat{H}_B$. Thus once again the equilibrium $\bar{p}'$ of E' satisfies $\bar{p}' < \bar{p}$ and S is worse off than in E's equilibrium, despite the transfer.

To conclude the proof of the theorem it suffices to show that at the price $\hat{\hat{p}}$ in E' at which H is just as well off as at the equilibrium $\bar{p}$ in E, there is excess supply of good B, $B'(\hat{\hat{p}}) < 0$. Hence $\hat{\hat{p}} < \bar{p}' < \hat{p} < \bar{p}$ and H is also worse off. Since we shall derive precisely such a result (actually its opposite) in Theorem 2, we leave this to the reader. Q.E.D.

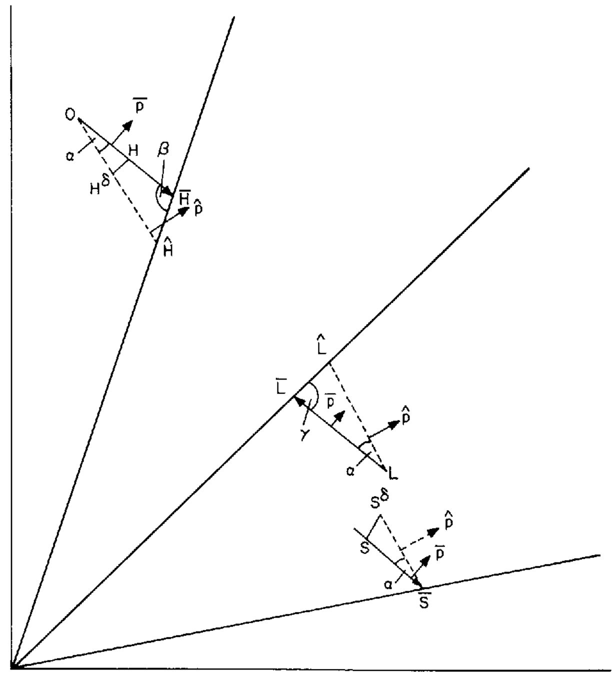

Fig. 4. Initial endowment vectors in E: H, L, S. Initial endowment vectors in E': H^{δ}, L, S^{δ}. Consumption vectors in E equilibrium: $\bar{H}$, $\bar{L}$, $\bar{S}$. Consumption vectors in E' at prices $\hat{p}$: $\hat{H}$, $\hat{L}$, $\hat{S}=\bar{S}$.

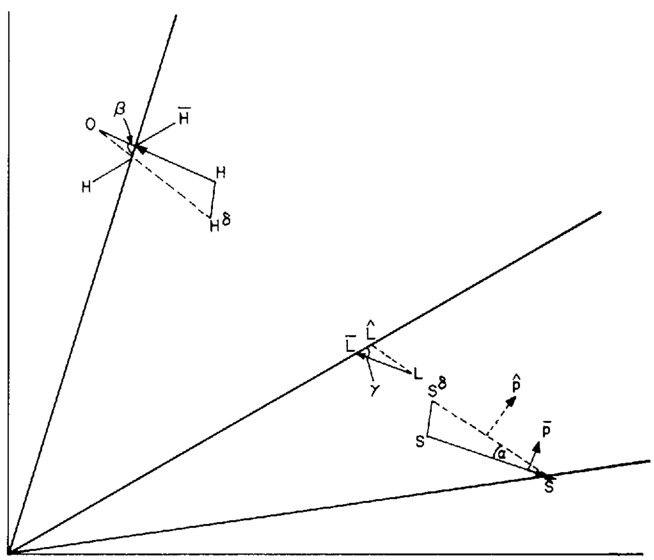

Fig. 5. Initial endowment vectors in E: H, L, S. Initial endowment vectors in E': H^{δ}, L, S^{δ}. Consumption vectors in E equilibrium: $\bar{H}$, $\bar{L}$, $\bar{S}$. Consumption vectors in E' at prices $\hat{p}$: $\hat{H}$, $\hat{L}$, $\bar{S}$.

5. The strong transfer paradox

In order to display a stronger form of the paradox in which *H*'s utility actually rises after he makes the transfer, let us drop assumption (A.4), replace it with several others, and investigate the transfer from *H* to *L*. This parallels the discussion in Chichilnisky, p. 517, last paragraph.

(B.1) $cS_A < S_B$.

(B.2) $aH_B > H_A$.[4]

(B.3) $L_A > L_B$.

The reader can easily convince himself that assumptions (A.1)–(A.3) and (B.1)–(B.3) together form a consistent set of postulates. The set E^2 of economies E satisfying assumptions (A.1)–(A.3) and (B.1)–(B.3) is clearly open. Hence the transfer of a sufficiently small bundle of commodities from H to L creates another economy $E' \in \mathrm{E}^2$.

Theorem 2 (Strong Transfer Paradox). Let the economy E satisfy assumptions (A.1)–(A.3) and (B.1)–(B.3). Then the transfer of a sufficiently small bundle of commodities from H to L necessarily lowers the utility of L (in the new equilibrium, compared to the old equilibrium) and raises the utility of H, despite the fact that both E and the new economy E′ are globally Walrasian stable.

Proof. It follows from assumptions (A.1)–(A.3) and Lemma 1 that both E and E' are globally Walrasian stable. As before it must be true that after the transfer, but at the old equilibrium prices $\bar{p}$, L is better off and H is worse off. However, at these prices there is necessarily an excess demand in E' for good B, $B'(\bar{p}) > 0$, since wealth has been shifted from a low marginal propensity-to-consume-B agent to a higher marginal propensity-to-consume-B agent. Since E' is globally stable, it follows that $\bar{p}' < \bar{p}$. Observe from fig. 6 that the lower is $\bar{p}'$, the worse off is L and the better off is H. Our strategy is to show that at the prices $\hat{p}$ in E' at which H will just buy his original equilibrium consumption bundle $\bar{H}$, L is already worse off and there is still excess demand for B, $B'(\hat{p}) > 0$. Hence the equilibrium $\bar{p}'$ in E' must satisfy $\bar{p}' < \hat{p}$ and so make H strictly better off and L strictly worse off than they were at price $\bar{p}$ in E. We must show that $\hat{\hat{S}}_B - \bar{S}_B > \bar{L}_B - \hat{\hat{L}}_B$. ($\bar{S}_B$ is the B component of $\bar{S}$, etc.)

Observe first, in fig. 6, that from the fact that $\bar{p}$ is an equilibrium for E we must have that $(\bar{H} - H) + (\bar{S} - S) = (L - \bar{L})$. Notice that the transfer $H - H^{\delta}$

[4]We could prove, exactly as in Lemma 2, that (A.1)–(A.3) and (B.1) imply both (B.2) and (B.3).

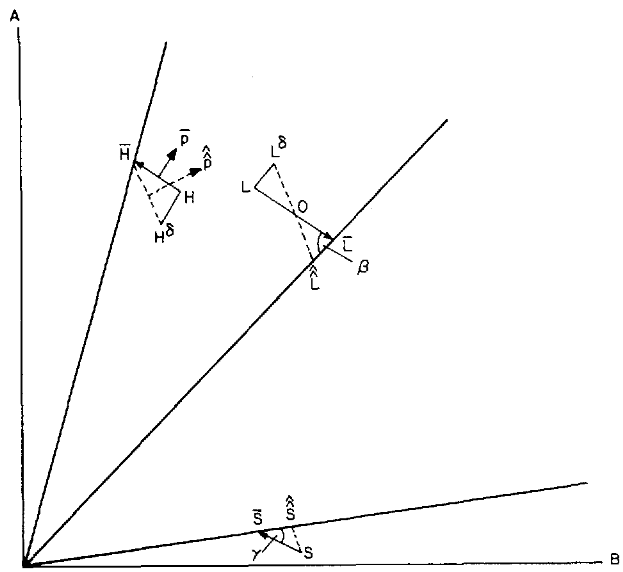

Fig. 6. Initial endowment vectors in E: H, L, S. Initial endowment vectors in E': H^δ, L^δ, S. Consumption vectors in E equilibrium: $\bar{H}$, $\bar{L}$, $\bar{S}$. Consumption vectors in E' at prices $\hat{\hat{p}}$: $\bar{H}=\hat{\hat{H}}$, $\hat{\hat{L}}$, $\hat{\hat{S}}$.

$=L^\delta-L$. Since all the angles corrcspond, the triangles $\Delta\bar{H}HH^\delta$ and ΔOLL^δ are identical. Hence we must have that $\bar{L}-O=S-\bar{S}$. Finally, since the angles $\angle\hat{\hat{L}}O\bar{L}$ and $\angle\hat{\hat{S}}S\bar{S}$ are the same and angle γ ($\angle S\bar{S}\hat{\hat{S}}$) < angle β ($\angle O\bar{L}\hat{\hat{L}}$) it follows that $\hat{\hat{S}}_B-\bar{S}_B>\bar{L}_B-\hat{\hat{L}}_B$.[5] Q.E.D.

6. Summary and alternative proofs

We have given a diagramatic argument to show that with three agents, the receipt of a gift may make the recipient worse off, even in an economy whose equilibria are globally Walrasian stable. It may now be worth setting out briefly the economic intuition behind this. Consider the case of a transfer from H to S. Clearly, at the original prices, S's income rises. S is already importing good B, and also has at the initial situation a higher marginal propensity to consume B. The increase in the income leads her to demand still more B; as the supply of this is fixed, and H's consumption of B drops little, there is a rise in demand for B which turns relative prices against S, and tends to make S worse off.

[5]To see this, draw the line parallel to $\bar{S}\hat{\hat{S}}$ starting at point $\bar{L}$ and note that it intersects $O\hat{\hat{L}}$ to the left of $\hat{\hat{L}}_B$.

The question is whether the price change is big enough to offset the commodity transfer; this is where the third agent matters. He must be an exporter of good B, so that when the price of B increases his wealth increases and he demands more of everything, including B. Note the crucial role of Leontief indifference curves: because the third party can now strictly afford his previous consumption bundle he chooses to buy more B even though the price of B is higher — the income effect always dominates the price substitution effect (which is zero for Leontief preferences). The argument is completed by showing (with a diagram) that the rise in price of good B just necessary to reduce the recipient's demand to her previous level, or to allow the donor's demand to rise again to her previous level (again note the income effect) is not sufficient to clear the market if the third party's marginal propensity to consume B is higher than the donor's or recipient's respectively. Hence a still further increase in the relative price of good B is required to clear the market and that causes the paradox.

An example may make matters easier to grasp. Suppose that the United States, an exporter of food and importer of oil, makes a gift to Saudi Arabia, which exports oil and imports food, and has a higher marginal propensity to consume food. Further, let us suppose that China, which imports oil and exports food, has a still higher marginal propensity to consume food and comprises the rest of the world. Then we have proved that if each of the three countries can be represented by a single consumer with Leontief preferences in competitive equilibrium then the American gift will make Saudi Arabia worse off and the United States and China better off.

It is worth emphasizing that the crucial factors in our argument are the relative marginal propensities to consume, the *net* trade positions of the agents in the initial equilibrium, and the Leontief preferences. The absolute levels of wealth — questions of big country vs. small country — play no role whatsoever. Nor does the form of the transfer matter — as we said in the introduction, any exchange which increases the recipient's wealth at the original prices will produce the same qualitative effect. Finally it is possible to show that no matter what the initial endowments are, if they satisfy the simple boundary conditions (A.1)–(A.3) then for some pair of the agents any small transfer will have a paradoxical effect. Far from being accidental, the paradoxes are the norm when the preferences are Leontief.

Let us now make very clear the role of the third agent by examining geometrically the effect of a transfer in an economy with only two agents (and two goods) with arbitrary tastes. Fig. 7 shows the initial endowments H, S of the two agents in E, and their consumption bundles $\bar{H}$, $\bar{S}$ at the equilibrium $\bar{p}=P_A/P_B$ of E. Let H^δ, S^δ be their initial endowments, after the transfer, in the economy E'. Without loss of generality we can assume that S, the transfer recipient, was exporting good B in the original equilibrium $\bar{p}$. Let us assume that there is some price $\hat{p}$ at which S would be precisely as well off

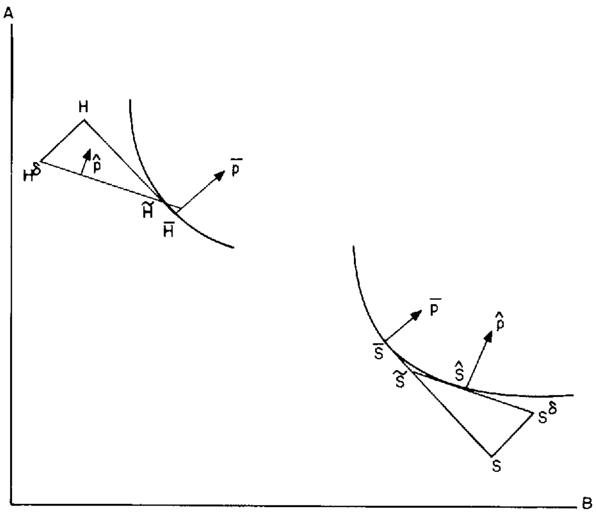

Fig. 7. Initial endowment vectors in the economy E: H, S. Initial endowment vectors in the economy E': H^δ, S^δ. Equilibrium price in E: $\bar{p}$. Consumption bundles in equilibrium $\bar{p}$ in E: $\bar{H}$, $\bar{S}$. Price vector in E': $\hat{p}$. Demand by S in E' at prices $\hat{p}$: $\hat{S}$. Points where $(E,\bar{p})$ budget sets intersect the corresponding $(E',\hat{p})$ budget set: $\tilde{H}$, $\tilde{S}$.

in E' as he was at price $\bar{p}$ in E. Evidently we must have $\hat{p} \geqq \bar{p}$. Let $\hat{S}$ be S's demand at prices $\hat{p}$ in E' and let $\hat{H}_B$ be H's demand in E' at prices p. If S's demand is continuous and the transfer was small, S must still be an exporter of good B, $\hat{S}_B < S^\delta_B$.[6]

Call $\tilde{S}$ the point where S's budget set in E' given by $\hat{p}$ intersects his old budget in E given by $\bar{p}$ and define the corresponding point $\tilde{H}$ where H's $(E',\hat{p})$ budget set intersects his old $(E,\bar{p})$ budget set. Assuming that the demand functions are single valued it follows from revealed preference that $\hat{S}_B \geqq \tilde{S}_B$ and $\hat{H}_B \geqq \tilde{H}_B$ where the inequalities are strict if $\hat{p} > \bar{p}$. Note now that the triangles $\Delta HH^\delta\tilde{H}$ and $\Delta SS^\delta\tilde{S}$ are congruent since all their angles are equal and the transfer sides are equal. Hence in particular $(\tilde{H}_B - H^\delta_B) + (\tilde{S}_B - S^\delta_B) = 0$. From this it follows that the aggregate excess demand $B'(\hat{p}) = (\hat{H}_B - H^\delta_B) + (\hat{S}_B - S^\delta_B) > 0$ if $\hat{p} > \bar{p}$. Assuming Walrasian stability implies that the E' equilibrium price $\bar{p}'$ must satisfy $\bar{p}' < \bar{p}$, which would imply that S could strictly afford $\hat{S}$, implying that he is better off in E' than E and therefore that there is no paradox.

Note that if the preferences were Leontief in fig. 7, then we would have $\hat{p} = \bar{p} = \bar{p}'$, that is the transfer would not affect the final allocation! The two

[6]We are also assuming that $\hat{p}$ moves continuously as a function of the transfer.

agent Leontief economy is as close to paradoxical as it is possible to come without violating stability. If the reader is puzzled by this, let him observe that the aggregate endowment is a point R^2 which can be decomposed in only one way into a sum of the two linearly independent vectors generating each agent's consumption expansion ray. Similarly in an economy with n goods, n Leontief consumers would make for an economy whose equilibrium is independent of the distribution of endowments. To obtain the paradox (using Leontief preferences) requires at least one more agent than goods. It is an open question whether there is in general a connection between the number of agents necessary for the transfer paradox to occur and the number of non-trivial goods in the economy. What is clearly necessary is that at least some of the agents display an income effect in their demands. For example, with utilities $u(x_1,\ldots, x_n, x_{n+1})$ given by $u^i(x)=v^i(x_1,\ldots,x_n) + x_{n+1}$ there is no income effect in the demand for the first n goods and in an economy made up entirely of such agents there can be no transfer paradox.

We conclude by giving an alternative proof of Theorems 1 and 2 based on the ideas we have just introduced.

Alternative proof of the transfer paradoxes. Let $w=(w_A, w_B)$ be the aggregate endowment of the economy E. Let the expansion rays of the agents H, L, S be denoted by $\{\alpha_H e_H: \alpha_H \geqq 0\}$, $\{\alpha_L e_L: \alpha_L \geqq 0\}$, $\{\alpha_S e_S: \alpha_S \geqq 0\}$ respectively. Note that any two of the vectors e_H, e_L, e_S are linearly independent; in particular, the middle vector e_L can be expressed uniquely as a sum

$$e_L = \beta_H e_H + \beta_S e_S, \tag{1}$$

where both β_H and β_S are strictly positive.

At the initial equilibrium $\bar{p}$ we can write the agent's consumption vectors $\bar{H}$, $\bar{L}$, $\bar{S}$ by $\bar{H}=\bar{\alpha}_H e_H$, $\bar{L}=\bar{\alpha}_L e_L$, $\bar{S}=\bar{\alpha}_S e_S$; note that

$$w=\bar{\alpha}_H e_H + \bar{\alpha}_L e_L + \bar{\alpha}_S e_S. \tag{2}$$

Observe that if $\hat{\alpha}_L > \bar{\alpha}_L$ and $w=\hat{\alpha}_H e_H + \hat{\alpha}_L e_L + \hat{\alpha}_S e_S$, then from (1) it follows that $\hat{\alpha}_H < \bar{\alpha}_H$ and $\hat{\alpha}_S < \bar{\alpha}_S$. Any rearrangement of the initial endowments which leaves the aggregate endowment w unchanged and makes L better off must make H and S worse off (and conversely).

To prove the weak transfer paradox (advantageous reallocation paradox) recall that H made a gift to S and that L was a net exporter of good B in the equilibrium $\bar{p}$. If after the transfer the same prices $\bar{p}$ were maintained it is clear, since S has a higher marginal propensity to consume B than H, that there would be an excess demand for B. The stability of E now guarantees that the new equilibrium $\bar{p}'$ will have a higher relative price of B. Hence L

must be better off and so from (1) H and S are worse off. Reversing the transfer makes both donor and recipient better off and L worse off. This proof, although it is very short, hides one very important fact which is plain in our original demonstration, that S must be an importer of good B in the original equilibrium $\bar{p}$ [in order for the economy to be stable, i.e., to satisfy (A.1)–(A.3)], as was shown in Lemma 2.

To prove the strong transfer paradox, recall that H made a gift to L and that S was a net exporter of good B. Again because L has a higher marginal propensity to consume good B than H, we must have that the new equilibrium prices $\bar{p}'$ involve a higher relative price of good B than $\bar{p}$, hence that S becomes better off. From (1) it follows that H must also be better off and L worse off. The donor has gained and the recipient lost as a result of the transfer. To conclude the proof we need to choose the endowments so that (S_A, S_B) lies below his consumption ray (so that in equilibrium he exports B) and so that the economy is stable i.e., so that (A.1)–(A.3) hold. The reader can check to see that is possible, but only if L's endowment lies above his consumption ray and H's endowment lies below his. Q.E.D.

These last proofs are extremely simply but they make the original net trade positions of the donor and recipient seem incidental details needed to guarantee stability. In fact, as our diagrammatic proof makes clear, the recipient becomes worse off only because the transfer induces the terms of trade to turn against her: she must therefore be a net importer of the good for which she has a higher marginal propensity to consume than the donor. Similarly, the donor becomes better off because the terms of trade in her favor: she must therefore be a net exporter of the good for which she has a lower marginal propensity to consume than the recipient.

References

Balasko, Y., 1978, The transfer problem and the theory of regular economies, International Economic Review 19, no. 3, 687–694.

Chichilnisky, G., 1980, Basic goods, the effects of aid and the international economic order, Journal of Development Economics 7, no. 4, 505–519.

Guesnerie, R. and J.-J. Laffont, 1978, Advantageous reallocations of initial endowments, Econometrica 46, no. 4, 835–842.

Leontief, W., 1939, Note on the pure theory of transfer, in: Explorations in economics, Taussig Festschrift Volume, 84–92.

Polemarchakis, H.M., 1982, On the transfer paradox, C.O.R.E. discussion paper, Université Catholique de Louvain, Louvain.

Samuelson, P.A., 1952, The transfer problem and transport costs: The terms of trade when impediments are absent, Economic Journal 62, no. 2, 278–304.