MONETARY EQUILIBRIUM WITH MISSING MARKETS

BY

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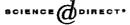
Monetary equilibrium with missing markets

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In Honour of Martin Shubik

Abstract

We add inside and outside money to the standard GEI model. If there enough gains to trade via money, then monetary equilibrium (ME) exists and money has positive value, even when GEI fails to exist. The nonexistence of GEI shows up as a liquidity trap in terms of the ME. In sharp contrast to GEI, the ME are generically determinate not only in terms of real, but also financial, variables.
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1. Introduction

1.1. Money in general equilibrium

In this paper, we introduce a two-period[1] general equilibrium model with uncertainty and rational expectations, in which money plays a central role. Compared to the standard general equilibrium model of Arrow–Debreu (GE), our model has four additional features: missing assets, in the sense that some imaginable contracts are not available for trade; missing market links, in the sense that not all pairs of instruments in the economy trade directly against each other; inside and outside fiat money; and a banking sector, through which agents can borrow and lend money. (Missing assets and missing market links can both be explained as a consequence of an underlying transactions cost, which we discuss in Section 17.)

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[1] Though our formal model has only two-periods, this is essentially for ease of notation. For an arbitrary finite number of periods, all our basic results remain intact. Major differences occur, however, in an infinite period setting (see Dubey and Geanakoplos, 2000).

We show that if there are enough missing market links relative to the ratio of outside to inside money, then monetary equilibrium (ME) exists and money has positive value. We thereby extend the one-period analysis in Dubey and Geanakoplos (1992, 2003). Our extension is significant because, in a multi-period general equilibrium with incomplete market (GEI) setting,[2] it is well known that incomplete assets may cause the breakdown of equilibrium. Putting outside and inside money into the GEI model overcomes the nonexistence problem. Our model can also serve as a framework in which to analyze fiscal and monetary policy, though we do not pursue that discussion here.

The nonexistence of GEI shows up in our model as a liquidity trap. If the government pumps in more and more bank money into an economy with no GEI, it succeeds only in increasing the stock of real money balances carried over by the agents, without appreciably reducing interest rates, increasing output, or even increasing commodity prices. Thus our model provides an explanation of the liquidity trap that is fully consistent with rational expectations.

Of course many others have sought to build a general equilibrium model with money.[3] Our approach is novel in combining all the following elements: heterogeneity in commodities and assets and agents, multiple periods, uncertainty, rational expectations, positive value of money in a finite horizon, real and nominal determinacy of equilibrium, non-neutrality of money, and the connection between gains to trade and the outside–inside money ratio.

An important and realistic feature of our model is that all exchange must be physically carried out between any two instruments. If an agent wants to buy a tomato with money, then he must turn over the money. If he wishes to buy the tomato with a credit card, he must turn over a slip of paper showing his promise to deliver money later. In exchange he obtains the tomato, but must pay out money in the future as promised. If he wishes to agree today to trade a tomato tomorrow against an orange the day after tomorrow, he must exchange a piece of paper today stating his promise against the corresponding piece of paper containing the other side's promise. Then he must deliver the tomato when promised.

This leads naturally to the idea that market actions form prices. The price of α in terms of β is simply the total amount of β chasing α at the market, giving rise to a strategic market game.[4] Indeed our ME existence proof is based on the Nash fixed point argument on the space of actions (not the price space, as in GE). Since we work with a continuum of agents, we recast the ME in terms of more familiar budget sets in which agents regard prices as fixed. But the fundamental aspect of a game, that every choice of agents' strategies engenders an outcome, is fully honored in our model.

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[2] GEI $\equiv$ general equilibrium with incomplete asset markets, and will refer to both the model as well as its equilibrium. (The meaning will be clear from the context.)

[3] To give an indicative, but by no means exhaustive, list: Bewley (1980, 1984), Dréze and Polemarchakis (1999, 2000, 2001), Grandmont (1983), Grandmont and Laroque (1975), Grandmont and Younes (1972, 1973), Hayashi (1974), Levine (1989), Lucas (1980), Magill and Quinzii (1992), Ostroy and Starr (1974), Shubik and Wilson (1977), and Woodford (1990).

[4] Strategic market games were introduced by Shapley and Shubik in 1973 in an Arrow–Debreu complete markets framework. They examined markets in which money traded against every other commodity. Later, Amir et al. (1990) considered markets in which all commodities traded directly against each other. We have extended that approach, allowing, for example, assets to trade directly against commodities, as in a credit card purchase.

GEI is a special case of our ME and obtains precisely when there are no private endowments of money, i.e. no outside money. In this situation all interest rates on bank loans at any ME are also zero, and commodity prices are indeterminate. Indeed, when the assets promise money (unindexed to commodity prices), the indeterminacy also pertains to real allocations (see Balasko and Cass, 1989; Geanakoplos and Mas-Colell, 1989).

When there are zero private endowments of money, our ME (since they coincide with GEI) inherit the nonexistence and indeterminacy problems of GEI. But when these endowments are positive, sharp contrasts occur. Every economy has at least one ME. Moreover, as we show in a companion paper (Dubey and Geanakoplos, 1994), ME are generically determinate, not only in terms of real but also financial variables, such as the level of prices and interest rates. To put it dramatically, the moment we introduce a "dime" of private outside money into the economy, both the nonexistence and the indeterminacy problems disappear.[5]

2. The model

2.1. The economy

The set of states of nature is $S^* = \{0, 1, \ldots, S\}$. State 0 occurs in period 0, and then nature moves and selects one of the states in[6] $S = \{1, \ldots, S\}$ which occur in period 1.

The set of commodities is $L = \{1, \ldots, L\}$. Thus, the commodity space may be viewed as $\mathbb{R}_+^{S^* \times L}$ whose axes are indexed by $\{0, 1, \ldots, S\} \times \{1, \ldots, L\}$. The pair $s\ell$ denotes commodity ℓ in state s. We view all commodities as perishable. Durable commodities are not thereby excluded, since we allow private production, including inventorying.

The set of agents is $H = \{1, \ldots, H\}$. Agent h has initial endowment $e^h \in \mathbb{R}_+^{S^* \times L}$ of commodities and utility function $u^h : \mathbb{R}_+^{S^* \times L} \to \mathbb{R}$. We assume that no agent has the null endowment of commodities in any state, i.e. for $s \in S^*$ and $h \in H$

$$e_s^h = (e_{s1}^h, \ldots, e_{sL}^h) \neq 0;$$

and, further,

$$\sum_{h \in H} (e_{s1}^h, \ldots, e_{sL}^h) \gg 0$$

i.e. every named commodity is actually present in the economy.

With incomplete markets, production is thought to be problematic. But the difficulty pertains only to jointly owned production, when the conflicting desires of different owners must be reconciled. (For our treatment of this issue, see Dubey and Geanakoplos, 1995.) In this paper, we sidestep the conflict simply by assuming that each production technology is owned by a single agent. (We note in passing that in the complete markets model of

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[5] Dréze and Polemarchakis (2001) show that introducing banks without outside money makes for indeterminacy of equilibria, even if the banks are privately owned.

[6] We use the notation $X = \{1, \ldots, X\}$. It will always be clear from the context whether X refers to the set or the element in the set.

Arrow–Debreu, in which production sets are convex, the hypothesis of exclusive ownership of private production can be made without loss of generality.) Production may involve input from many sources, for example the labor of many different individuals. What is important is that the control of each firm is not for sale. For ease of notation, we suppose then that each firm has a single owner who is not allowed to sell shares of the firm. To the extent that there are assets with payoffs that are highly correlated with the returns from his production, the firm owner can simulate the selling of shares by selling short these assets.

The incompleteness of asset markets potentially has an enormous effect on production choices. An owner may not want to choose a risky production plan unless he can protect himself by holding a particular kind of asset. If there is no such asset available, then he may choose a less adventurous production plan.

Formally, each agent h has a private production set $\Omega^h \subset \mathbb{R}_+^{S^* \times L} \times \mathbb{R}_+^{S^* \times L}$. For any $\omega \in \Omega^h$, with inputs and outputs $\omega \equiv (z, y) = (z_s, y_s)_{s \in S^*}$, the vector $z_0 \in \mathbb{R}_+^L$ gives the inputs (in period 0) and $\{y_s\}_{s \in S} \in \mathbb{R}_+^{S \times L}$ the corresponding state-dependent outputs (in period 1). We assume $z_s = 0$ for all $s \in S$ and $y_0 = 0$. (Thus, production takes time in our model.) We make the standard assumptions that if $z = 0$ then $y = 0$ (impossibility of free production); that $0 \in \Omega^h$ (possibility of no production); that Ω^h is convex; and that Ω^h admits free disposal (i.e. if $(z, y) \in \Omega^h$ and $\tilde{z} \geq z$, $\tilde{y} \leq y$, then $(\tilde{z}, \tilde{y}) \in \Omega^h$).

Durable goods like tobacco fit into our model as perishables which, if not consumed, can be put into production and emerge intact next period.

Let $\hat{B}$ be the maximum amount of any commodity $s\ell$ that can be produced in the economy with the endowments and production possibilities on hand; and let 1 denote the unit vector in $\mathbb{R}^{S^* \times L}$. Then we assume that each u^h is continuous, concave, strictly increasing in each variable. Without loss of generality,

$$\exists D^* > 0 \text{ such that } u^h(0, \ldots, 0, D^*, 0, \ldots, 0) > u^h(\hat{B}1), \tag{*}$$

for D^* in an arbitrary component.[7]

2.2. Assets

The set of assets is $J = \{1, \ldots, J\}$. They are traded in period 0, and call for deliveries in period 1. The seller of one unit of asset $j \in J$ promises to deliver a state contingent vector of commodities and money. Thus, we may view asset j as an $(L+1) \times S$ dimensional vector A^j, whose sth components $(A_{s1}^j, \ldots, A_{sL}^j, A_{sm}^j)$ specify the amount $A_{s\ell}^j$ of commodity $\ell \in L$, and the money A_{sm}^j, due in state $s \in S$. We assume that

$$A^j \neq 0, \qquad A^j \geq 0.$$

Agents have no endowments of assets. An asset sale is therefore a short sale. No limit is imposed on these sales. (In GEI, this unboundedness can destroy the existence of equilibrium.)

All asset deliveries must be made in money. When the asset promises include commodities, the seller is obliged to deliver the money equivalent, obtained by multiplying the

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[7] These assumptions can be relaxed, and are made for ease of presentation. See Dubey and Geanakoplos (2003).

quantities of promised commodities by their spot prices in the relevant state. But this is for ease of notation. We could have given the seller the option of delivering part of the promised commodities and the balance in money equivalent.

When the asset j promises delivery solely in money, unindexed to any commodity, i.e. $A_{s\ell}^j = 0$, $\forall \ell \in L$, $s \in S$, then we call j a nominal asset. The most important nominal asset is the so-called riskless asset: $A_{sm}^j = 1$, $\forall s \in S$. The buyer of this asset in effect is loaning, and the seller is borrowing, against a promise to deliver one dollar for sure in the future.

If asset j promises delivery only in commodities, then we call it *real*. If the deliveries are both in commodities and money, we call it *mixed*.

2.3. Outside money

Our model is designed to capture the multiple facets of money. We will suppose that money is the stipulated medium of exchange. All commodities and assets can be traded for money, and (as we have noted) all assets deliver exclusively in money.

Money is fiat; unlike commodities it gives utility to no agent. Also unlike commodities, it cannot be privately produced. It is perfectly durable. Its value resides in the fact that it can be used for transactions, and as a store of value (by carrying it forward for future use).

Money enters the economy in two ways. It may be present in the private endowments of agents. Let

$$m_s^h \equiv \text{private endowment of money for } h \text{ in state } s \in S^*$$

We can interpret m_s^h as a government transfer to agent h or as h's private inheritance from the (unmodeled) past. The vector $(m_s^h)_{s\in S}^{h\in H}$, is called *outside money*, because it enters the system free and clear of any offsetting obligations.

2.4. Inside money

A crucial ingredient of our model is a government bank which stands ready to loan an exogenously specified quantity of money at interest rates that are endogenously determined at equilibrium. The money borrowed from the bank also enters the system.

Formally speaking, we may regard a bank loan as a purchase by the bank of a special kind of bank bond[8] from the borrower. For simplicity, we allow only two kinds of bank bonds. Short-term bank bonds promise US$ 1 at the end of the same state in which they are traded. Long-term bank bonds, which can only be traded in period 0, promise US$ 1 at the end of every state in period 1. Let $N \equiv \{\bar{0}, 0, 1, \ldots, S\}$ index the bank bonds, where $\bar{0}$ is for the long-term bank bond, and s for the short-term bank bond in state $s \in S^*$. Let p_{snm} denote the price of bank bond n, in terms of money, in state s. Then p_{snm} corresponds to interest rate r_n with $p_{snm} = 1/(1 + r_n)$, for all $n \in N$. Thus, an agent who borrows z dollars on the short-term loan in state s [or on the long-term loan in state 0] owes $(1 + r_s)z$ [or $(1 + r_{\bar{0}})z$] dollars at the end of state s [at the end of every $t \in S$].

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[8] We call them bank bonds because the bank trades them. In our first model the bank only buys them. Later it will also sell them.

We suppose the bank puts up a fixed stock of (inside) money on each bank bond. Let

$M_n \equiv$ bank money put up for bank bond $n \in N$.

The quantity M_n is effectively the money loaned against the promise denoted by bank bond n. The vector $(M_n)_{n \in N}$ is called *inside money*, because it enters the system accompanied by an offsetting obligation, signalling its eventual departure.

Agents are permitted to buy bank bonds, as well as sell them, in which case they earn the same return on their money as the bank is getting.

2.5. Markets

Let $I \equiv L \cup \{m\} \cup J \cup N$. Thus, I is the set of all instruments (commodities, money, assets and bank loans) in the economy.

A market always involves a bilateral exchange between a pair of instruments at a particular point of time. Many markets are missing in any modern day economy. For example, while commodities trade against money, we often find that they do not directly trade against each other. Money is unique in that it can be traded against everything and, indeed, it is precisely on account of this that it has value. But trading exclusively via money is not without its difficulties: there may not be enough money to support desired levels of simultaneous trade. When this happens the demand for money must be rationed through high interest rates.

One real world institution which has emerged to ameliorate these cash constraints is the credit card. A credit card purchase of a commodity is in effect the exchange of a promise to deliver money in the future for the commodity today. We incorporate credit cards into our model by postulating that some assets can be traded directly against commodities. By trading an asset against money we also allow for credit card withdrawals of money. Furthermore, we put no limit on the quantity of credit card purchases an agent can make. (The agent will, however, be obliged to keep all his promises, and that obligation will limit his promises to what can be obtained from his future revenue.)

A *market* is denoted by a triple $s\alpha\beta$, where $s \in S^*, \alpha\beta \in I \times I$ and $\alpha \neq \beta$. We shall always identify $s\alpha\beta$ and $s\beta\alpha$. This symbol represents a market in state s in which α and β can directly trade with each other. Let $\mathcal{M}$ be the collection of all markets in the economy.

2.6. Market timing

In order to facilitate comparison of our model with the canonical GEI model, we have restricted ourselves to a two-period setting. Since agents must put up money for purchases of some assets and commodities, we need to introduce an earlier moment in time when they can borrow money from the bank and a later moment to repay. Thus, we subdivide each period s into three stages:

Stage 1: Bank bonds are traded, and production from previous period materializes.
Stage 2: Commodity and asset markets meet and old assets deliver.
Stage 3: Bank bonds are repaid, and consumption-investment occurs.

2.7. Assumptions on $\mathcal{M}$

Clearly asset markets meet only in period 0, i.e.

$$s\alpha\beta \in \mathcal{M} \quad \text{and} \quad \alpha \in J \Rightarrow s = 0.$$

Bank loans can only be taken out in states as designated:

$$snm \in \mathcal{M}, n \in N \Leftrightarrow n = s; \quad \text{or} \quad n = \bar{0} \text{ and } s = 0.$$

For simplicity, and perhaps not unrealistically, we also assume that each available instrument trades against money:

$$s\ell m \in \mathcal{M}, \quad \text{for all } s \in S^* \text{ and } \ell \in L,$$
$$0jm \in \mathcal{M}, \quad \text{for all } j \in J.$$

We finally make the assumption that commodities do not trade directly against each other in period 1:

$$s \in S, \ell \in L, k \in L \Rightarrow s\ell k \notin \mathcal{M}$$

This last assumption may be dispensed with (see Section 11.1).

3. Market actions

Next consider the market actions of an agent h. It will be convenient to think of this as a vector $q^h \in \mathbb{R}_+^{S^* \times I \times I}$, where

$$q^h_{s\alpha\beta} = \begin{cases} \text{quantity of } \alpha \text{ sent by } h \text{ (in state } s\text{) to trade against } \beta, & \text{if } s\alpha\beta \in \mathcal{M} \\ 0, & \text{if } s\alpha\beta \notin \mathcal{M} \end{cases}$$

As was already emphasized in the introduction, all transactions have a physical interpretation in our model, e.g. goods are traded for money and vice versa. Thus, the money receipts from a sale cannot be used for purchases at any other market that meets contemporaneously. On the other hand, our model permits each agent to take out bank loans at the beginning of every state (before trade takes place). The agent can use the borrowed money for purchases, and repay the resulting loan out of the receipts from his sales. Nevertheless, the interest rate on the loan may be so high that the agent is indeed liquidity constrained.

The government may also act in the same way. We denote by $Q_{s\alpha\beta}$ the quantity of α sent to trade against β in state s by the government. When $\alpha = m$ and $\beta \in N$, we have already specified $Q_{s\alpha\beta}$ by M_β.

In addition, the government can in principle also intervene with positive $Q_{s\alpha\beta}$ on other markets. When it sells assets, we suppose that it fully honors the corresponding deliveries. If for some $\ell \in L$, the government sets $Q_{sm\ell}$ or $Q_{s\ell m}$ positive, then it purchases or sells commodity ℓ. (Think of the purchase of labor for public projects, or the sale of grain out of government stocks.) Thus, our model permits a mix of monetary and fiscal policies, though we do not pursue that discussion here. (See, however, Dubey and Geanakoplos, 1996.)

4. Prices

We define $0 < p_{s\alpha\beta} < \infty$, where

$$p_{s\alpha\beta} = \begin{cases} \text{price of } \alpha \text{ in terms of } \beta, & \text{if the market } s\alpha\beta \in \mathcal{M} \\ 1, & \text{if } s\alpha\beta \notin \mathcal{M} \end{cases}$$

and naturally require

$$p_{s\alpha\beta} = (p_{s\beta\alpha})^{-1}.$$

Market actions determine prices:

$$p_{s\alpha\beta} = \frac{Q_{s\beta\alpha} + \sum_{h \in H} q_{s\beta\alpha}^h}{Q_{s\alpha\beta} + \sum_{h \in H} q_{s\alpha\beta}^h} = (p_{s\beta\alpha})^{-1}.$$

5. The budget set

The agents regard the prices $p \in \mathbb{R}_{++}^{S^* \times I \times I}$ as fixed. Given p, the *choice set* available to agent h is denoted by Σ_p^h.

Denote the choices of agent h by $\sigma^h \equiv (q^h, x^h, \omega^h)$ where $x^h \in \mathbb{R}_+^{S^* \times L}$ is his consumption, $\omega^h = (z^h, y^h) \in \Omega^h$ is his private production, and q^h is the vector of all his market actions as discussed.

It will be convenient, in presenting the budget set, to use the following notation: $q_{s\alpha C}^h, q_{sC\alpha}^h$ are the vectors with components $q_{s\alpha\beta}^h, q_{s\beta\alpha}^h$ for $\beta \in C \subset I$; $p_{s\alpha C}$ and $p_{sC\alpha}$ are interpreted similarly. We will use 1 for the vector with all components 1 (whose dimension will be clear from the context), and · for dot product. Also $A_{s\tilde{L}}^j$ is the vector $(A_{s1}^j, \ldots, A_{sL}^j, A_{sm}^j)$ (i.e. we have set $\tilde{L} \equiv L \cup \{m\}$). When $s = 0$, the components of $A_{0\tilde{L}}^j$ are understood to be zero.

The constraints on $\sigma^h = (q^h, x^h, \omega^h)$, given fixed prices p, are as follows. (Here, $\Delta(\gamma)$ denotes the difference between the right-hand side and left-hand side of inequality (γ) and s' denotes the predecessor state of s, so that $s' = 0$ if $s \in S$. For $s = 0$, $s' \equiv 0'$ does not exist and we take all quantities involving $0'$ to be zero.)

First, we require $\omega^h \equiv (z^h, y^h) \in \Omega^h$. Furthermore, in state $s \in S^*$:

Stage 1

(s(i)): Buy and sell bank bonds, with expenditures ≤ money on hand:

$$1 \cdot q_{smN}^h \le m_s^h + \tilde{m}_{s'}^h.$$

(Here, $\tilde{m}_{s'}^h$ is nonnegative and represents the money at the end of state s' and carried into state s from the past. It is not an action variable and is determined residually.)

Stage 2

(s(ii))$_m$: Money spent on purchases and deliveries ≤ money left in (s(i)) + money borrowed:

$$1 \cdot q^h_{sm(I \setminus N)} + \sum_{j \in J} (p_{s\tilde{L}m} \cdot A^j_{s\tilde{L}})(1 \cdot q^h_{s'jl}) \leq \Delta(s(i)) + p_{sNm} \cdot q^h_{sNm}$$

$(s(ii))_{\ell \in L}$: Commodity ℓ sold $\leq \ell$ endowed $+\ell$ produced:

$$1 \cdot q^h_{s\ell I} \leq e^h_{s\ell} + y^h_{s\ell}$$

Stage 3

$(s(iii))$: Money inventoried into next state = money leftover in $(s(ii))_m$ + money obtained from sales, asset deliveries and bank deposits − money returned on bank loans:

$$\tilde{m}^h_s \equiv \Delta(s(ii))_m + p_{s(I \setminus N)m} \cdot q^h_{s(I \setminus N)m} + \sum_{j \in J} (p_{s\tilde{L}m} \cdot A^j_{s\tilde{L}})(p_{s'Ij} \cdot q^h_{s'Ij})$$

$$+ p_{sms} q^h_{sms} + p_{s'm\bar{s}'} q^h_{s'm\bar{s}'} - q^h_{ssm} - q_{s'\bar{s}'m} \geq 0.$$

$(s(iv))_{\ell \in L}$: ℓ consumed and used for production inputs $\leq \ell$ leftover in $(s(ii))_\ell + \ell$ purchased:

$$x^h_{s\ell} + z^h_{s\ell} \leq \Delta(s(ii))_\ell + p_{sI\ell} \cdot q^h_{sI\ell}.$$

First note that the set Σ^h_p of all σ^h that satisfy the above constraints is clearly convex, for any fixed $p \gg 0$; and so is its projection B^h_p onto the consumption components x^h. The set B^h_p is *the budget set* of trader h, given prices p.

Suppose $(q^h, x^h, \omega^h) \in \Sigma^h_p(e^h, m^h)$, where the latter makes explicit the dependence of the choice set on e^h, m^h. Then, for any $0 < \lambda < 1$, we have $(\lambda q^h, \lambda x^h, \lambda \omega^h) \in \Sigma^h_p(\lambda e^h, m^h) \subset \Sigma^h_p(e^h, m^h)$. We call this the *scaling property* of Σ^h_p.

5.1. Netting bank loans

We allow the agent to pay the net that he owes or is owed. (See $(s(iii))$.) By contrast, we could have required agents to repay their loans at the end of any state out of their money on hand *before* receiving returns on their deposits. But nowadays on Wall Street, "netting" is commonplace. If an agent buys and sells the same asset (perhaps at different times), he is deemed afterwards to have traded just the difference. In our budget set, netting is done on the short loan in state 0, and across *both* the short and long-term loans in every state $s \in S$.

6. Netting asset deliveries

Netting is socially important. It economizes on the aggregate amount of money that is necessary, since only the net needs to be delivered. Without netting, nominal asset sales would necessarily be bounded, since there is a finite stock of money in the economy. But netting permits arbitrarily large sales of different assets, so long as their net delivery is

bounded. We shall nevertheless show that so long as netted assets trade only against money or commodities, the existence of ME carries through.[9]

We could imagine a future innovation on Wall Street in which there is one clearing house for the deliveries on all assets. In the presentation of the budget set so far we maintained the hypothesis that the agents must be able to physically deliver the promised money (see $(s(ii))_m$) on each asset prior to receiving any deliveries. But we will incorporate netting on asset deliveries and show how the budget set needs to be modified.

Consider a pool of assets $\tilde{J} \subset J$. Imagine a central clearing house which keeps track, for each agent, of how much money he owes or is owed on $\tilde{J}$. In short, agent h is called upon to pay the *net*:

$$\sum_{j \in \tilde{J}} (p_{s\tilde{L}m} \cdot A^j_{s\tilde{L}})(1 \cdot q^h_{0j\tilde{L}} - p_{0\tilde{L}j} \cdot q^h_{0\tilde{L}j}) \equiv N^h_s(p, q^h, \tilde{J})$$

in each state $s \in S$ on the pool $\tilde{J}$. We use the subscript $\tilde{L}$ on the market actions q^h_j, rather that I, because we assume:

assets $j \in \tilde{J}$, whose deliveries are netted, trade only against money and commodities.

If this net is positive, h must pay the clearing house; if it is negative he receives money from the clearing house. After making its collections and disbursals, the clearing house itself nets to zero.

To define the budget-set under netting we need to choose a stage in each state $s \in S$ in the budget set dedicated to the settlement of deliveries on $\tilde{J}$. To preserve the interpretation of a separate pool of assets, the settlement on $\tilde{J}$ must not be aggregated with that of another disjoint pool $\hat{J}$ of assets. Were we to do so and write: "$N^h_s(p, q^h, \tilde{J}) + N^h_s(p, q, \hat{J}) \leq$ money on hand," this would in effect create a new pool $\tilde{J} \cup \hat{J}$. More importantly the settlement on $\tilde{J}$ must not be combined with liquidity constraints on markets, for this would have the effect of pooling across those markets. To sum up, we can partition J into pools, and a priori assign distinct stages in the budget set tree for their deliveries to be settled.

For concreteness, we shall give the proof for the case when all the netted assets in $\tilde{J}$ form one pool, and their deliveries are made, after commodity trades, in each $s \in S$. Thus, in the budget set, we add a stage $s(i)_+$ after $s(i)$:

$(s(i)_+)$: $N^h_s(p, q^h, \tilde{J}) \leq \Delta(s(i)) + \rho_{sNm} \cdot q^h_{sNm}$

and in $s(ii)_m$ we replace $s(i)$ with $s(i)_+$, and on its right-hand-side we delete money borrowed and we replace $\sum_{j \in J}$ with $\sum_{j \in J \setminus \tilde{J}}$.

7. Monetary equilibrium

The monetary economy $\mathcal{E}$ is described by

$$\mathcal{E} = ((u^h, e^h, \Omega^h, m^h)_{h \in H}, (A^j)_{j \in J}, \mathcal{M}, Q, \tilde{J}).$$

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[9] Second, in case there is the possibility of default (which we have explicitly ruled out), netting reduces its likelihood. If h_1 sells an asset j to h_2, and h_2 simultaneously sells asset j to h_3, then h_2 might collect from h_1 and choose not to deliver to h_3. If there were netting, then h_2 would be netted out, and h_1 would effectively owe the debt to h_3, eliminating h_2's default.

We say that $(p, (\sigma^h)_{h \in H})$, where $\sigma^h = (q^h, x^h, \omega^h)$, is a monetary equilibrium of $\mathcal{E}$ (and denote it ME) iff:

For all $s\alpha\beta \in \mathcal{M}$,

$$p_{s\alpha\beta} \left(Q_{s\alpha\beta} + \sum_{h \in H} q^h_{s\alpha\beta} \right) = Q_{s\beta\alpha} + \sum_{h \in H} q^h_{s\beta\alpha} \tag{1}$$

For all $h \in H$,

$$\sigma^h \in \Sigma^h_p \tag{2}$$

$$\hat{\sigma}^h = (\hat{q}^h, \hat{x}^h, \hat{\omega}^h) \in \Sigma^h_p \Rightarrow u^h(\hat{x}^h) \leq u^h(x^h). \tag{3}$$

Condition (1) says that all markets clear, and (2) and (3) say that all agents optimize in their budget sets.

Recall that, by assumption, p is strictly between 0 and ∞ in each component. Thus, our definition of monetary equilibrium stipulates that money has positive value.

8. Intratemporal gains to trade

Since money is fiat, it can only have value if it is actually used in trade. We shall therefore assume any allocation achievable without money must be far from Pareto efficient.

Debreu (1951) introduced the coefficient of resource utilization to measure how far a given allocation is from Pareto-optimal. His measure identifies the fraction of the aggregate resources that can be given up while leaving behind enough to distribute so as to maintain the same utility levels as before. In Dubey and Geanakoplos (1992, 2003), we proposed an alternative measure of the gains to trade. The idea was not to tax, as in Debreu, the *aggregate* resources, but instead to consider the maximum tax on *traded* resources that would still leave room for Pareto improvement. We extend that one period definition to the multistate setting of our present model.

Let $x^h \in \mathbb{R}^{S^* \times L}_+$ for each $h \in H$. For any $\gamma \geq 0$, we will say that $(x^1, \ldots, x^H) \in (\mathbb{R}^{S^* \times L}_+)^H$ is *not γ-Pareto-optimal in state s* if $\exists$ trades $\tau^1_s, \ldots, \tau^H_s$ in $\mathbb{R}^L$ (in state s) such that

$$\sum_{h \in H} \tau^h_s = 0 \tag{4}$$

$$x^h_s + \tau^h_s \in \mathbb{R}^L_+, \quad \text{for all } h \in H \tag{5}$$

$$u^h(\bar{x}^h(\gamma, \tau^h_s)) > u^h(x^h), \quad \text{for all } h \in H \tag{6}$$

where

$$\bar{x}^h(\gamma, \tau^h)_{t\ell} = \begin{cases} x^h_{t\ell}, & \text{if } t \in S^* \backslash \{s\} \\ x^h_{s\ell} + \min\{\tau^h_{s\ell}, \tau^h_{s\ell}/(1+\gamma)\}, & \text{for } \ell \in L \text{ and } t = s \end{cases}$$

Note that when $\gamma > 0$, $\bar{x}^h(\gamma, \tau^h_s)_{s\ell} < x^h_{s\ell} + \tau^h_{s\ell}$, if $\tau^h_{s\ell} > 0$, and $\bar{x}^h(\gamma, \tau^h_s)_{s\ell} = x^h_{s\ell} + \tau^h_{s\ell}$, if $\tau^h_{s\ell} \leq 0$.

Thus, the trades contemplated to "γ-Pareto-improve" involve a tax of $\gamma/(1+\gamma)$ on trade.
If, at the allocation $(x^1, \ldots, x^H)$, we can find $p_s \in \mathbb{R}_+^L$ such that: $p_s \cdot \tau_s^h \leq 0$ implies $u^h(\bar{x}^h(\gamma, \tau_s^h)) \leq u^h(x^h)$, for all $h \in H$, then $(x^1, \ldots, x^H)$ is γ-Pareto-optimal in state s.
Note finally that 0-Pareto-optimal coincides with the standard notion of Pareto-optimal.

Definition. The gains to trade $\gamma_s(x)$ in state $s \in S$ at a point $x \in \mathbb{R}_+^{(S^* \times L)H}$ is defined as the supremum of all γ for which x is not γ-Pareto-optimal in state s.

9. Intratemporal outside–inside money ratio

The moment we enter any state $s \in S$ in period 1, the stock of outside money (owned free and clear without any offsetting obligations) is equal to the fresh endowment of money in state s plus the money inventoried from period 0, less what is already owed on the long loan to the bank. We shall see that this stock is never more than

$$\hat{m}_s \equiv \sum_{h \in H} m_0^h + \sum_{h \in H} m_s^h - \min_{t \in S} \frac{\sum_{h \in H} m_0^h + \sum_{h \in H} m_t^h}{M_{\bar{0}} + M_t} M_{\bar{0}}$$

The stock of inside money injected in state s is M_s.
The maximal ratio of outside money to inside money in state $s \in S$ is therefore given by

$$\mu_s(m, M) = \frac{\hat{m}_s}{M_s}.$$

10. Gains to Trade Hypothesis

For any state $s \in S$, define the set X_s of allocations that involve no trade in state s:

$$X_s = \left\{ (x^1, \ldots, x^H) \in \mathbb{R}_+^{(S^* \times L)H} : \exists \omega^h = (z^h, y^h) \in \Omega^h, \ \forall h \in H; \right.$$

$$\sum_{h \in H} x_0^h + \sum_{h \in H} z_0^h = \sum_{h \in H} e_0^h, \sum_{h \in H} x_t^h = \sum_{h \in H} e_t^h + \sum_{h \in H} y_t^h \text{ for all } t \in S, \text{ and}$$

$$\left. x_s^h = e_s^h + y_s^h \text{ for all } h \in H \right\}.$$

Thus, if $\Omega^h = \{0\}$, for all h, then,

$$(x^1, \ldots, x^H) \in X_s \Rightarrow x_s^h = e_s^h, \quad \text{for all } h \in H.$$

We are ready to state the assumption that there are enough gains to trade at each point of X_s, for all $s \in S$.

Gains to Trade Hypothesis. For all $s \in S$ and every $x \in X_s$, $\gamma_s(x) > \mu_s(m, M)$.

This hypothesis requires that there be gains to trade in every state $s \in S$ in period 1. (The hypothesis is not necessary for $s = 0$.) It also rules out the case of only one commodity per state, i.e. it implies $\#L > 1$. Observe that if no allocation in X_s is Pareto-optimal in state s, i.e. the endowment (enhanced by private production) is not Pareto-optimal in state s, then as $M_s \to \infty$, leaving the economy otherwise fixed, the Gains to Trade Hypothesis is automatically satisfied.

11. The existence of equilibrium

We are now ready to write our main theorem. The theorem shows that if the potential gains to trade are larger than the maximal outside–inside money ratio, then the economy will find a way to use money to exploit some of these gains. And that will inevitably give money positive value.

Theorem 1. *Consider a monetary economy which satisfies the Gains to Trade Hypothesis. Suppose that government actions consist solely of putting up bank money, with $M_n > 0$, for all $n \in N$. Suppose $\sum_{h \in H} m_0^h > 0$. Then a monetary equilibrium exists.*

Corollary. *Existence of ME also holds in the above model if the long loan is missing, i.e. $0\bar{0}m \notin \mathcal{M}$. (In this case, we interpret $M_{\bar{0}}$ to be zero in the Gains to Trade Hypothesis.)*

This theorem comes as a bit of a surprise for several reasons. First, as we mentioned earlier, money is fiat, the time horizon is finite, and agents own positive endowments of the money free and clear, with no balancing debts. The proof shows that agents will voluntarily borrow money from the bank, driving up interest rates precisely to the point that they owe (in the aggregate) not only what they borrowed, but also all of their private endowments of money. The backward induction paradox (Hahn, 1965) is resolved because in the last trading periods agents will indeed accept money in exchange for goods in order to pay back their bank loans. (See also Dubey and Geanakoplos (1992, 2003) for a proof that money has positive value in a one-period model.)

We distinguish our model from the Lerner model. In Lerner's model (1947) money has positive value because it is assumed that the stock of private fiat money is equal to the total of (exogenously specified) tax debts. (See, for example, Balasko and Shell (1983) for a formal version.) In contrast, in our model the private endowments of money correspond to "outside" money—they are accompanied by no offsetting debts. Moreover there is no a priori lock step between money endowments and taxes. (Indeed our model has no taxes, but they could easily be added in any quantity which does not exceed the private stock of money. Existence of equilibrium would remain unaffected. For our treatment of taxes in a one-period model, see Dubey and Geanakoplos (2003).)

The second reason why the universal existence of monetary equilibrium is surprising is that the potential sales and purchases of assets are unbounded, and hence the action space is not compact. In the model of general equilibrium with incomplete markets this allows for the nonexistence of equilibrium, as Hart (1975) has shown. The method of analysis for GEI suggested by Radner (1972) proceeds by postulating a priori individual bounds on the

amount of sales and purchases of assets. By this ad hoc compactification of the choice space, existence of GEI equilibrium is guaranteed. Hart's example demonstrated that no matter how far the bounds are relaxed, they still remain binding. At least one agent will go very long in some assets, and very short in others. Since the money payoffs of the assets are not necessarily linearly independent (recall they may depend on spot prices), the agents' net receipts (receipts minus deliveries) may still be bounded in every state, so no contradiction results.

Our method of proof proceeds in a similar fashion: we start by putting bounds $1/\varepsilon$ on asset trades. We also assume that an external agent puts up ε units of α on every market $s\alpha\beta$. Using a standard fixed point argument, we obtain an ε-ME. As $\varepsilon \to 0$, we show that prices $p_{s\alpha\beta}$ stay bounded, otherwise the Gains to Trade Hypothesis is violated. As importantly, we also find that the asset sales constraints are no longer binding. Hence, they can be dropped altogether. This holds no matter whether the assets are real or nominal or mixed.

Trades are naturally bounded for assets that do not have netting. This is so even though we allow such assets to trade against each other. The point is that agents who sold such assets would be called upon to obtain money for deliveries, and that would bring pressure on future interest rates, given that the stock of money in each state $s \in S$ is fixed, putting a brake on asset sales. (See the proofs for further details.)

Next, consider assets that are netted. Their purchases require money or commodities in advance. Though deliveries are netted, purchases and sales are not netted. If any asset price is bounded away from zero, then obviously the asset trades cannot grow large, because the stock of commodities $\sum_{h \in H} e_{0\ell}^h$ or money $M_0 + M_{\bar{0}} + \sum_{h \in H} m_0^h$ at period 0 is fixed. Real asset prices must indeed stay bounded away from zero, otherwise (as in the Pigou effect) the agents with private endowments of commodities or money will, "with one dime," be able to buy a huge amount of real goods via the asset, contradicting market clearing at the asset-constrained equilibrium. If an asset is nominal, and its price goes to zero, commodity prices in the states in which the asset delivers must all go to infinity. We shall show that this in turn contradicts the Gains to Trade Hypothesis.

11.1. The cashless economy

The set X_s contains all the allocations which could be achieved if money were valueless in state s, since we have assumed that commodities do not trade directly against each other and that assets deliver only in money. But if some commodities did trade directly, or if some assets directly delivered commodities, then the set X_s would have to be enhanced to reflect the extra activity of the cashless economy. Thus, with more market links, our existence theorem holds provided the Gains to Trade Hypothesis is maintained on a proportionately larger domain. Indeed if all commodities were directly linked, then the domain would be all-inclusive, and money would have no value, i.e. ME would not exist.

12. The demand for money, the term structure of interest rates, and the government budget

Money has been called the grease that turns the wheels of commerce. This can be seen in Theorem 1: when there is enough bank money, the outside–inside money ratios

are low, the Gains of Trade Hypothesis is easily satisfied, and consequently equilibrium exists.

Our model gives a fully general equilibrium approach to money. Money has two prices: the interest rate charged for borrowing it (or, more generally, the inverse of asset prices), and the inverse of the money prices of commodities. These prices are determined by the demand for money, which in turn arises from the interplay of many factors.

First there is a transactions demand for money: consumers need money to buy goods, and producers to buy inputs. In period 0 other motivations also enter the picture.

There is a precautionary demand for money. If interest rates r_s become very high in some future state $s \in S$, then agents will try to acquire money in advance in period 0. Either they will borrow more on $M_{\bar{0}}$, or else try to sell goods at period 0 for money. But this latter policy tends to reduce the price of goods at period 0. This in turn motivates agents to borrow further on $M_{\bar{0}}$ to buy goods in period 0, since these loans need not be repaid until after the sales of relatively expensive goods in period 1.

There is also a speculative demand for money. Inventorying money (obtained via bank loans or the sale of commodities and assets) from period 0 into 1, is tantamount to holding an implicit asset which competes in equilibrium with other assets. As the implicit asset becomes more attractive, the speculative demand for money rises.

Finally, there is an inflation demand for money $M_{\bar{0}}$. If prices in period 1 are very high, as they likely would be if all M_s were very high, then agents could borrow on $M_{\bar{0}}$, purchase goods at time 0, then sell expensive goods at time 1 to repay their loans. This would drive up $r_{\bar{0}}$.

Our model gives scope for the full interplay of all these factors and thus in principle it can encompass a number of monetary theories. Special assumptions would be needed to derive structural results. But even at our current level of generality, we notice a few interesting facts.

Theorem 2. *At any* ME (i) $r_s \geq 0$, $\forall s \in S^*$, (ii) $(1 + r_{\bar{0}}) \geq \min_{s \in S}(1 + r_0)(1 + r_s)$, *with strict inequality unless all* r_s, $s \in S$ *are the same,* (iii) $r_0 \leq \sum_{h \in H} m_0^h / M_0$ *and* $r_s \leq \mu_s(m, M)$, $\forall s \in S$, (iv) $M_0 r_0 + M_{\bar{0}} r_{\bar{0}} + M_s r_s = \sum_{h \in H}[m_0^h + m_s^h]$, $\forall s \in S$, *and* (v) *if* $\sum_{h \in H} \sum_{s \in S^*} m_s^h > 0$ *then* $r_{\bar{0}} > 0$.

The most significant of these conclusions is embodied in equality (iv). On its left, we have the interest revenue of the government, and on the right its fiscal expenditures (by way of gifts of m_s^h to agents). Thus, this equation asserts that the government is balancing its budget over the long run. Note, however, that it is the market forces that adjust interest rates to make this so. The government is not constrained[10] in its issue of M_s, m_s^h.

We also see that although there are $S+2$ interest rates, there are only 2 degrees of freedom. Still these two degrees of freedom are enough to leave the term structure of interest rates at period 0 endogenously determined in equilibrium and subject to the effect of policy.

In a companion paper (Dubey and Geanakoplos, 1994), we prove that equilibrium is generically determinate, so that the forces of supply and demand determine the term structure

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[10] This is to be contrasted with other models, such as Lerner (1947), in which taxes are mechanically matched to government expenditures, overlooking the fact that the Treasury can borrow from the Federal Reserve.

of interest rates, once the government commits to $(M_n)_{n\in N}$. We also show that money is not neutral: by changing the $(M_n)_{n\in N}$, the central bank can induce real effects in production and consumption.

If we had replaced our two-period model with a tree of date–events, with T terminal nodes, then the degrees of freedom would be the total number of loans minus T.

Inequalities (iii) confirm that the interest rates will never exceed the maximal outside–inside money ratios. Inequality (ii) is a no-arbitrage condition. Combining (ii) and (iv) yields (v).

13. Interest rates as bank policy

So far the central bank was committed to quantities of money $M_{\bar{0}}, M_0, M_1, \ldots, M_S$, leaving interest rates to form endogenously in equilibrium to clear the loan markets. We could reverse this scenario, and imagine that the bank fixes interest rates $r_{\bar{0}}, r_0, r_1, \ldots, r_S$, and stands ready to buy or sell bank bonds in order to clear the loan markets. In other words, it must choose either $Q_{smn} \equiv M_n > 0$ and $Q_{snm} = 0$, or $Q_{smn} = 0$ and $Q_{snm} > 0$, or $Q_{smn} = Q_{snm} = 0$, depending on whether $(1 + r_n) \sum_{h\in H} q^h_{smn} < \sum_{h\in H} q^h_{snm}$, or the reverse strict inequality holds, or equality holds.

Theorem 3. *Fix positive interest rates, $r_n > 0, \forall n \in N$, which satisfy the no-arbitrage condition* (ii) *of Theorem 2. Suppose $\gamma_s(x) > r_s$, for all $x \in X_s$ and $s \in S$. Then $\exists Q_{smn} \equiv M_n \geq 0, Q_{snm} \geq 0$ (all other $Q_{s\alpha\beta} = 0$) such that $\mathcal{E} = ((u^h, e^h, \Omega^h, m^h)_{h\in H}, (A^j)_{j\in J}, \mathcal{M}, Q, \bar{J})$ has an ME whose interest rates are equal to $(r_{\bar{0}}, r_0, r_1, \ldots, r_S)$.*

The equilibria of Theorems 1 and 3 overlap, but are not identical. If an ME of Theorem 1 has $r_s > 0, \forall s \in S^*$, then setting those interest rates as exogenous in Theorem 3, we recover the inside stocks $M_n = Q_{smn}, Q_{snm} = 0$, as endogenous. Conversely, if an ME of Theorem 3 corresponding to interest rates $(r_{\bar{0}}, r_0, r_1, \ldots, r_S)$ has $M_n = Q_{smn} > 0$ and $Q_{snm} = 0, \forall n \in N$, then setting this Q as exogenous in Theorem 1, we recover the interest rates.

But there might be an ME in Theorem 1 with $r_s = 0$ for some $s \in S^*$ which is not covered by Theorem 3. Similarly there might be ME in Theorem 3 with the banking sector selling some bank bonds (instead of buying them), i.e. $Q_{snm} > 0$ for some $n \in N$, which is not covered by Theorem 1.

14. ME versus GEI

ME always exist when $\sum_{h\in H} m^h_0 > 0$, if the Gains to Trade Hypothesis holds. Yet GEI do not. What precisely is their connection? We begin by recalling the formal definition of GEI for the underlying economy $E = ((u^h, e^h)_{h\in H}, A)$. For simplicity, all $\Omega^h = \{0\}$. Let $p \in \mathbb{R}^{S^* \times L}_{++}$ denote commodity prices, $\pi \in \mathbb{R}^J_{++}$ denote asset prices, and $\varphi \in \mathbb{R}^J$ denote asset trades. As usual, x refers to consumption.

Definition. $(p, \pi, (x^h, \varphi^h)_{h \in H})$ is a GEI for E if

(1) $\sum_{h \in H} x^h = \sum_{h \in H} e^h$,

(2) $\sum_{h \in H} \varphi^h = 0$,

(3) $(x^h, \varphi^h) \in B^h(p, \pi) \equiv \Big\{ (x, \varphi) \in R_+^{S^* \times L} \times R^J : p_0 \cdot (x_0 - e_0^h) + \pi \cdot \varphi \leq 0$ and

$p_s \cdot (x_s - e_s^h) \leq \sum_{j \in J} \Big[\sum_{\ell \in L} p_{s\ell} A_{s\ell}^j + A_{sm}^j \Big] \varphi_j, \quad \forall s \in S \Big\}$,

(4) $(x, \varphi) \in B^h(p, \pi) \Rightarrow u^h(x) \leq u^h(x^h)$.

Consider now the monetary economy $\mathcal{E}$ derived from the underlying GEI economy E by supposing that $\mathcal{M}$ consists of all money-commodity and money-asset markets, with all asset deliveries netted. Call $\mathcal{E}$ the *short-loan model* if the long loan market is missing, and *canonical* if it is not.

Compared to an ME, the GEI ignores all monetary phenomena. All trades, deliveries, etc. are processed by one giant clearing house. Receipts from sales at any market are available for simultaneous purchases at other markets.

Theorem 4. *Suppose $\sum_{h \in H} m_s^h = 0$, for all $s \in S^*$ (i.e. there is no private endowment of money). Then, in the short-loan model, the ME of $\mathcal{E}$ are the GEI of E. Similarly, if one of the assets j is the riskless nominal asset, then in the canonical model, ME are GEI. Finally, suppose in addition that all assets in J are real or nominal, but never mixed. Then in the short loan and canonical models, GEI are also ME in relative prices and final consumption.*

Our ME model thus includes GEI as a special (limiting) case.

Consider a fixed underlying economy E, and fixed m_s^h with $\sum_{h \in H} m_s^h > 0$, for all $s \in S$. Now, let $M_n \to \infty$, for all $n \in N$. What can be said about the limit? With the private money positive, some interest rates must be positive (by Theorem 2(iv)). With interest rates positive, all the bank money must be spent. (Why borrow at positive interest, if not to spend?) Thus, at least some prices must go to infinity.

We say that the sequence of ME $(p(n), (q(n), x(n))$ converges if $x(n) \to x$, and $p(n)/\|p(n)\|$ and $q(n)/\|q(n)\|$ converge (where $\|z\| = \sum_i |z_i|$). We would expect the normalized ME to converge to a GEI. In the numeraire asset case they do.

Corollary. *Suppose there is $\ell \in L$ such that all assets deliver exclusively in ℓ. Suppose that the vectors $(A_{s\ell}^j)_{s \in S}$, for $j \in J$, are linearly independent. Fix model $(E, \{m_s^h\})$ with $\sum_{h \in H} m_s^h > 0$, for all $s \in S$. Also assume that, for $s \in S$, no allocation in X_s is Pareto-optimal in state s. Consider the short-loan monetary economy built on E. Let $M_s \to \infty, \forall s \in S^*$. Take a sequence of ME, one for each vector $(M_s)_{s \in S^*}$. Then any convergent subsequence of the ME has a GEI of E as a limit.*

The Corollary to Theorem 4, together with our existence theorem, gives an alternative proof of the existence of GEI when asset payoffs are in a single numeraire good.

But what happens when there is no GEI?

15. Liquidity trap

The old-fashioned equilibrium theory, against which Keynes inveighed, held that monetary policy is ineffective, since commodity prices adjust to changes in money supply in such a way that real magnitudes are unaffected. Keynes, however, deemphasized the response of prices to changes in the stock of money. Accordingly, he concludes that the other price of money, namely the money rate of interest, would normally do the adjusting to changes in money supply. Keynes believed, therefore, that increases in the stock of money typically lower interest rates and thereby stimulate investment.

But Keynes acknowledged that there was an important possibility that monetary policy would affect neither the commodity price level, nor the interest rates. Consumers might simply hold increases in the stock of money in their portfolios as extra real money balances. He called such a situation the "liquidity trap". His explanation was that when the interest rates are sufficiently low, consumers expected them to go up. As a result they are loath to put any of their money into assets like bonds which suffer losses in value when interest rates rise. Real money balances absorb all the extra inside money. Needless to say, this explanation depends on the irrationality of investor expectations. (Bond prices should already reflect expectations of future interest rates.)

After Keynes a long series of authors commented on the irrationality of beliefs assumed by Keynes, and sought other ways to formalize the liquidity trap (see Grandmont and Laroque, 1976; Hool, 1976; Tobin, 1961, among others). None of these papers, however, is consistent with rational expectations either. Grandmont–Laroque and Hool, for example, explicitly work in a temporary equilibrium framework, in which future expectations are taken as exogenous. In contrast, our model rigorously adheres to rational expectations.

Consider again the thought experiment of Section 14, in which all the M_n go to infinity, with m_0^h fixed and $m_s^h = 0$, $\forall h \in H$ and $s \in S$. But this time suppose the underlying economy has no GEI. One might expect that increases in the stock of bank money would lead to proportional increases in prices, since (by Theorem 2(iv)) some interest rates must be positive and all the corresponding bank money must be spent. But this does not happen. Agents hold all the extra real money balances, and monetary policy is ineffective.

As the stock of bank money is increased, agents borrow and spend almost all of the extra money on buying assets. They defray these loans by selling other assets.[11] Since there is nearly no extra activity on the commodity markets, commodity prices remain relatively stable. At the individual level there is tremendous extra activity on the asset markets. But at the *aggregate* level there is almost no new *net* activity on the asset markets. From an aggregate point of view, nothing much happens when monetary stocks are increased except that larger real money balances are inventoried from period 0 to period 1.

The reason agents spend the extra borrowed money buying assets is that, in the situation posited in Theorem 5, assets are incomplete and their payoffs differ only slightly. By buying and selling nearly identical assets in large quantities, it is possible to create net payoffs which are very different from the original assets. If these synthetic payoffs are not directly available through some asset, and if these payoffs can be used to insure holders against some risk

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[11] Recall that all asset sales are, by definition, short sales; and there is no a priori limit on their magnitude.

that is important, then the agents will in fact rationally operate on bigger and bigger scales in the asset markets. This is the reason GEI fails to exist.

The precise scenario described in Theorem 5 is extreme and nongeneric. If asset payoffs were slightly different, then GEI would exist; and as $M_s \to \infty$, eventually the activity on asset markets would be arrested, and money would begin to flow back to commodity markets. From this "turning point" onward, price levels would start to increase at the same rate as M_s and real money balances would tend to a finite limit. But if the underlying GEI economy were close to a GEI economy which had no equilibrium, then this turning point would be reached for large M_s, so we would still get a robust liquidity trap, though not the *bottomless* trap of Theorem 5.

A liquidity trap is a sign of inefficiency. The synthetic asset, that is created by combining huge purchases and sales of nearly identical assets, costs buyers much more (after calculating interest borrowing costs) than sellers receive. Trade in this synthetic security is therefore inefficient, and the risk it represents is not as completely hedged as it could be if the synthetic asset payoffs were directly marketed.

Thus, our liquidity trap arises directly as a consequence of the incompleteness of assets. If nothing else, our liquidity trap provides an interesting interpretation of the breakdown of GEI. Recall that Hart (1975) constructed an example of an underlying economy with real assets ($A^j_{sm} = 0, \forall s \in S$) that has no GEI. (Naturally the assets could not be numeraire, otherwise we would contradict the Corollary to Theorem 4.) The same kind of counterexample could be created even if there was one riskless nominal asset.

Theorem 5.

(a) *Consider a short-loan model with real assets. Suppose the underlying economy has no GEI. Then as $M = (M_0, M_1, \ldots, M_S) \to \infty$, $M_0/\|p_{0Lm}\| \to \infty$, and asset trades $\to \infty$.*

(b) *Similarly, consider a canonical model. Suppose that the underlying economy, after adding a riskless financial asset to it, has no GEI. Then as $M' = (M_{\bar{0}}, M_0, M_1, \ldots, M_S) \to \infty$ in a relatively bounded manner,[12] $M_{\bar{0}}/\|p_{0Lm}\| \to \infty$ and asset trades $\to \infty$.*

16. Limited market access and intertemporal gains to trade: the case for zero short-term interest rates

Theorem 3 guarantees the existence of ME for arbitrary interest rates $r_n > 0$. One wonders if there are regimes in which all the short-term interest rates can be set to zero, without jeopardizing the existence of an ME. In this event, the long-term interest rate will have to mop up *all* of the outside money, and so the value of money would derive solely from the demand for the long loan.

We present a special scenario, which is meant to be suggestive rather than general. Suppose $\sum_{h \in H} m_0^h > 0$ and $\sum_{h \in H} m_s^h = 0$ for all $s \in S$. Suppose all assets deliver in a numeraire commodity (as in the Corollary to Theorem 4). Imagine that markets are not

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[12] Means the ratios of all components of M' remain bounded

universally open to all households: there is *limited market access*. (Our model could easily have accommodated this at the cost of more notation.) In particular, imagine there is a household (a poor unknown entrepreneur) who is extremely productive (infinite marginal productivity at zero levels of inputs) but has zero endowment of commodities and no access[13] to assets $j \in J$. He does, however, have access to the long bank loan. Assume $\gamma_s(x) > 0$, for all $x \in X_s$ and $s \in S$. Then an ME exists with $r_{\bar{0}} > 0$ set arbitrarily and all $r_s, s \in S^*$, set equal to zero. To see this, consider the model of Theorem 1 with exogenous M_n and endogenous r_n. Fix $M_{\bar{0}}$ to ensure that $r_{\bar{0}} M_{\bar{0}} = \sum_{h \in H} m_0^h$. Let $M_s = M$, for all $s \in S^*$ and let $M \to \infty$ with $M_{\bar{0}}$ fixed. Then for large enough M, ME exists, and all short-term interest rates $r_s, s \in S^*$, are zero. (For the proof, see Section 18.)

The point is that the value of money can sometimes be sustained by *intertemporal* gains to trade (embodied in our example by the entrepreneur). This is important, because when all the short interest rates are zero, the budget set becomes simpler, since the timing of events within the period no longer matters.

17. Transactions costs

We can generalize, and also motivate, the notions of missing assets, missing markets, and limited market access by introducing transactions costs. To each $s\alpha\beta \in S^* \times I \times I$ and agent $h \in H$ we associate a set-up cost $c_{s\alpha\beta}^h \geq 0$ representing the fixed utility cost to agent h of selling any amount $q_{s\alpha\beta}^h > 0$ of instrument α against β in state s. (Note that proportional costs can be subsumed by our production technology if we label goods by individuals.) We might think of these costs as broker's fees or search costs or bargaining costs.

If we take $c_{s\alpha\beta}^h > u^h(\hat{B}I)$, then agent h is effectively excluded from the market $s\alpha\beta$. Similarly, if this inequality holds for all $h \in H$, then effectively the market $s\alpha\beta \notin \mathcal{M}$. Finally, if the inequality holds for all $h \in H$ and all $\beta \in I$, for a given $\alpha = j$, then asset j is effectively missing.

The presence of fixed transactions costs complicates our analysis because it involves a crucial nonconvexity. For this reason, we only informally report on it here. For a fuller discussion, see Dubey and Geanakoplos (1996). There we show that if we replace our finite set of agents with a finite-type continuum of agents, then ME still exists with a positive value of money provided the costs $c_{sm\beta}^h$, $c_{s\beta m}^h$ are not too big, relative to $u^h(\hat{B}I)$ and to $c_{s\alpha\beta}^h$ for $\alpha \neq m$ and $\beta \neq m$. (Of course, the ME may not be type-symmetric: different agents of the same type could be taking different (but indifferent!) actions.)

The presence of transactions costs is important in and of itself, and not just for motivating missing markets. Indeed, at intermediate levels of costs $c_{s\alpha\beta}^h$, agent h will not be excluded from selling α against β, but only discouraged from selling it often in a short period of time. For example, if c_{sjm}^h is high, then (in order to arrange an even flow of income over time or across different states of nature) agent h would not want to buy a large quantity of asset j and sell it off piece by piece. He would prefer to find an asset that paid dividends, or failing to find that, he would prefer to sell off large chunks of the asset infrequently, inventorying

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[13] Typically, households with low wealth (collateral) and visibility cannot sell assets directly to the public, only established entities can.

the money so obtained in between sales to carry out day to day transactions (assuming $c^h_{sm\ell}$ low). This is precisely the motivation behind the Baumol (1952) and Tobin (1956) model of transactions demand for money.

18. Proofs

Proof of Theorem 1. It may help to describe the outline of the proof. For every $\varepsilon > 0$ we define an "ε-ME" and show that it exists. An ME is then obtained as a limit of ε-ME, $\varepsilon \to 0$. $\square$

An ε-ME may be thought of as a strategic equilibrium of a "generalized game" Γ_ε. First we replace each $h \in H$ by a continuum $(h - 1, h]$ of identical players, i.e. each t in the interval $(h - 1, h]$ has the characteristics

$$e^t \equiv e^h, \qquad u^t \equiv u^h.$$

All players in $(0, H]$ move according to the stages in the budget set, and at each stage their moves are simultaneous. At any stage they can only observe prices formed in the past. There is an "external agent" who puts up ε units for sale on each side of every market. Also, he fully delivers on his ε sale of assets. Note, however, that we do *not* quite have a classical game in extensive form, on account of the fact that no agent can default.[14] An ε-ME will correspond to a type-symmetric strategic equilibrium of the generalized game that we do have, i.e. one in which all players in $(h - 1, h]$ use the same strategy, for $h \in H$.

The external agent has the role of a "strategic dummy," i.e. he is optimizing nothing, and just behaves as described.

Now we begin the proof formally.[15] Let Σ be the ambient Euclidean space in which the choices of each agent h lie. Put $\Sigma(\varepsilon) = \{\sigma^h \in \Sigma : 0 \leq \sigma^h_i \leq 1/\varepsilon$ for every component $i\}$, and $\tilde{\Sigma}(\varepsilon) = $ the H-fold Cartesian product of $\Sigma(\varepsilon)$. Also denote $\bar{A} = \max\left\{\sum_{\ell \in \tilde{L}} A^j_{s\ell} : j \in J, s \in S\right\}$; $\bar{M} = \max\left\{M_0 + M_{\bar{0}} + M_s + \sum_{h \in H}(m^h_0 + m^h_s) : s \in S\right\}$; $|\mathcal{M}|, |J| = $ cardinality of these sets; $f(\varepsilon) = \varepsilon^2/2\bar{M}\bar{A}|\mathcal{M}|$.

Given $(\sigma^1, \ldots, \sigma^H) \equiv \sigma \in \tilde{\Sigma}(\varepsilon)$, define $p_{s\alpha\beta}(\varepsilon, \sigma)$ for $s\alpha\beta \in \mathcal{M}$ by

$$p_{s\alpha\beta}(\varepsilon, \sigma) = \frac{\varepsilon_{s\beta\alpha} + Q_{s\beta\alpha} + \sum_{h \in H} q^h_{s\beta\alpha}}{\varepsilon_{s\alpha\beta} + Q_{s\alpha\beta} + \sum_{h \in H} q^h_{s\alpha\beta}}$$

where

$$\varepsilon_{s\alpha\beta} = \begin{cases} f(\varepsilon), & \text{if } \alpha \in J \\ \varepsilon, & \text{otherwise} \end{cases}$$

For $s\alpha\beta \notin \mathcal{M}$, set $p_{s\alpha\beta}(\varepsilon, \sigma) = 1$.

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[14] This is not a serious matter. By allowing for default and adding default penalties, we would indeed end up with a proper market game. Then, taking the penalties to be sufficiently harsh, the strategic equilibria of the game coincide with the equilibria of our generalized game. (For our general treatment of default, see Dubey et al. (1999).)

[15] We remind the reader that, for simplicity of notation, we are taking the government's actions $Q_{s\alpha\beta} = 0$, except for their supplies of bank money $Q_{0m\bar{0}} \equiv M_{\bar{0}}$, $Q_{sms} \equiv M_s$ for $s \in S^*$.

In general, we need not have $\sigma^h \in \Sigma^h_{p(\varepsilon,\sigma)}$. If it turns out that $\langle \sigma, p(\varepsilon, \sigma) \rangle$ is "consistent," i.e. if $\sigma^h \in \Sigma^h_{p(\varepsilon,\sigma)}$, for all h, then we have a physically compatible system: the quantities of any commodity or money sent to market by any agent cannot exceed what he has on hand at the time; no agent defaults on asset deliveries or bank loans; and, by our formulae for $p_{s\alpha\beta}$ the market $s\alpha\beta$ also clears, i.e. sends out what it receives (taking the external agent into account).

Suppose throughout, from now on, $0 < \varepsilon < \varepsilon^* < \bar{M}/(|\mathcal{M}| + |J|)$ for some fixed $\varepsilon^* > 0$.

It is clear that if $\langle \sigma, p(\varepsilon, \sigma) \rangle$ is consistent, then the total amount of commodities in the system is bounded above (as the external agent only creates ε units of commodities in each market)[16]. The total amount of money is also bounded. To check this, observe that $p_{s\ell m} \le (\bar{M} + |\mathcal{M}|\varepsilon)/\varepsilon \le 2\bar{M}/\varepsilon$ for $\ell \in L$ (as the external agent creates ε units of money in at most $|\mathcal{M}|$ markets). Consequently $p_{s\ell m}A^j_{s\ell}$ is bounded above, and he never has to create and deliver more than $|\mathcal{M}|f(\varepsilon)\bar{A}2\bar{M}/\varepsilon \le \varepsilon$ units of money on his sale of $f(\varepsilon)$ units of any asset $j \in J$. Thus, the total money in the system is at most $\bar{M} + |\mathcal{M}|\varepsilon + |J|\varepsilon \le 2\bar{M}$.

Let B^* (or M^*) denote the smallest upper bound[17] on the total amount of any commodity (or money) in the system at any admissible $\langle \sigma, p(\varepsilon, \sigma) \rangle$, for $\varepsilon < \varepsilon^*$. Let $E^* = \max\{B^*, M^*\}$. Clearly $B^* > \hat{B}$, where $\hat{B}$ is as in Section 2.1.

To obtain an ε-ME we construct a point-to-set map ψ_ε on the compact, convex set $\tilde{\Sigma}(\varepsilon)$, as follows. First recall that $\sigma^h = (q^h, x^h, \omega^h)$ and let $x^h(\sigma^h)$ denote the projection of σ^h onto its second component. Set

$$\psi^h_\varepsilon(\sigma) = \arg\max{}_{\tilde{\sigma}^h \in \Sigma^h_{p(\varepsilon,\sigma)} \cap \Sigma(\varepsilon)} u^h(x^h(\tilde{\sigma}^h))$$

and

$$\psi_\varepsilon(\sigma) = \psi^1_\varepsilon(\sigma) \times \cdots \times \psi^H_\varepsilon(\sigma).$$

It can be checked that Σ^h_p is convex and is also upper and lower semi-continuous in p as long as $p \gg 0$; and that $p(\varepsilon, \sigma) \gg 0$ and is continuous in σ (for fixed $\varepsilon > 0$). Therefore, $\Sigma^h_{p(\varepsilon,\sigma)} \cap \Sigma(\varepsilon)$ continuous in σ. (This intersection is nonempty, e.g. it contains 0.) Clearly, $u^h(x^h(\tilde{\sigma}^h))$ is continuous and concave in $\tilde{\sigma}^h$. It follows from the maximum principle that ψ_ε satisfies all the conditions of Kakutani's fixed point theorem.

Choose a fixed point $\sigma(\varepsilon) = \{\sigma^h(\varepsilon)\}_{h \in H}$ with $\sigma^h(\varepsilon) = (q^h(\varepsilon), x^h(\varepsilon), \omega^h(\varepsilon))$ and denote the attendant prices $p(\varepsilon, \sigma(\varepsilon)) \equiv p^\varepsilon$. Note that $(\sigma(\varepsilon), p^\varepsilon)$ is clearly consistent.

Let $\varepsilon \to 0$, and choose a subsequence of ε such that each component of $\sigma(\varepsilon)$ and p^ε converges (possibly to infinity or zero), and also all possible ratios of prices converge (possibly to infinity or zero).

We will examine this subsequence of $(p^\varepsilon, \sigma(\varepsilon))$ in the steps below.

Let 1_i be the unit vector in $\mathbb{R}^{S^* \times L}_+$ which has 1 for the ith component and 0 elsewhere, and let $1 = \Sigma_i 1_i$. Let

$$u^h_* = u^h(B^* 1),$$

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[16] The money created by the external agent by way of asset deliveries comes too late to be used for market purchases by agents. They can use it only to repay bank loans.

[17] (We should write $B^*(\varepsilon)$, $E^*(\varepsilon)$, etc. to be exact, but the ε will be suppressed. This should cause no confusion.) Note that $B^* \to \hat{B}$ as $\varepsilon \to 0$.

$$\xi_-^h = \min\left\{ u^h(x + 1_i) - u^h(x) : x \le B^*1, \text{ and } i \text{ is a component of } \mathbb{R}_+^{S^* \times L} \right\}.$$

Clearly both are well-defined, and $\xi_-^h > 0$. Further, by the concavity of u^h, we have: $x \le B^*1, \Delta < 1 \Rightarrow u^h(x + \Delta e_i) - u^h(x) \ge \xi_-^h \Delta$.

Finally, if ε^* is chosen sufficiently small, then our assumption on utilities (see Section 2.1) implies that

$$u^h(0, \dots, D^*, 0, \dots, 0) > u_*^h,$$

where D^* could occur in any component. From now on the above inequality will be assumed.

Step 1. $\exists \underline{p} > 0$ such that $p_{s\ell m}^\varepsilon > \underline{p}$ for sufficiently small ε, all $\ell \in L$ and $s \in S^*$.

Proof. Suppose some $p_{s\ell m}^\varepsilon \to 0$. Take $h \in H$ with $m_0^h > 0$, and let h do nothing except spend m_0^h to purchase $m_0^h/p_{s\ell m}^\varepsilon \to \infty$ of $s\ell$ (by inventorying m_0^h into period 1, if $s \in S$). Then he can consume more than D^* of $s\ell$, a contradiction, since no agent could be getting more than u_*^h utiles at the ε-ME. □

Step 2. $\exists \bar{r}$ such that, for sufficiently small ε, $r_0^\varepsilon < \bar{r}$, and $r_s^\varepsilon < \bar{r}$, for all $s \in S^*$.

Step 3. $\exists B > 0$ such that $q_{0\bar{0}m}^h(\varepsilon), q_{ssm}^h(\varepsilon) < B$ for sufficiently small ε, all $h \in H$ and $s \in S^*$.

Proof of Steps 2 and 3. By (s(iii)) of the budget set conditions, no $q_{0\bar{0}m}^h$ or q_{ssm}^h (for $s \in S^*$) can exceed the money on hand. But the latter is at most E^*, proving Step 3. Step 2 is now evident from the formulae (remembering that $M_{\bar{0}} > 0$ and each $M_s > 0$):

$$1 + r_{\bar{0}}^\varepsilon = \frac{\varepsilon + \sum_{h \in H} q_{0\bar{0}m}^h(\varepsilon)}{\varepsilon + M_{\bar{0}} + \sum_{h \in H} q_{0m\bar{0}}^h(\varepsilon)}, \qquad 1 + r_s^\varepsilon = \frac{\varepsilon + \sum_{h \in H} q_{ssm}^h(\varepsilon)}{\varepsilon + M_s + \sum_{h \in H} q_{sms}^h(\varepsilon)}. \qquad \square$$

Step 4. For sufficiently small ε, $r_{\bar{0}}^\varepsilon \ge 0$ and $r_s^\varepsilon \ge 0$, for $s \in S^*$.

Proof. First take r_s^ε for $s \in S^*$. Clearly $1 + r_s^\varepsilon > 0$. If some $r_s^\varepsilon < 0$, then let h deviate from $\sigma^h(\varepsilon)$ as follows: increase $q_{ssm}^h(\varepsilon)$ by a small $\Delta > 0$ (which is feasible, if $(1/\varepsilon) > B$, by Step 3), obtain $\Delta/(1 + r_s^\varepsilon)$ more of M_s, spend $(\Delta/(1 + r_s^\varepsilon)) - \Delta > 0$ more to buy and consume an extra amount of any commodity he likes in state s, and return Δ more on M_s. This improves his payoff, a contradiction.

The same argument shows $r_{\bar{0}}^\varepsilon \ge 0$ (except that he must now inventory the Δ into period 1 to return on the long loan). □

Step 5. $\exists \bar{B}$ such that $q_{0j\alpha}^h(\varepsilon) < \bar{B}$ for sufficiently small ε, all h in H and all markets $0j\alpha$ (with $j \in J, \alpha \in I$).[18]

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[18] For simplicity, we now suppose that $\check{J} = \phi$, i.e. there are no netted assets, and deal with $\check{J}$ at the end of the proof.

Proof. Take any $j \in J$ and let $A_{s\ell}^j > 0$ for $s\ell \in S \times \tilde{L}$. Suppose $q_{0j\alpha}^h(\varepsilon) \to \infty$. Then the amount of money owed for delivery by h is at least:

$$q_{0j\alpha}^h p_{s\ell m}^\varepsilon A_{s\ell}^j$$

where $p_{smm}^\varepsilon \equiv 1$. By Step 1, the $p_{s\ell m}^\varepsilon$ are bounded away from 0, so this amount will eventually exceed E^*, contradicting condition (s(ii))$_m$ of the budget set. □

Steps 3 and 5 show that $q_{s\alpha\beta}^h(\varepsilon)$ is bounded from above for all $\alpha \in N \cup J$. Clearly if $\alpha \in \tilde{L}$, $q_{s\alpha\beta}^h(\varepsilon) \le E^*$. Hence, all actions $q^h(\varepsilon)$ are bounded independent of ε. Therefore, $\lim_{\varepsilon \to 0}(q^h(\varepsilon), x^h(\varepsilon), \omega^h(\varepsilon)) \equiv (q^h, x^h, \omega^h)$ is finite in every component.

Step 6. For sufficiently small ε, and all $h \in H$, $\sigma^h(\varepsilon)$ maximizes $u^h(x^h(\sigma^h))$ on $\Sigma_{p^\varepsilon}^h$ (not just on $\Sigma_{p^\varepsilon}^h \cap \Sigma(\varepsilon)$).

Proof. Since $\sigma^h(\varepsilon)$ is bounded in each component, the constraint of $1/\varepsilon$ is not binding on σ^h for small ε, and then the conclusion follows by the concavity of $u^h(x^h(\sigma^h)))$ on $\Sigma_{p^\varepsilon}^h$. □

We next show that $\lim_{\varepsilon \to 0} p_{s\alpha\beta}^\varepsilon \equiv p_{s\alpha\beta}$ is positive and finite for every $s\alpha\beta \in \mathcal{M}$.

Step 7. $\exists R > 0$ such that $p_{s\ell m}^\varepsilon / p_{skm}^\varepsilon < R$ and $p_{0\ell m}^\varepsilon / p_{skm}^\varepsilon < R$ for sufficiently small ε, all $s \in S^*$ and $\ell, k \in L$.

Proof. Suppose some $p_{s\ell m}^\varepsilon / p_{skm}^\varepsilon \to \infty$. Take h with $e_{s\ell}^h > 0$. Let him set apart $\Delta(\varepsilon)e^h$ of his endowment and scaling down his actions by $(1 - \Delta(\varepsilon))$, for small $\Delta(\varepsilon) > 0$ (this is feasible by the scaling property of his action space). Then his utility decreases by at most $\Delta(\varepsilon)(u_*^h - u^h(0))$ (since his payoff is a concave function of his actions), and he has at least $\Delta(\varepsilon)e_{s\ell}^h > 0$ of good $s\ell$ at hand. Let h (i) borrow more money on M_s, increasing $q_{ssm}^h(\varepsilon)$ by $\Delta(\varepsilon)p_{s\ell m}^\varepsilon e_{s\ell}^h$ (by Step 3, this is possible for small ε and $\Delta(\varepsilon)$); (ii) spend the extra money obtained on M_s (namely $\Delta(\varepsilon)p_{s\ell m}^\varepsilon e_{s\ell}^h/(1+r_s^\varepsilon)$) to purchase and consume more of sk; and (iii) sell $\Delta(\varepsilon)e_{s\ell}^h$ more of $s\ell$ for money. The proceeds of (iii) will defray the extra loan. And, choosing $\Delta(\varepsilon)$ small enough to ensure $\Delta(\varepsilon)p_{s\ell m}^\varepsilon e_{s\ell}^h/[(1+r_s^\varepsilon)p_{skm}^\varepsilon] < 1$, the increase in h's utility is at least

$$\Delta(\varepsilon)\left(\xi^h \cdot \frac{p_{s\ell m}^\varepsilon e_{s\ell}^h}{(1+r_s^\varepsilon)p_{skm}^\varepsilon} - [u_*^h - u^h(0)]\right)$$

which becomes positive, since $p_{s\ell m}^\varepsilon / p_{skm}^\varepsilon \to \infty$ and since (by Step 2) r_s^ε is bounded above, a contradiction. If $p_{0\ell m}^\varepsilon / p_{smk}^\varepsilon \to \infty$ for some $s \in S$, we repeat the above proof, except that h now inventories money obtained from selling 0ℓ into state s, and then buys sk. □

Step 8. For $s \in S^*$, $s\alpha\beta \in \mathcal{M}$ and $\alpha, \beta \in L \cup J'$

$$p_{s\alpha\beta}^\varepsilon \nrightarrow \infty$$

where $J' \equiv$ the set of assets that are not nominal.

Proof.

Case I. $\alpha, \beta = \ell, \ell' \in L$. Suppose $p_{s\ell\ell'}^{\varepsilon} \to \infty$. Then take h with $e_{s\ell}^h > 0$ and let him scale his actions down by $1 - \Delta(\varepsilon)$, as in the proof of Step 7. Let him sell $\Delta(\varepsilon)e_{s\ell}^h$ more at the market $s\ell\ell'$ (which is feasible for small enough ε and $\Delta(\varepsilon)$ by Step 3), obtaining $p_{s\ell\ell'}^{\varepsilon}\Delta(\varepsilon)e_{s\ell}^h$ of $s\ell'$. The change in his utility is at least

$$\Delta(\varepsilon)\left(\xi_-^h p_{s\ell\ell'}^{\varepsilon}e_{s\ell}^h - [u_*^h - u^h(0)]\right)$$

which becomes positive since $p_{s\ell\ell'}^{\varepsilon} \to \infty$, a contradiction.

Case II. $\alpha, \beta = j, j' \in J'$. Suppose $p_{0jj'}^{\varepsilon} \to \infty$. Take any $h \in H$. Since $e_s^h \neq 0$, for all $s \in S$, we see (in view of Steps 1 and 7) that there exists a constant $C > 0$ such that

$$\frac{p_{sLm}^{\varepsilon} \cdot e_s^h}{A_{sm}^j + p_{sLm}^{\varepsilon} \cdot A_{sL}^j} > C$$

for all $s \in S$ (for small enough ε). Moreover, since j' is not a nominal asset, there is a state $\underline{s}$ and a commodity $k \in L$ with $A_{\underline{s}k}^{j'} > 0$. As in Case I, let h set aside $\Delta(\varepsilon)e^h$ of his endowment by scaling his actions by $1 - \Delta(\varepsilon)$. Let $\Delta'(\varepsilon)$ satisfy the equation $(1 + \bar{r})\Delta'(\varepsilon) = C\Delta(\varepsilon)$, where $\bar{r}$ is an upper bound on interest rates in accordance with Step 2. Now let h sell $\Delta'(\varepsilon)$ more of asset j at the market $0jj'$ to obtain $p_{0jj'}^{\varepsilon}\Delta'(\varepsilon)$ more of asset j'. (This marginal increase in his action, and others later in the proof, are all feasible for small ε and $\Delta(\varepsilon)$ by Step 3.) Further let him in each state $s \in S$: (i) increase $q_{ssm}^h(\varepsilon)$ by $\Delta'(\varepsilon)(A_{sm}^j + p_{sLm}^{\varepsilon} \cdot A_{sL}^j)(1 + r_s^{\varepsilon})$; (ii) use the additional money obtained on M_s to make the deliveries entailed by the extra sale of asset j; (iii) sell $\Delta(\varepsilon)e_s^h$ more of his endowment for money; (iv) use the proceeds of (iii) to repay the loan of (i) (which is feasible since $\Delta(\varepsilon)p_{sLm}^{\varepsilon} \cdot e_s^h > C\Delta(\varepsilon)[A_{sm}^j + p_{sLm}^{\varepsilon} \cdot A_{sL}^j] = (1+\bar{r})\Delta'(\varepsilon)[A_{sm}^j + p_{sLm}^{\varepsilon} \cdot A_{sL}^j]$ by our choice of $\Delta'(\varepsilon)$, and since $r_s^{\varepsilon} \leq \bar{r}$); (v) increase $q_{\underline{s}\,\underline{s}m}^h(\varepsilon)$ by an additional (i.e. over and above the increase in (i)) amount $\Delta'(\varepsilon)p_{0jj'}^{\varepsilon}A_{\underline{s}k}^{j'}p_{\underline{s}km}^{\varepsilon} \equiv \Delta^*(\varepsilon)p_{\underline{s}km}^{\varepsilon}(say)$; (vi) use the money obtained in (v) to buy at least $(\Delta^*(\varepsilon)p_{\underline{s}km}^{\varepsilon}/((1+\bar{r})p_{\underline{s}\ell m}^{\varepsilon}) \geq \Delta^*(\varepsilon)/R(1+\bar{r})$ of a commodity $\underline{s}\ell$ that he likes[19] where R is as in Step 7. The deliveries obtained by h on his extra purchase of asset j' defrays the loan incurred in (v), hence the above deviation is feasible for him. The resulting gain in utility is at least $(\xi_-^h\Delta^*(\varepsilon)/R(1+\bar{r})) - \Delta(\varepsilon)(u_*^h - u^h(0))$ which becomes (after substituting $\Delta'(\varepsilon)$ into $\Delta^*(\varepsilon)$, and $\Delta(\varepsilon)$ into $\Delta'(\varepsilon)$):

$$\Delta(\varepsilon)\left(\frac{\xi_-^h C}{R(1+\bar{r})^2} \cdot p_{0jj'}^{\varepsilon}A_{\underline{s}k}^{j'} - [u_*^h - u^h(0)]\right)$$

This is positive for small enough ε since $p_{0jj'}^{\varepsilon} \to \infty$, a contradiction.

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[19] In the context of purely Theorem 1, he could buy commodity $\underline{s}k$, since we have assumed each agent likes every commodity. But we do the more general argument to indicate how this assumption could be relaxed.

Case III. $\alpha, \beta \equiv j, \ell$ where $j \in J'$, $\ell \in L$. Suppose $p^\varepsilon_{0j\ell} \to \infty$. Here, let h sell $p^\varepsilon_{0j\ell} \Delta'(\varepsilon)$ more of asset j at the market $0j\ell$, where $\Delta'(\varepsilon)$ is as in Case II and deviate as in (i), (ii), (iii), (iv) of Case II, after scaling his actions by $1 - \Delta(\varepsilon)$. Then he can repay on his extra sale of asset j, obtaining $\Delta'(\varepsilon) p^\varepsilon_{0j\ell} = (C/(1 + \bar{r})) p^\varepsilon_{0j\ell}$ units of commodity 0ℓ, which exceeds D^* as $p^\varepsilon_{0j\ell} \to \infty$, a contradiction.

Case IV. $\alpha, \beta = \ell, j$ for $\ell \in L$, $j \in J'$. Suppose $p^\varepsilon_{0\ell j} \to \infty$, i.e. $p^\varepsilon_{0j\ell} \to 0$. This is contradicted in Step 11 below.   □

Step 9. For $\ell \in L$ and $j \in J$ (not just $j \in J'$), $p^\varepsilon_{0j\ell} \not\to \infty$.

Proof. As in Case III of Step 8.   □

Step 10. For $j \in J'$, $p^\varepsilon_{0jm}/\|p^\varepsilon_{0Lm}\| \not\to \infty$, where $\|p^\varepsilon_{0Lm}\| \equiv \Sigma_{k \in L} p^\varepsilon_{0km}$.

Proof. Suppose $p^\varepsilon_{0jm}/\|p^\varepsilon_{0Lm}\| \to \infty$. Let h scale down his actions by $1 - \Delta(\varepsilon)$ and sell $\Delta'(\epsilon)$ more of asset j at the market $0jm$, where $\Delta'(\varepsilon) \equiv C\Delta(\varepsilon)(1 + \bar{r})^{-1}$ is as in the proof of Step 8. Let him (i) increase $q^h_{00m}(\varepsilon)$ by $\Delta'(\varepsilon) p^\varepsilon_{0jm}$; (ii) spend the money obtained in (i) to buy and consume $\Delta'(\varepsilon) p^\varepsilon_{0jm}/[(1 + r^\varepsilon_0) p^\varepsilon_{0\ell m}]$ of some good 0ℓ; (iii) deviate in each state $s \in S$ exactly as in the proof of Step 8. (Note that the extra sale of $\Delta'(\varepsilon)$ of asset j will defray the loan of (i).) Then his change in payoff is at least (for $\Delta(\varepsilon)$ suitably small):

$$\Delta(\varepsilon) \left(\xi^h_- \frac{C}{(1 + \bar{r})} \frac{p^\varepsilon_{0jm}}{(1 + \bar{r}) p^\varepsilon_{0\ell m}} - [u^h_* - u^h(0)] \right)$$

which becomes positive, since $p^\varepsilon_{0jm}/p^\varepsilon_{0\ell m} \to \infty$, if $p^\varepsilon_{0jm}/\|p^\varepsilon_{0Lm}\| \to \infty$, a contradiction.   □

Step 11. Let $\ell \in L$, $j \in J$ and $p^\varepsilon_{0j\ell} \to 0$. Then j is a nominal asset, and (for all $s \in S$):

$$A^j_{sm} > 0 \Rightarrow p^\varepsilon_{sLm} \to \infty.$$

Proof. Suppose j is not nominal, i.e. $A^j_{sk} > 0$ for some $sk \in S \times L$. Choose h with $e^h_{0\ell} > 0$. Let h (i) scale his actions down by $1 - \Delta(\varepsilon)$ losing at most $\Delta(\varepsilon)[u^h_* - u^h(0)]$ utility; (ii) sell $\Delta(\varepsilon) e^h_{0\ell}$ more of 0ℓ on the market $0\ell j$; (iii) increase $q^h_{s\bar{s}m}(\varepsilon)$ by $[\Delta(\varepsilon) e^h_{0\ell}/p^\varepsilon_{0j\ell}] p^\varepsilon_{skm} A^j_{sk}$; (iv) use the money obtained in (iii) to purchase and consume $(\Delta(\varepsilon) e^h_{0\ell} p^\varepsilon_{skm} A^j_{sk})/[p^\varepsilon_{0j\ell}(1 + r^\varepsilon_s) p^\varepsilon_{s\ell m}]$ more of some commodity $s\ell$. The delivery he receives on the extra purchase of asset j defrays the extra loan in (iii). Thus, his change in payoff is at least

$$\Delta(\varepsilon) \left(\xi^h_- \frac{e^h_{0\ell} A^j_{sk} p^\varepsilon_{skm}}{p^\varepsilon_{0j\ell}(1 + r^\varepsilon_s) p^\varepsilon_{s\ell m}} - [u^h_* - u^h(0)] \right)$$

which becomes positive, since $p^\varepsilon_{0j\ell} \to 0$; and (by Step 2) r^ε_s is bounded; and (by Step 7) $p^\varepsilon_{skm}/p^\varepsilon_{s\ell m}$ is bounded. This is a contradiction. Hence, j is nominal.

Next suppose $A_{sm}^j > 0$ and $\|p_{sl,m}^\varepsilon\|$ is bounded. Let h deviate as before, replacing $p_{skm}^\varepsilon A_{sk}^j$ by A_{sm}^j throughout. We get the same contradiction, since p_{slm}^ε is bounded.   □

Step 12. Suppose, for $j \in J$, $p_{0jm}^\varepsilon / \|p_{0Lm}^\varepsilon\| \to 0$. Then, again, j is a nominal asset, and (for all $s \in S$):

$$A_{sm}^j > 0 \Rightarrow p_{sLm}^\varepsilon \to \infty.$$

Proof. This involves a minor modification of the previous proof. Choose h with $e_{0\ell}^h > 0$. Let (i) be as before. Replace (ii) and (iii) by: let h sell $\Delta(\varepsilon)e_{0\ell}^h$ more of 0ℓ for money (as usual); increase $q_{0\bar{0}m}^h(\varepsilon)$ by $\Delta(\varepsilon)p_{0\ell m}^\varepsilon e_{0\ell}^h$ (the sale above will defray this loan); purchase $(\Delta(\varepsilon)p_{0\ell m}^\varepsilon e_{0\ell}^h)/[(1 + r_0^\varepsilon)p_{0jm}^\varepsilon]$ more of asset j out of the extra loan. Again let h borrow more money in state s to spend on consuming $s\ell$, making sure that the extra loan is defrayed by the delivery on his additional purchase of asset j. The same contradictions obtain, as in the previous proof, replacing $\Delta(\varepsilon)e_{0\ell}^h / p_{0j\ell}^\varepsilon$ by $(\Delta(\varepsilon)p_{0\ell m}^\varepsilon e_{0\ell}^h)/[(1 + r_0^\varepsilon)p_{0jm}^\varepsilon]$, and noting that (by Step 7) $p_{0\ell m}^\varepsilon / p_{0jm}^\varepsilon \to \infty$ under the current scenario.   □

Step 13. $r_s \le \mu_s(m, M)$, for all $s \in S$.

Proof. This will be organized through a series of claims (all meant for small enough ε). Throughout denote $\bar{m}_s \equiv \sum_{h \in H} m_s^h$, for all $s \in S^*$ and $r_n = \lim_{\varepsilon \to 0} r_n^\varepsilon$, for $n \in N$.   □

Claim I. $(1 + r_0^\varepsilon) \ge \min_{s \in S}(1 + r_0^\varepsilon)(1 + r_s^\varepsilon)$.

Suppose the claim is false. Now, by Step 4, the external agent borrows $\varepsilon/(1 + r_n^\varepsilon) \le \varepsilon$ dollars on any loan $n \in N$ (while also depositing ε dollars on it). Therefore, agents' net borrowing on M_n is between M_n and $M_n + \varepsilon$. It follows that $\exists h \in H$ who borrows a positive amount on M_0. Let h borrow Δ dollars less on M_0 and (as is feasible, for small Δ and ε, by Step 3) Δ dollars more on $M_{\bar{0}}$. This enables him to act exactly as before in all other respects in state 0. But he owes $(1 + r_0^\varepsilon)\Delta$ less on M_0 at the end of state 0. So let him, instead of repaying $(1 + r_0^\varepsilon)\Delta$ on M_0, inventory it into every state $s \in S$ in period 1 and then deposit it on M_s. He will earn $(1 + r_0^\varepsilon)(1 + r_s^\varepsilon)\Delta > (1 + r_0^\varepsilon)\Delta$ at the end of each state $s \in S$, i.e. will have surplus money $\Delta[(1 + r_0^\varepsilon)(1 + r_s^\varepsilon) - (1 + r_0^\varepsilon)] \equiv \Delta_s^\varepsilon > 0$, without having affected his consumption. In view of this he could (again invoking Step 3) borrow $\Delta_s^\varepsilon/(1 + r_s^\varepsilon)$ more on some M_s and spend it to consume more in state s, improving his utility, a contradiction. This proves Claim I. (Indeed we see that strict inequality holds when the r_s^ε, $s \in S$, are not equal. For otherwise, the same arbitrage works with $\Delta_t^\varepsilon > 0$ for some $t \in S$, and $\Delta_s^\varepsilon \ge 0$, $\forall s \in S$.)

Claim II. $r_0^\varepsilon M_{\bar{0}} + r_0^\varepsilon M_0 + r_s^\varepsilon M_s \le \bar{m}_0 + \bar{m}_s + (|\mathcal{M}| + |\mathcal{J}|)\varepsilon$, $\forall s \in S$.

To check this, consider the path $(0, s)$ in the date–event tree. On this path agents' net borrowing is (as we just saw) at least $M_0 + M_{\bar{0}} + M_s$. The external agent creates (as we saw much earlier) no more than $(|\mathcal{M}| + |\mathcal{J}|)\varepsilon$ dollars. Therefore, the total outside money

available on path $(0, s)$ to agents is at most $\bar{m}_0 + \bar{m}_s + (|\mathcal{M}| + |\mathcal{J}|)\varepsilon$, which must cover their interest payment $r_{\bar{0}}^\varepsilon M_{\bar{0}} + r_0^\varepsilon M_0 + r_s^\varepsilon M_s$ on the path $(0, s)$, proving Claim II.

Claim III. *There is some $t \in S$ with $(1+r_{\bar{0}}^\varepsilon) \geq (1+r_0^\varepsilon)(1+r_t^\varepsilon)$, and $r_t \leq (\bar{m}_0+\bar{m}_t)/(M_0 + M_t) \leq \mu_t(m, M) < \gamma_t(\tilde{x}), \forall \tilde{x} \in X_t$.*

To verify Claim III, observe that $\mu_s(m, M) \equiv [\bar{m}_0 + \bar{m}_s - \min_{t\in S}\{(\bar{m}_0 + \bar{m}_t)/(M_{\bar{0}} + M_t)\}M_{\bar{0}}]/M_s \geq [\bar{m}_0 + \bar{m}_s - \{(\bar{m}_0 + \bar{m}_s)/(M_{\bar{0}} + M_s)\}M_{\bar{0}}]/M_s = (\bar{m}_0 + \bar{m}_s)/(M_{\bar{0}} + M_s)$, $\forall s \in S$. By Claim I, $\exists t \in S$ such that $(1 + r_{\bar{0}}^\varepsilon) \geq (1 + r_0^\varepsilon)(1 + r_t^\varepsilon)$, i.e. $r_{\bar{0}}^\varepsilon \geq r_0^\varepsilon + r_t^\varepsilon$ and, by Claim II, $r_{\bar{0}}^\varepsilon M_{\bar{0}} + r_0^\varepsilon M_0 + r_t^\varepsilon M_t \leq \bar{m}_0 + \bar{m}_t + (|\mathcal{M}| + |\mathcal{J}|)\varepsilon$. Substituting $r_0^\varepsilon + r_t^\varepsilon$ in place of $r_{\bar{0}}^\varepsilon$ in the latter inequality yields $r_t^\varepsilon \leq [\bar{m}_0 + \bar{m}_t + (|\mathcal{M}| + |\mathcal{J}|)\varepsilon]/(M_{\bar{0}} + M_t)]$. Hence, for small enough ε, $\gamma_t(\tilde{x}) > r_t^\varepsilon$, $\forall \tilde{x} \in X_t$ (since, by the gains to trade hypothesis, $\gamma_t(\tilde{x}) > \mu_t(m, M)$, $\forall \tilde{x} \in X_t$).

Claim IV. *Let $t \in S$. If $r_t \leq \mu_t(m, M)$, then $p_{tLm}^\varepsilon \not\to \infty$.*

(By Step 7, either all prices in a state stay bounded or else all go to infinity.)

To verify Claim IV, suppose $p_{tLm}^\varepsilon \to \infty$. Recalling the formula for $p_{t\ell m}^\varepsilon$, we must have[20] $\sum_{h\in H} q_{t\ell m}^h(\varepsilon) \to 0$, and therefore $\lim_{\varepsilon\to 0}x(\varepsilon) \equiv x \in X_t$. (Not only commodity trades become negligible in state t, but net asset deliveries—which cannot exceed the money in system—also count for nothing since their purchasing power is going to zero.) Let $\tilde{p}_{tLm}^\varepsilon = p_{tLm}^\varepsilon/\|p_{tLm}^\varepsilon\|$ and $\hat{p}_{tLm} = \lim_{\varepsilon\to 0}\tilde{p}_{tLm}^\varepsilon$.

For each agent $h \in H$, define a utility of trade τ in state t by $v^h(\tau) = u^h(x^h + \tau^*(\tau, r_t))$ where $\tau^*(\tau, r_t) \in \mathbb{R}^{S^*\times L}$ is given by $\tau_{s\ell}^* = 0$ if $s \in S^*\backslash\{t\}$, $\tau_{t\ell}^* = \tau_\ell$ if $\tau_\ell < 0$, $\tau_{t\ell}^* = \tau_\ell/(1 + r_t)$ if $\tau_\ell \geq 0$. Then, using the fact that we are taking limits of ε-ME and that $p_{tLm}^\varepsilon \to \infty$, it is easy to verify that no-trade constitutes a Walras equilibrium of the pure exchange L-goods economy in state t with utilities v^h, endowments x^h and prices $\hat{p}$. But then there are no gains-to-trade (i.e. no Pareto-improvement) with utilities v^h. This translates easily[21] into $\gamma_t(x) \leq r_t$. But $r_t \leq \mu_t(m, M)$. Hence, $\gamma_t(x) \leq \mu_t(m, M)$. This contradicts the gains-to-trade hypothesis, proving Claim IV.

Claim V. *Let t be as in Claim III. Then $r_{\bar{0}}M_{\bar{0}} + r_0M_0 + r_tM_t = \bar{m}_0 + \bar{m}_t$ (where $r_n \equiv \lim_{\varepsilon\to 0}r_n^\varepsilon$ for all $n \in N$).*

Taking limits in Claim II, we already have $r_{\bar{0}}M_{\bar{0}} + r_0M_0 + r_tM_t \leq \bar{m}_0 + \bar{m}_t$. By Step 7 and Claim IV, we see that commodity prices p_{0Lm}^ε and p_{tLm}^ε are bounded on the path $(0, t)$. Moreover, arguing as in Steps 8–12, and using the fact that p_{tLM}^ε is bounded, it follows that if $A_t^j \neq 0$ then $p_{0j\alpha}^\varepsilon$ is bounded away from zero, for all $0j\alpha \in \mathcal{M}$; which in turn implies that the ε-external agent owns at most $K\varepsilon$ units of any asset that makes nonzero deliveries in state t (for some constant K). Thus, the money obtained by the ε-external agent from his sale of commodities and receipt of asset deliveries is at most $\tilde{\kappa}\varepsilon$ (for some other constant $\tilde{\kappa}$)

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[20] Since total money in the system is bounded, prices go to infinity only if the sale of commodities goes to zero.

[21] Or else see Lemma 2 in Dubey and Geanakoplos (2003).

on the path $(0, t)$. So at least $\bar{m}_0 + \bar{m}_t + M_{\bar{0}} + M_0 + M_t - \bar{\kappa}\varepsilon$ amount of money is in agents' hands on the path $(0, t)$ at the ε-ME. All this money must be owed to the banks (since, if any agent was left with worthless surplus money he could have improved his consumption by buying more goods with an incremental bank loan and repaying it using the surplus). Hence,[22] $\bar{m}_0 + \bar{m}_t - \bar{\kappa}\varepsilon + M_{\bar{0}} + M_0 + M_t \leq (1 + r_{\bar{0}}^{\varepsilon})(M_{\bar{0}} + \varepsilon) + (1 + r_0^{\varepsilon})(M_0 + \varepsilon)(1 + r_t^{\varepsilon})(M_t + \varepsilon) - [(1 + r_{\bar{0}}^{\varepsilon}) + (1 + r_0^{\varepsilon}) + (1 + r_t^{\varepsilon})]\varepsilon$. Taking limits we get the reverse inequality: $\bar{m}_0 + \bar{m}_t \leq r_{\bar{0}}M_{\bar{0}} + r_0 M_0 + r_t M_t$, establishing the equality of Claim V.

Completion of the Proof of Step 13. Suppose that $r_s > (m_0 + m_s - \min_{t' \in S}[(\bar{m}_0 + \bar{m}_{t'})/(M_{\bar{0}} + M_{t'})]M_{\bar{0}})/M_s$ for some $s \in S$. This implies

$$M_{\bar{0}}\min_{t' \in S}\left[\frac{\bar{m}_0 - \bar{m}_{t'}}{M_{\bar{0}} - M_{t'}}\right] + M_s r_s > \bar{m}_0 + m_s. \tag{*}$$

But by Claim II, we have

$$M_0 r_0 + M_{\bar{0}} r_{\bar{0}} + M_s r_s \leq \bar{m}_0 + \bar{m}_s. \tag{**}$$

Subtracting (**) from (*) yields

$$M_{\bar{0}}\left(\min_{t' \in S}\left[\frac{\bar{m}_0 + \bar{m}_{t'}}{M_{\bar{0}} + M_{t'}}\right] - r_{\bar{0}}\right) > M_0 r_0,$$

i.e.

$$r_{\bar{0}} < \min_{t' \in S}\left[\frac{\bar{m}_0 + \bar{m}_{t'}}{M_{\bar{0}} + M_{t'}}\right] - \frac{M_0 r_0}{M_{\bar{0}}} \leq \frac{\bar{m}_0 + \bar{m}_t}{M_{\bar{0}} + M_t} - \frac{M_0 r_0}{M_{\bar{0}}}$$

where t is as in Claim III. But then

$$M_{\bar{0}} r_{\bar{0}} + M_0 r_0 + M_t r_t < M_{\bar{0}}\left(\frac{\bar{m}_0 + \bar{m}_t}{M_{\bar{0}} + M_t + M_0 r_0}\right) - M_0 r_0 + M_t r_t$$

$$= M_{\bar{0}}\left(\frac{\bar{m}_0 + \bar{m}_t}{M_{\bar{0}} + M_t}\right) + M_t r_t.$$

Since $r_t \leq (\bar{m}_0 + m_t)/(M_{\bar{0}} + M_t)$ by Claim III, we see (by substituting $(\bar{m}_0 + \bar{m}_t)/(M_{\bar{0}} + M_t)$ for r_t in the last term) that

$$M_{\bar{0}} r_{\bar{0}} + M_0 r_0 + M_t r_t < \bar{m}_0 + \bar{m}_t,$$

which contradicts Claim V. This proves Step 13.

Step 14. $p_{sLm}^{\varepsilon} \nrightarrow \infty$, for $s \in S^*$.

Proof. First take $s \in S$. The result follows from Step 13 and Claim IV of Step 13.
  If $p_{0Lm}^{\varepsilon} \rightarrow \infty$, then by Step 7, $p_{sLm}^{\varepsilon} \rightarrow \infty$, for all $s \in S$, a contradiction. Thus, $p_{sLm}^{\varepsilon} \nrightarrow \infty$, for all $s \in S^*$. □

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[22] Taking into account that the external agent puts up ε on both sides of each loan market, the RHS is the total money owed to banks by agents.

Step 15. $p_{s\alpha\beta}^{\varepsilon}$ is bounded for all $s\alpha\beta \in \mathcal{M}$. Hence, $p = \lim p^{\varepsilon}$ is positive and finite in each component.

Proof. By Step 8, the only case left to check is when either $\alpha = m$ or $\beta = m$ or $\alpha \in J \backslash J'$, $\beta \in J'$ or vice versa. But these follow from Steps 7, 9–14. (Since p_{sLm}^{ε} is bounded for $s \in S$, nominal assets can be treated like real assets.)   □

Step 16. The limit $((q^h, x^h, \omega^h)_{h \in H}, p)$ is an ME.

Proof. This is evident from Step 15, and the continuity of u^h.   □

Proof of Theorem 1 with Netting. The only place in the proof where we used the fact that there is no netting was to show (in Step 5) that $q_{0j\alpha}^h(\varepsilon) \nrightarrow \infty$ for any $j \in J$ and $0j\alpha \in \mathcal{M}$. The idea was that since each asset called for the delivery of either money or commodities, no agent could be too short and still be able to deliver. But when asset deliveries are pooled, this argument no longer holds, since an agent could buy a large amount of some assets, and sell a large amount of some other assets, and yet be called upon to make a small amount of net money deliveries. We now show how to fill this gap.

If asset j is not nominal, i.e. $j \in J'$, then $0 < \lim_{\varepsilon \to 0} p_{0j\ell}^{\varepsilon}$, for all $\ell \in \tilde{L}$, as argued in Steps 11 and 12. If j is nominal and $p_{0j\ell}^{\varepsilon} \to 0$, for some $\ell \in \tilde{L}$, then we must have

$$A_{sm}^{j} > 0 \Rightarrow p_{s\ell m}^{\varepsilon} \to \infty, \text{ for all } \ell \in L.$$

Otherwise any h with $e_{0\ell}^h > 0$ can give up a little bit of 0ℓ to purchase a huge amount of asset j, and use the money delivered by j in state s to consume[23] a huge amount of commodities in state s, a contradiction. But if $p_{sLm}^{\varepsilon} \to \infty$, we contradict the gains-to-trade hypothesis in state s. (The net deliveries on $\tilde{J}$ in state s are still bounded by E^* and count for nothing since $p_{sLm}^{\varepsilon} \to \infty$. So commodity trade in state s goes to zero even with netted assets.)

To sum up, $0 < \lim_{\varepsilon \to 0} p_{0j\ell}^{\varepsilon}$, for all $j \in J$ and all $\ell \in \tilde{L}$. But since the total amount of $\ell \in \tilde{L}$ is bounded above by E^* in an ε-ME, we conclude that asset sales $q_{0j\ell}^{\varepsilon}(\varepsilon)$ stay bounded, for all $h \in H$ and $\ell \in \tilde{L}$ and $j \in \tilde{J}$. But since assets in $\tilde{J}$ trade only against $\tilde{L}$, we are done.   □

Proof of Theorem 2. That $r_0 \leq \sum_{h \in H} m_0^h / M_0$ is evident from the fact that there is no more than $\sum_{h \in H} m_0^h$ outside money to repay on loans by the end of state 0. The other part of (iii), as well as (i) and (ii) are established as in the proof of Theorem 1 (see Steps 4 and 13), replacing ε-ME by ME. The proof of equality (iv) is also implicit in the proofs of Claims II, IV of Step 13. Indeed consider any path $(0, s)$ for $s \in S$. Then the total money on this path is $\bar{m}_0 + \bar{m}_s + M_{\bar{0}} + M_0 + M_s$. At any ME, no agent will end up holding worthless surplus cash at the end of state s (otherwise, he could have borrowed, spent and consumed

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more; and returned the incremental loan on M_s with this surplus). So all the money must be owed to the bank, implying

$$\bar{m}_0 + \bar{m}_s + M_{\bar{0}} + M_0 + M_s \leq (1 + r_{\bar{0}})M_{\bar{0}} + (1 + r_0)M_0 + (1 + r_s)M_s.$$

On the other hand, since no agent can default, total money on $(0, s) \geq$ money owed on $(0, s)$, proving the reverse inequality. This establishes equality in the above display, $\forall s \in S$, proving (iv).

Finally (v) follows from (ii) and (iii). $\qquad\square$

Proof of Theorem 3. Define an ε-ME as before, except that the ε-external agent does *not* act on the bank loan markets. Instead the central bank issues either bonds $Q_{snm} > 0$ or inside money $Q_{smn} > 0$ (depending on whether $(1 + r_n) \sum_{h \in H} q^h_{smn} > \sum_{h \in H} q^h_{snm}$ or the reverse strict inequality holds) in order to clear the loan market: $(1 + r_n)[Q_{smn} + \sum_{h \in H} q^h_{smn}] = Q_{snm} + \sum_{h \in H} q^h_{snm}$. It never issues both bonds and inside money (i.e. $Q_{snm} \times Q_{smn} = 0$). Then ε-ME exist as before. We shall show that, in conjunction with the no-arbitrage condition on interest rates, this implies that the total money in the system is bounded at ε-ME, so that the proof of existence can proceed as for Theorem 1.

Denote actions at the market $s\alpha\beta$ in the ε-ME under consideration by $Q^\varepsilon_{s\alpha\beta}$, $(q^h_{s\alpha\beta}(\varepsilon))_{h \in H}$. Since the outside money available for bank deposits in period 0 is $\bar{m}_0 \equiv \sum_{h \in H} m^h_{0m0}$, we have $\sum_{h \in H} q^h_{0m0}(\varepsilon) + \sum_{h \in H} q^h_{0m\bar{0}}(\varepsilon) \leq \bar{m}_0$. Since for $n = 0$ or $\bar{0}$, if $Q^\varepsilon_{0nm} > 0$ then $(1 + r_n) \sum_{h \in H} q^h_{0mn}(\varepsilon) = Q^\varepsilon_{0nm} + \sum_{h \in H} q^h_{0nm}(\varepsilon)$, we conclude $Q^\varepsilon_{00m} + Q^\varepsilon_{0\bar{0}m} \leq (1 + r_{\bar{0}})m_{\bar{0}}$ (using the obvious fact that $r_{\bar{0}} \geq r_0$). This shows that the outside money in agents' hands at the start of period $s \in S$ is bounded by $B_1 \equiv (1 + r_0)\bar{m}_0 + \bar{m}_s$.

We next argue that $Q^\varepsilon_{0m\bar{0}}$ is bounded. For suppose $Q^\varepsilon_{0m\bar{0}} \to \infty$. By the no-arbitrage condition, $(1 + r_{\bar{0}}) > (1 + r_t)$ for some $t \in S$ since $r_0 > 0$. Thus, even if all of $Q^\varepsilon_{0m\bar{0}} + B_1$ were deposited to earn the interest r_t, agents would not have enough money to repay their long loan, since $(1 + r_{\bar{0}})Q^\varepsilon_{0m\bar{0}} - (1 + r_t)Q^\varepsilon_{0m\bar{0}} - (1 + r_t)B_1 \to \infty$. Hence, $Q^\varepsilon_{0m\bar{0}}$ is bounded. But then so is Q^ε_{0m0}, since at most $Q^\varepsilon_{0m\bar{0}} + \bar{m}_0$ is available in aggregate at the end of period 0 to repay the debt $(1 + r_0)Q^\varepsilon_{0m0}$. For the same reason, the total deposit on r_s is also bounded by $Q^\varepsilon_{0m\bar{0}} + B_1$ in any state $s \in S$. (Agents have no more money in their hands at the start of state s.) Arguing as in state 0, this bounds Q^ε_{ssm} by $(1 + r_s)(Q^\varepsilon_{0m\bar{0}} + B_1)$, for all $s \in S$. But then the total outside money at the end of state s to repay the loan on r_s is at most $(1 + r_{\bar{0}})\bar{m}_0 + \bar{m}_s$, which is bounded. This must not be less than $(1 + r_s)Q^\varepsilon_{sms}$, bounding Q^ε_{sms}, $\forall s \in S$. Thus, the total money in the system is bounded. $\qquad\square$

Proof of Theorem 4. By Theorem 2, $\sum_{h \in H} m^h_s = 0$, for all $s \in S^* \Rightarrow r_s = 0$, for all $s \in S^*$ and $r_{\bar{0}} = 0$. The rest then follows immediately from the definitions of ME and GEI. It is evident that with zero interest rates, an ME is a GEI, in both the short loan and canonical models. We need only check that given a GEI, the bank stocks $M_{\bar{0}}, M_0, \ldots, M_S$ are high enough to support the levels of trade at the GEI, so that it is obtained as an ME. To this end, scale down all commodity prices by the same factor (say $1/K$) in states $s \in S^*$. If an asset j is real, scale its price down by $1/K$ as well. If it is nominal, leave its price fixed. With hoarding of bank money in plenty, we will have an ME which coincides with the GEI in

real terms (production and trade of commodities), though not necessarily in prices or trades of financial assets.   □

Proof of Corollary to Theorem 4. As long as asset trades are uniformly bounded over the equilibria, we can take convergent subsequences such that (1) all the net trades of the agents converge, and (2) $(p_{0Im})/\|p_{0Im}\|$ converges, and (3) $p_{sLm}/\|p_{sLm}\|$ converges, for all $s \in S$. Since by Theorem 2(iv) interest rates converge to 0, these limiting net trades and price ratios would constitute a GEI. Note also that by Theorem 1, ME do indeed exist for large enough M_s, since the Gains to Trade Hypothesis is automatically satisfied.

We now show that asset trades must stay bounded. Observe first that p_{sLm} must be bounded away from zero, as in the proof of Theorem 1, otherwise any agent with $m_0^h > 0$ would be able to buy the whole economy in state s (i.e. more than D^* of each commodity) contradicting the existence of ME for large M_s. Furthermore, since each asset delivers in full, in each state s the holder of a unit of asset j will be able to consume at least $A_{s\ell}^j/(1+r_s)$ units more of commodity $s\ell$ (by borrowing on M_s to purchase $s\ell$, and repaying the loan with the asset deliveries). Since $r_s \to 0$ and the numeraire asset payoffs are linearly independent, if some agent h goes arbitrarily long or short in assets as $(M_s)_{s \in S^*} \to \infty$, there will be some state s for which the "net receipts"

$$\left| \sum_{j \in J} \frac{(p_{0Ij} \cdot q_{0Ij}^h) A_{s\ell}^j}{(1+r_s)} - \sum_{j \in J}(1 \cdot q_{0jI}^h) A_{s\ell}^j \right| \to \infty$$

Since, as in the proof of Step 7 of Theorem 1, relative prices $p_{skm}/p_{sk'm}$ remain bounded for k and k' in L, agent h will be able to buy the whole economy in state s, or else will owe more than he can repay, a contradiction.   □

Proof of Theorem 5. First we prove (a). For any fixed $M = (M_0, M_1, \ldots, M_S)$, an ME $\equiv [(q^h, x^h, \omega^h)_{h \in H}, p](M)$ exists. Let $M \to \infty$ and suppose on some subsequence that all individual asset trades stay bounded. From that subsequence select a further subsequence along which all ratios of components of $((q^h(M), x^h(M), \omega^h(M))_{h \in H}, p(M))$ also converge, possibly to 0 or ∞. Arguing as in the proof of the Corollary to Theorem 4, the limit of this last subsequence is a GEI, which is a contradiction. Hence, we conclude that asset trades $\to \infty$. In order that arbitrarily large asset purchases be feasible, we must have that $M_0/\|p_{0Jm}(M)\| \to \infty$. But (as in the proof of Theorem 1), asset prices $p_{0Jm}(M)$ and commodity prices $p_{0Lm}(M)$ are relatively bounded. Hence, $M_0/\|p_{0Lm}(M)\| \to \infty$. This proves (a).

The proof of (b) is exactly the same, except that now we must replace $M_0/\|p_{0Lm}\| \to \infty$ with $(M_0 + M_{\bar{0}})/\|p_{0Lm}\| \to \infty$. However, since M_0 and $M_{\bar{0}}$ are relatively bounded, we conclude that $M_{\bar{0}}/\|p_{0Lm}\| \to \infty$.   □

Proof for Section 16. Since $\gamma_s(x) > 0$, for all $x \in X_s$, and all $s \in S$, it follows from Theorem 1 that ME exist for large enough M. Denote prices and interest rates in ME of the Mth economy by $p_{sLm}(M)$, $r_n(M)$. Then $\|p_{0Lm}(M)\|/\min_{s \in S}\|p_{sLm}(M)\|$ stays bounded as $M \to \infty$ (arguing as in Step 7 of the proof of Theorem 1). Now suppose $\|p_{0Lm}(M)\| \to \infty$ as $M \to \infty$. Then the entrepreneur, who is operating at nearly zero-input levels, will have

the prospect of very lucrative profits in every $s \in S$, since the future sale price of his output never crashes. Therefore, he will be anxious to borrow much more than $M_{\bar{0}}$ on the long loan (at its bounded interest rate $r_{\bar{0}}(M) \leq \bar{m}_0/M_{\bar{0}}$), and that loan market will not clear, a contradiction. Hence, $\|p_{0Lm}(M)\|$ stays bounded as $M \to \infty$.

Now, by (iv) of Theorem 2, all $r_s(M)$ are the same, say $r_s(M) \equiv r(M)$, $\forall s \in S$. Suppose $r(M) > 0$ (on some subsequence) as $M \to \infty$. Since there can be no hoarding at positive interest rates, all of $M_s \equiv M$ is spent on commodity purchases or asset deliveries in every $s \in S$. However, assets being numeraire, all asset sales are bounded (see the proof of the Corollary to Theorem 4); so the total delivery on assets in state s is a bounded multiple of $\|p_{sLm}(M)\|$. We conclude that $\|p_{sLm}(M)\| \to \infty$, $\forall s \in S$ as $M \to \infty$. But since $\|p_{0Lm}(M)\|/\min_{s \in S}\|p_{sLm}(M)\|$ is also bounded away from zero (otherwise an agent can consume more than D^* in state 0 via the long loan, repaying it by the sale of his endowment in every $s \in S$), it follows that $\|p_{0Lm}(M)\| \to \infty$ as $M \to \infty$, a contradiction. This proves that $r(M) = 0$ for large enough M.

Finally if $r_0(M)$ stays positive, all $M_0 \equiv M$ is spent on commodity or asset purchases in period 0. Since asset sales are bounded and since (as in Step 12 of the proof of Theorem 1) so is $\|p_{0Jm}(M)\|/\|p_{0Lm}(M)\|$, we deduce that $\|p_{0Lm}(M)\| \to \infty$, again a contradiction.

Thus, $r_s(M) = 0$, $\forall s \in S^*$ for large enough M. □

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