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The Value of Money in a Finite-Horizon Economy: A Role for Banks

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Frank Hahn (1965), among others, has argued that it is difficult to prove, or even to justify, a positive price for fiat (i.e., paper) money in economic equilibrium. The problem is particularly serious when there are only a finite number of time periods. In the last period nobody will want to hold a positive amount of it, if its price is positive. Hence, if there is a positive supply of money, equilibrium requires its price to be zero in the last period. But then in the second-to-last period nobody will want to consume it, or hold it, since it brings no reward in the last period. Again its price must be zero if its supply is positive. Working backward to the beginning, it follows that the value of money is always zero if it is in positive supply, in a finite-horizon economy.

This argument is not affected by the stipulation that all trade must be conducted between goods and money, provided that preferences for commodities are strictly monotonic. If the world is expected to come to an end in exactly 1 billion years, then no trade can take place through money in all the intervening years, no matter how horrible the initial distribution of commodities.

There are at least four ways around these problems. One could postulate an infinite-horizon model, so that there is no last period. This was done by Samuelson (1958), who made the agents finitely-lived, and also by Grandmont and Younes (1973), who had infinitely-lived agents that faced a transactions constraint. Alternatively, one could model the present, leaving the future as part of exogenous expectations. As long as these expectations suppose positive prices of money (Grandmont and Younes 1972) in the future, no matter what the conditions today, or at least levels of prices that are not too much higher than those today (Hool 1976), money will have positive price in today's temporary equilibrium. Third, one could postulate the existence of an external agent who stands ready to trade commodities for money at prearranged prices. Finally, following Lerner (1947), one could postulate the existence of a government that is owed in taxes (payable only in money) an amount precisely equal to the cash balances of all the individuals. In the first two approaches, money has value in any period because it is a store of value, and will by hypothesis have value in the next period; in the last two approaches, money has value because an external agent gives something (either commodities or relief from taxes) in

exchange for it. In the Lerner model the delicate balance of money and debt implies that the net supply of outside money is zero.

We propose to add another explanation for the value of money by introducing a competitive banking sector. The "banking sector" is endowed with fiat money and wishes only to get more money. Agents also have endowments of money that they own free and clear, with no offsetting debts to an outside government or bank. In contrast to the first two approaches to money, its value to the agents can be derived from its transaction role alone, quite apart from any store-of-value role it might also have. Moreover, unlike the external agent mentioned in the last two approaches above, the banking sector does not have anything to offer in exchange for money. It does, however, have the power to lend money to voluntary borrowers, and to enforce the collection of the ensuing debts.

Consider an Arrow-Debreu economy in which there is some commodity that every agent has in positive supply. Suppose that one of the agents is endowed only in this good and cares only about consumption of this one good (in which he is never satiated), and that the rest have no utility for the good. Curiously, there can be no Arrow-Debreu equilibrium. If the good has positive price, then the first agent will demand exactly his endowment and the others nothing, so supply will exceed demand. If the price is zero, the first agent will demand an infinite amount of it, and so demand will exceed supply. If we think of the first agent as the banking sector, and the privileged commodity as fiat money, then we see that adding the banking sector to the standard general-equilibrium model can create still more problems: instead of getting an equilibrium in which money has a price of zero, we get an economy with no equilibrium at all!

Our model is therefore further enriched by postulating a cash-in-advance constraint along the lines of Clower 1965. All trades of commodities must be in exchange for money. Agents who wish to spend more than their endowments of cash can borrow money from the banking sector, at some endogenously determined interest rate. They repay the bank out of the cash receipts from the sale of goods. The banking sector is endowed with an exogenously fixed stock of money, M. In equilibrium, which exists under quite general conditions, the demand for money is equal to M, and the value of money is positive. By adding a banking sector and a cash-in-advance constraint to the traditional general-equilibrium model, we are able to show what neither modification alone can generate: that money has value because of its transactions role.

The crucial idea behind our analysis is that agents, who do not initially owe the bank anything, are driven by their own optimizing behavior to borrow and incur debts until their endowments are owed to the bank. More precisely, the money market for cash borrowing will clear only when the interest rate reaches at least a level θ so high that the money owed back to the bank, $(1 + \theta)M$, is at least equal to the sum $M + \sum_{\alpha \in H} m^\alpha$ of bank money and the private endowments $\{m^\alpha\}_{\alpha \in H}$ of fiat money owned by the agents. Although the result is superficially similar to the Lerner model, in that all outside money is finally owed to an external agency, each agent in our model begins with outside money, which he considers real wealth. There is no artificial requirement that the stock of money precisely balance government taxes.

When $\sum_{\alpha \in H} m^\alpha > 0$, the set of monetary equilibria is determinate; i.e., there are (generically) only a finite number of equilibria. Supply and demand determine not only the relative prices but also the level of prices and the interest rate. As $M/\sum_{\alpha \in H} m^\alpha \to \infty$, the interest rate goes to 0, and the monetary equilibrium commodity allocations converge to the Arrow-Debreu equilibrium allocations of the underlying nonmonetary economy.

Our existence theorem for monetary equilibrium provides a completely new proof of the existence of Arrow-Debreu equilibrium, by taking the limit as $M \to \infty$ (see Dubey and Geanakoplos 1989b). When $\sum_{\alpha \in H} m^\alpha = 0$, and agents honor all their debts, the price level is indeterminate, but all monetary equilibrium allocations are Arrow-Debreu allocations.

By contrast, in the Lerner model, the monetary equilibrium commodity allocations are always indeterminate. (See Balasko and Shell 1983.) Adding a cash-in-advance constraint implies that are also disjoint from the Arrow-Debreu commodity allocations.

In our model the banking sector "earns" a positive rate of interest on worthless paper by exploiting the agents' need to transact through money. If one thinks of the banking sector as a central bank, then it is easy to see that the bank could acquire real commodities in exchange for printed paper (exactly the opposite flow envisaged in the third approach to monetary equilibrium mentioned above, in which the government gives goods for money). Even when the banking sector is content to acquire only paper, its gains still cause a deadweight loss for the agents on account of the interest rate. Since a positive interest rate puts a wedge between buying and selling prices, it inhibits trade unless the gains to be made are sufficiently large. We are thus led to introduce a measure $\gamma^*(x)$ for the available gains to trade at

any allocation x, which may be of some value in its own right. Our existence theorem for monetary equilibrium (which implies a positive price for money) depends on the hypothesis that the gains to trade $\gamma^*(e)$ at the initial endowment e exceed $\sum_{\alpha \in H} m^\alpha / M$.

Our model displays some of the rudimentary properties of a full-fledged monetary economy when the private stocks of money are positive. Injections of bank money are not neutral. They tend to cause inflation, but to lower the interest rate. On the other hand, gifts of fiat money to agents also cause inflation, but raise the interest rate. The former injections are analogous to open-market operations and the latter to fiscal policy. In this simple model, injections of bank money tend to push the economy toward the Pareto frontier, but the resulting inflation may make agents with large endowments of fiat money worse off. As Friedman (1969) and Bewley (1980) suggested, albeit in different contexts, monetary equilibria for economies with $\sum_{\alpha \in H} m^\alpha > 0$ become efficient only when $M \to \infty$ and $\theta \to 0$. When $\sum_{\alpha \in H} m^\alpha = 0$, changes in the stock of bank money M are neutral. But the moment $\sum m^\alpha$ exceeds zero, equilibrium price levels become determinate, and money is no longer neutral, as explained above (except, of course, with respect to the joint scaling up of bank money and private endowments, which is tantamount to changing dollars into cents).

A more realistic model of a monetary economy would involve many possible periods of trade, and uncertainty. Money as a durable good would then have a store-of-value role as well as a transactions role to play. If there were other assets in the economy, money would have to compete with them in each agent's portfolio, and we could speak of the speculative demand for money. If we added production to the model, we could examine the effects of increasing the stock of money on firms' output. And if we further allowed for durable goods other than money, we could discuss the velocity of money as well. (None of these features is present in the current work; the interested reader is referred to Dubey and Geanakoplos 1989a and 1989c.)

Shubik and Wilson (1977) first introduced a banking sector into a general-equilibrium model. They computed the equilibrium in an example, but did not give a general statement or proof of the existence of equilibrium. Nor did they note the possibility that money would have positive value even if the stock of privately held money was positive. They did, however, introduce bankruptcy penalties to represent the idea that the banking sector might only imperfectly be able to enforce the repayment of its loans. In

their formulation, an agent suffered a loss of utility proportional to the number of dollars of his unpaid bank loan.

We investigate the existence of equilibrium when there are finite bankruptcy penalties. If the penalties are sufficiently high, then bankruptcy will not occur, and we are back in the situation described above. For lower penalty rates, bankruptcy must occur in equilibrium. Yet we are able to show that, no matter what level of penalty rates, so long as the rate is multiplied by the number of dollars owed, a monetary equilibrium will exist, given the above condition on gains to trade. Some agents will fail to repay the banks, but the equilibrium interest rate will be so high that the banking sector will recover the same amount of money in the aggregate as it did when there was no chance of default. The interpretation we make of the banking sector is that banks have no information about the reliability of particular customers and so stand ready to lend as much as is desired (up to their total holdings) to any customer at the going rate of interest. In the regime of low penalties, with bankruptcies occurring in equilibrium, scaling both the stock of bank money and the private endowments of money will have non-neutral effects (as has been noted by Shubik and Wilson) when the penalty rates are multiplied by the nominal debts.

We also consider real penalties, where the punishment depends on the size of the unpaid debt deflated by some price index. Here we find that there is a minimum threshold to the harshness of the penalty that is necessary to maintain the integrity of the banking system and the value of money. This threshold level is, however, low enough so that there can be considerable bankruptcy in equilibrium. Once we permit the harshness of the penalties to depend on the level of prices, it is only natural to extend the analysis to allow for them to depend on other macro variables, such as the volume of trade. We conclude by developing a much more general scheme for penalizing the nonpayment of debts and showing that the crucial property that must hold to guarantee the existence of monetary equilibrium, when there are sufficient gains to trade, is that the penalty becomes harsh as the volume of trade goes to zero.

1 The Economy with Fiat Money

Let $H = \{1,\ldots,h\}$ and $L = \{1,\ldots,\ell\}$ be the sets of agents and commodities, respectively. Thus, the commodity space may be viewed as $\mathbb{R}^L_+$, whose axes are indexed by the elements of L. Agent $\alpha \in H$ has initial endowment

$e^\alpha \in \mathbb{R}^L_+$ and utility function u^α: $\mathbb{R}^L_+ \to \mathbb{R}$. We assume that

$$e^\alpha = (e^\alpha_1, \ldots, e^\alpha_\ell) \neq 0,$$

$$\sum_{\alpha \in H} e^\alpha \gg 0,$$

and, further, each u^α is concave and continuously differentiable (we assume smoothness throughout only for ease of exposition), and

$$D_j u^\alpha \equiv \frac{\partial u^\alpha}{\partial x_j} > 0 \quad \text{for } j \in L, \alpha \in H, x \in \mathbb{R}^L_+.$$

There is fiat money in the economy, and it is the stipulated medium of exchange. Let $M > 0$ be the supply of money at the bank, and let $m^\alpha \geq 0$ be the private endowment of money for $\alpha \in H$.

To describe the budget set and the optimization problem faced by an agent α, it will be handy to represent the sequence of events as a tree (see figure 1). Let $\theta \in (-1, \infty)$ and $p \in \mathbb{R}^L_{++}$, where θ is the money rate of interest and p_j is the price of $j \in L$ (quoted in terms of money). Both θ and p are taken by α as exogenously fixed.

At the start, α borrows $c^\alpha \in \mathbb{R}_+$ units of money from the bank. Thus, α owes $\mu^\alpha = (1 + \theta)c^\alpha$ to the bank. Next α trades in the L commodities, using money for purchases. The choices available to him at this point are then given by

$$\left\{(b^\alpha, q^\alpha) \in \mathbb{R}^L_+ \times \mathbb{R}^L_+ : q^\alpha_j \leq e^\alpha_j \text{ for } j \in L; \text{ and } \sum_{j \in L} b^\alpha_j \leq c^\alpha + m^\alpha \equiv \frac{\mu^\alpha}{1+\theta} + m^\alpha\right\}.$$

Here b^α_j is the money spent by α for purchase of commodity j, and q^α_j is the

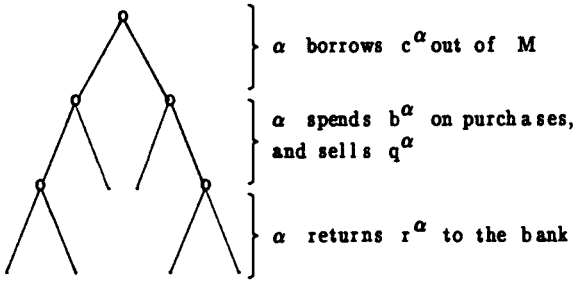

Figure 1

quantity of j that he puts up for sale. Thus, α winds up with the commodity bundle $x^\alpha \in \mathbb{R}^L_+$, where

$$x_j^\alpha = x_j^\alpha(b^\alpha,q^\alpha,p) = e_j^\alpha - q_j^\alpha + \frac{b_j^\alpha}{p_j} \tag{I}$$

for $j \in L$, and the amount of money

$$\bar{c}^\alpha = c^\alpha + m^\alpha - \sum_{j\in L} b_j^\alpha + \sum_{j\in L} p_j q_j^\alpha.$$

Finally α chooses to repay $r^\alpha \leq \bar{c}^\alpha$ on his loan. Consequently, his debt outstanding at the bank is

$$d^\alpha = d^\alpha(\mu^\alpha,r^\alpha) = \mu^\alpha - r^\alpha. \tag{II}$$

In summary, the *choice set* of agent α is

$$\Sigma^\alpha(\theta,p) = \Big\{(\mu^\alpha,b^\alpha,q^\alpha,r^\alpha) \in \mathbb{R}_+ \times \mathbb{R}^L_+ \times \mathbb{R}^L_+ \times \mathbb{R}_+ : q_j^\alpha \leq e_j^\alpha \text{ for } j \in L;$$

$$\sum_{j\in L} b_j^\alpha \leq \frac{\mu^\alpha}{1+\theta} + m^\alpha; r^\alpha \leq \frac{\mu^\alpha}{1+\theta} + m^\alpha - \sum_{j\in L} b_j^\alpha + \sum_{j\in L} p_j q_j^\alpha\Big\}.$$

The outcome functions x_j^α and d^α are continuous functions from

$$\Sigma^\alpha(\theta,p) \times \mathbb{R}^L_{++}$$

into $\mathbb{R}_+$, and for any fixed $(\theta,p) \in (-1,\infty) \times \mathbb{R}^L_{++}$ they are linear on $\Sigma^\alpha(\theta,p)$.

It may help to illustrate the attainable consumption bundles of α when $h = 2$ and $\ell = 2$ for fixed p and $\theta > 0$. (See dark lines in figure 2 and 3.)

Note that when $\theta = 0$, $m^\alpha = 0$, and α chooses to have $d^\alpha = 0$, figures 2 and 3 reduce to the classical Walrasian budget set.

Note also that, for any fixed (p,θ,d^α), the consumption set is convex if we allow free disposal.

We must now specify the utility U^α, to α, of the outcome (x^α,d^α). Were there no incentive to return money to the bank, the agent would always choose $r^\alpha = 0$ and b^α and d^α very large. Prices would be driven to infinity, and the value of money to zero. It almost goes without saying that, since agents are not rationed in borrowing from the bank, there must be some compulsion to return their loans if money is to have value. We model this

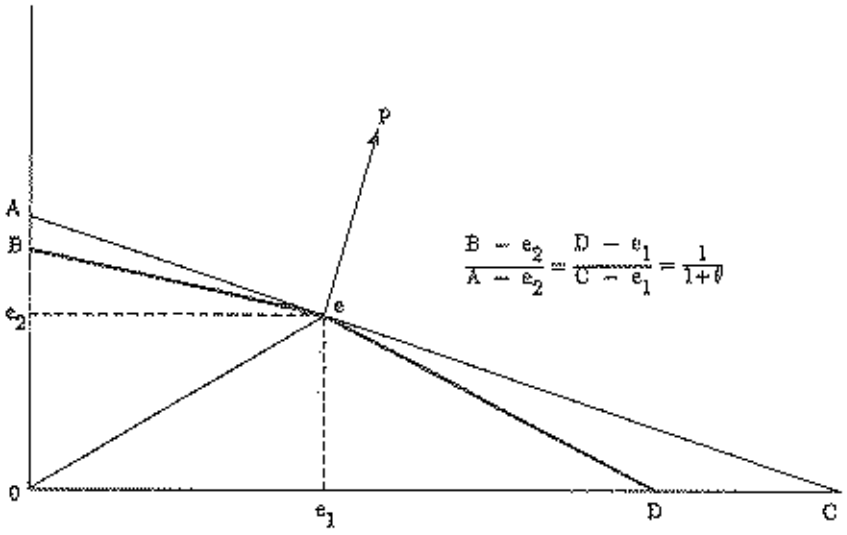

Figure 2
($m^s = 0, d^s = 0$.)

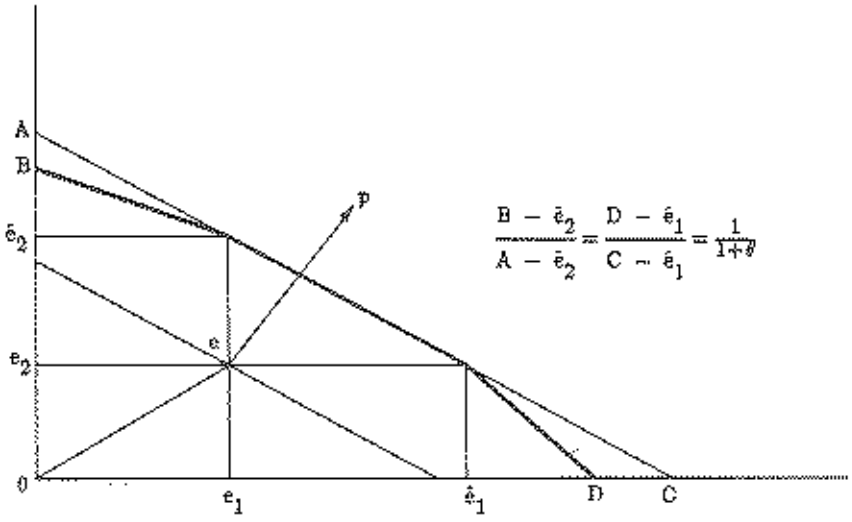

Figure 3
($m^s > 0, d^s \geq 0$.)

through U^α, in terms of an "extra-economic penalty" that is levied on α when $d^\alpha > 0$, and that has the effect of reducing α's utility. The simplest form for U^α is

$$U^\alpha(x^\alpha,d^\alpha) = u^\alpha(x^\alpha) - \lambda^\alpha d_+^\alpha,$$

where $c_+ \equiv \max\{c,0\}$ for any real number c. (Note that, for fixed θ and p, U^α is a concave function of the outcome variables (x^α,d^α), which in turn are linear in the choice variables $(\mu^\alpha,b^\alpha,q^\alpha,r^\alpha)$.)

The function U^α incorporates a bankruptcy penalty via a flat rate λ^α. The penalty increases in harshness directly with the size of the debt. By thinking of λ^α as infinity, or a fixed constant, or a function of prices, we can describe many different kinds of bankruptcy laws.

Case 0: Infinite Penalties

If we imagine $\lambda^\alpha = \infty$, or indeed sufficiently large λ^α, agent α will never choose to go bankrupt, and his budget set Σ^α is effectively modified by the requirement that $d^\alpha = 0$, i.e., $r^\alpha = \mu^\alpha$. This may be reformulated as follows. Let

$$z = x^\alpha - e^\alpha \in \mathbb{R}^L$$

denote the net trade. Then the net trades available to α are

$$\left\{z \in \mathbb{R}^L: e^\alpha + z \geq 0, p\cdot z^+ + \frac{1}{1+\theta} p\cdot z^- \leq m^\alpha\right\},$$

where $z_j^+ \equiv \max\{0,z_j\}$ and $z_j^- \equiv \min\{0,z_j\}$.

When $\theta = 0$, this is the conventional Walrasian budget set. For $\theta > 0$, the available net trades are reduced. There is a "float loss" caused by the fact that purchases require the use of money. Since purchases and sales occur simultaneously, in markets where goods trade for money and not directly for each other, the money receipts from sales are obtained too late to be used directly for the purchases. This makes it necessary for α to borrow $(p\cdot z^+ - m^\alpha)_+$ from the bank. To avoid bankruptcy (i.e., to ensure $d^\alpha = 0$), α must choose

$$(1+\theta)(p\cdot z^+ - m^\alpha)_+ \leq -p\cdot z^-.$$

(See figures 2 and 3.)

Case I: Nominal Penalties

Let $\lambda^\alpha > 0$ be a fixed scalar. In this case agents may choose to go bankrupt (i.e., choose $d^\alpha > 0$), and they are punished by a utility loss proportional to the number of dollars they fail to return, regardless of the prevailing price levels. This is not fanciful. Historically, bankruptcy laws and penalties (such as the threshold above which a misdemeanor becomes a felony) have not quickly adjusted to inflation. Such nominal penalties were, as we said, first introduced by Shubik and Wilson (1977).

Case II: Real Penalties

Let $a \in \mathbb{R}^L_+$, $a \neq 0$, and

$$U^\alpha(x^\alpha,d^\alpha,p) \equiv u^\alpha(x^\alpha) - \frac{\lambda^\alpha}{p \cdot a} d^\alpha_+.$$

Here we allow for bankruptcy penalties to adjust instantaneously to the price level. The vector $a \in \mathbb{R}^L_+ \setminus \{0\}$ serves as a "price deflator" with respect to which penalties are measured.

Case III: Endogenous Penalties

Once we have admitted the possibility that the penalty rates can depend on price levels, as in case II, it is only natural to consider penalties that depend on other macro-variables of the economy, such as interest rates, the total volume of trade, or the total volume of debt. To describe such penalties, let $A = \{(\theta,p,x) \in (-1,\infty) \times \mathbb{R}^L_+ \times (\mathbb{R}^L_+)^H\}$ denote the set of all interest rates, prices, and allocations, and let

$$\eta: A \to \Omega$$

be a map from A into a (Euclidean) signal space Ω. For each $\alpha \in H$, consider

$$\lambda^\alpha: (-1,\infty) \times \mathbb{R}^L_+ \times \Omega \to \mathbb{R}_{++}.$$

Both η and λ^α are assumed continuous. The term $\lambda^\alpha(\theta,p,\omega)$ is interpreted to be the penalty rate for α when interest rates θ and prices p prevail, and when other relevant macrovariables in the economy are according to ω. Thus, the utility to α is

$$U^\alpha(x^\alpha,d^\alpha,\theta,p,\omega) = u^\alpha(x^\alpha) - \lambda^\alpha(\theta,p,\omega)d^\alpha_+.$$

One scenario that is of central importance in the current case occurs if penalty rates are harsh when the total volume of trade in the economy is close to zero, and progressively more lenient as the volume expands.

2 Monetary Equilibrium

We say that $\{\theta,p,y\}$, where $y = (y^1,\ldots,y^h)$ and $y^\alpha \in \mathbb{R}^L_+$ for $\alpha \in H$, is a *monetary equilibrium* (M.E.) if there exist $(\mu^\alpha,b^\alpha,q^\alpha,r^\alpha)_{\alpha\in H}$ such that

(i) for $\alpha \in H$,

(a) $(\mu^\alpha,b^\alpha,q^\alpha,r^\alpha) \in \underset{(\mu,b,q,r)\in\sum^\alpha(\theta,p)}{\text{Arg max}} U^\alpha(x^\alpha(b,q,p), d^\alpha(\mu,r),\theta,p,\omega)$

(b) $y^\alpha = x^\alpha(b^\alpha,q^\alpha,p)$,

(ii) $\omega = \eta(\theta,p,y)$,

(iii) $\sum_{\alpha\in H} y^\alpha = \sum_{\alpha\in H} e^\alpha$,

(iv) $\sum_{\alpha\in H} \dfrac{\mu^\alpha}{1+\theta} = M$.

In other words, each agent optimizes (i), correctly anticipating the macrovariables ω (ii); commodity markets clear (iii); and the money market clears (iv).

By the very definition of an M.E. we guarantee that, if it exists, money will have positive value. (Commodity prices are not allowed to be infinite at an M.E.)

But the only role for fiat money in our model is to facilitate trade. When the endowments are Pareto optimal there are no gains to trade. In such a situation, money cannot be expected to have positive value, and M.E. will not exist.

Furthermore, the presence of money does not of itself make for frictionless trade. For instance, suppose the interest rate is positive. Then, as already discussed (see figures 2 and 3), a float loss is imposed which inhibits trade unless the gains to be made are sufficiently large.

It turns out that if the initial endowment of fiat money is positive, a positive interest rate is inevitable in any M.E. No agent will want to hold money at the end, since it is fiat and has no direct utility of consumption for him. Thus, all of $M + \sum_{\alpha\in H} m^\alpha$ will be returned to the bank at the end of trade. On the other hand, no agent will return more than he owes (otherwise he could spend further on purchases without going bankrupt, and improve his utility), so that at least $M + \sum_{\alpha\in H} m^\alpha$ is owed to the bank.

The upshot is that $\sum_{\alpha \in H} m^\alpha / M$ represents a "floor" on the money rate of interest, which is achieved precisely when all debts d^α are 0 for $\alpha \in H$.

3 A Measure of Gains from Trade

Positive endowments of money imply a positive money interest rate, and a positive money rate of interest inhibits trade. It follows that the question of sufficient gains to trade must arise in the analysis of existence of M.E. We now give a measure of the gains to trade available at any allocation $y \equiv (y^1, \ldots, y^h)$.

DEFINITION Given utilities $(u^\alpha)_{\alpha \in H}$ and $\gamma \in \mathbb{R}_+$, define the allocation $(y^1, \ldots, y^h)$ to be *γ-Pareto optimal* iff there do not exist feasible net trades $z^1, \ldots, z^h$ (i.e., $z^\alpha \in \mathbb{R}^L$, $y^\alpha + z^\alpha \in \mathbb{R}^L_+$ for all $\alpha \in H$; and $\sum_{\alpha \in H} z^\alpha = 0$) such that

$$u^\alpha_j(z) \equiv u^\alpha(y^\alpha + z^\alpha(\gamma)) \geq u^\alpha(y^\alpha)$$

for all $\alpha \in H$, with strict inequality for at least one α; where $z^\alpha_j(\gamma) = \min\{z^\alpha_j, z^\alpha_j/(1+\gamma)\}$ for $\alpha \in H, j \in L$. If $z^\alpha_j > 0$, then $z^\alpha_j(\gamma) < z^\alpha_j$, while $z^\alpha_j(\gamma) = z^\alpha_j$ for $z^\alpha_j < 0$. (Recall that positive (negative) components of z^α denote purchases (sales) by α.)

Figure 4 illustrates an allocation that is γ-Pareto optimal but not Pareto optimal.

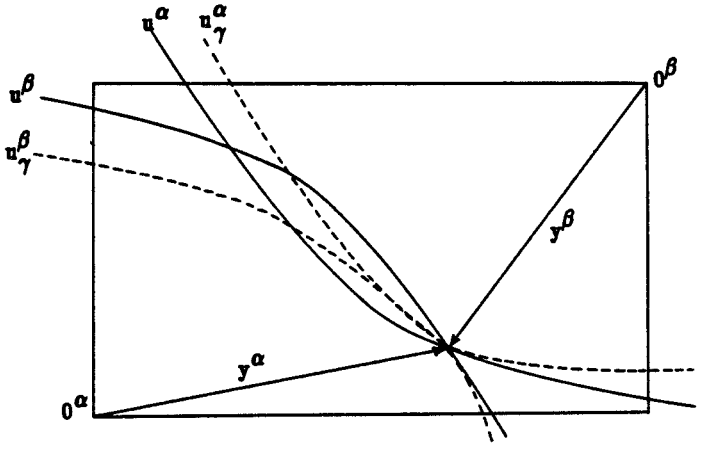

Figure 4

The indifference curve for u_j^α through y^α lies strictly inside the preferred set for u^α through y^α. Moreover, it has a kink at y^α.

Remark 1 If $(y^1,\ldots,y^h)$ is not Pareto optimal, then for small enough $\gamma > 0$, $(y^1,\ldots,y^h)$ is still not γ-Pareto optimal. On the other hand, for large enough γ, $(y^1,\ldots,y^h)$ is γ-Pareto optimal. Note that for any allocation $y = (y^1,\ldots,y^h)$ there is a minimum $\gamma^* \geq 0$ for which y is γ-Pareto optimal. *This $\gamma^* \equiv \gamma^*(y)$ can be thought of as a measure of how far y is from Pareto optimality.*

Remark 2 We could also define γ-Pareto optimality of $(y^1,\ldots,y^h)$ by requiring that there are no feasible trades $(z^1,\ldots,z^h)$ such that $u^\alpha(y^\alpha + z^\alpha(\gamma)) > u^\alpha(y^\alpha)$ for all $\alpha \in H$. By the usual arguments, this coincides with the other definition, since utilities are strictly monotonic.

Remark 3 The initial endowment allocation $e = (e^1,\ldots,e^h)$ is γ-Pareto optimal if and only if there exists $p \gg 0$ such that for all $\alpha \in H$

$$0 \in \operatorname*{Arg\,max}_{z \in R^L} \{u^\alpha(e^\alpha + z(\gamma)) : e^\alpha + z \geq 0, p \cdot z \leq 0\}.$$

Equivalently,

$$0 \in \operatorname*{Arg\,max}_{z \in R^L} \{u^\alpha(e^\alpha + z)) : e^\alpha + z \geq 0, (1+\theta)p \cdot z^+ + p \cdot z^- \leq 0\}.$$

To see why this is so, let

$$u_\gamma^\alpha(z) \equiv u^\alpha(e^\alpha + z(\gamma)) \text{ for } z \geq -e^\alpha.$$

Since the map from $z_j \to z_j(\gamma)$ is coordinate-wise concave, and u^α is strictly monotonic and concave, it follows that u_γ^α is concave. Moreover, it is trivial to check that u_γ^α is strictly monotonic. Remark 3 therefore follows from the standard first and second welfare theorems applied to the utilities u_γ^α.

4 The Existence of Monetary Equilibrium with Nominal or Real Penalties

The crucial condition we place on the economy is the following:

(CI) The initial endowment is not $\sum_{\alpha \in H} m^\alpha/M$-Pareto optimal; i.e., $\gamma^*(e) > \sum_{\alpha \in H} m^\alpha/M$.

Remark 4 If $\gamma = 0$, the notion of γ-Pareto optimality reduces to standard Pareto optimality. Thus, if the initial endowment of fiat money is zero (i.e., $m^\alpha = 0$ for $\alpha \in H$), then CI is tantamount to requiring that the initial endowment is not Pareto optimal.

Remark 5 It is evident that CI rules out the case where $\#L = 1$.

Our first result is now easily stated.

THEOREM 1 Consider case I with nominal penalties. Suppose condition CI holds. Then so long as $(\lambda^\alpha)_{\alpha \in H} \gg 0$, an M.E. exists.

To guarantee the existence of M.E. when the debts are measured in real terms, penalty rates cannot be allowed to be arbitrarily low. Let $Du^\alpha = (D_1 u^\alpha, \ldots, D_\ell u^\alpha)$ denote the gradient of u^α, and let $\cdot$ stand for dot product.

(CII) For $\alpha \in H$: $a_j > 0 \Rightarrow e_j^\alpha > 0$ for $j \in L$, and $(Du^\alpha(e^\alpha)) \cdot a < \lambda^\alpha$.

THEOREM 2 Consider case II with real penalties. Suppose conditions CI and CII hold. Then an M.E. exists.

THEOREM 3 Consider the case of either real or nominal penalties (case I, II). For all sufficiently large λ^α, no agent goes bankrupt at any M.E. In particular, equilibrium exists in case 0 of infinite λ^α under condition CI.

COROLLARY Let $m^\alpha = 0$ for $\alpha \in H$. For all sufficiently large λ^α, every M.E. yields Walrasian prices and allocations (in cases I and II, and obviously also in case 0).

Remark 6 An M.E. with nominal penalties can always be interpreted as an M.E. with real penalties suitably adjusted to reflect the price levels, and vice versa. However theorems 1 and 2 are not equivalent, since one must know the price levels to effect the transformation between nominal and real penalties, and the price levels are endogenous to the M.E.

5 General Bankruptcy Penalties

We could now turn to case III, which includes cases 0, I, and II, and develop the analogue of CII to ensure the existence of M.E. But in fact we have a still more general setup, of which the existence theorems 1 and 2 are corollaries. In addition to allowing for bankruptcy penalties that depend upon macrovariables, we no longer require that the penalty is linear or even separable from consumption.

Consider

$$U^\alpha: \mathbb{R}^L_+ \times \mathbb{R} \times (-1,\infty) \times \mathbb{R}^L_{++} \times \Omega \to R$$

and assume the following:

(i) U^α is continuous.

(ii) $U^\alpha(x^\alpha,d^\alpha,\theta,p,\omega)$ is concave in (x^α,d^α) for any fixed choice of θ, p, ω.

(iii) $U^\alpha(x^\alpha,d^\alpha,\theta,p,\omega) = u^\alpha(x^\alpha)$ if $d^\alpha \leq 0$.

(iv) (a) $\partial U^\alpha/\partial d^\alpha$ exists, is continuous, and is negative if $d^\alpha > 0$.

(b) $\dfrac{\partial U^\alpha}{\partial x_j}(x^\alpha,d^\alpha,\theta,p,\omega)$ exists, is continuous and positive; and

$$\sup\left\{\frac{\partial U^\alpha}{\partial x_j}(x^\alpha,d^\alpha,\theta,p,\omega)\colon j \in L,\ x^\alpha \text{ lies in a compact set},\ d^\alpha \in \mathbb{R},\ \theta \in (-1,\infty),\ \omega \in \Omega\right\} < \infty,$$

and

$$\inf\left\{\frac{\partial U^\alpha}{\partial x_j}(x^\alpha,d^\alpha,\theta,p,\omega)\colon j \in L,\ x^\alpha \text{ lies in a compact set},\ d^\alpha \in \mathbb{R},\ \theta \in (-1,\infty),\ \omega \in \Omega\right\} > 0.$$

(v) For any trader $\alpha \in H$, and any sequence $(x^\alpha(n), p(n))$ that is uniformly bounded, suppose that $(d^\alpha(n), \theta(n)) \to \infty$. Then

$$\liminf_n U^\alpha((x^\alpha(n), d^\alpha(n), \theta(n), p(n), \eta(\theta(n), p(n), x(n))) < u^\alpha(e^\alpha).$$

Conditions (i) and (ii) are technical. The remaining three conditions incorporate the essential properties of fiat money and bankruptcy penalties: fiat money has no direct utility of consumption (iii); unpaid bank debts are penalized and cause disutility (iv); if prices and consumptions stay bounded, then for sufficiently large debts and interest rates the bankruptcy penalties are harsh enough that an agent would prefer not to have traded at all (v).

This generalized setup clearly includes all the previous cases, without relying on the artificial separability between consumption and bankruptcy penalties. It might well be, for example, that the penalty represents a term in jail, and that could in turn affect the relative marginal utilities of consumption goods.

The analogue of condition CII in the general setting is as follows. Let

$$\lambda^\alpha(p) = \inf\left\{\left|\frac{\partial U^\alpha}{\partial d^\alpha}(e^\alpha, d^\alpha, \theta, p, \eta(\theta,p,e))\right| : d^\alpha > 0, \theta \geq 0\right\}$$

and

$$v^\alpha(p) = \sup_{d^\alpha,\theta} \inf_{a} \left\{ \frac{\left(\frac{\partial U^\alpha}{\partial x}(e^\alpha, d^\alpha, \theta, p, \eta(\theta,p,e))\right)\cdot a}{p\cdot a} : a \in \mathbb{R}^L_+ \setminus \{0\}, \right.$$

$$\left. a \leq e^\alpha, d^\alpha > 0 \text{ and } \theta \geq 0 \right\}.$$

Then require, for every sequence $p \to \infty$,

$$\text{(CIII)} \quad \limsup_{p\to\infty} \frac{\lambda^\alpha(p)}{v^\alpha(p)} > 1.$$

Note that $\lambda^\alpha(p)$ is the minimum utility loss to α if he owes a dollar to the bank and goes bankrupt (when no trade is taking place in the economy). On the other hand, $v^\alpha(p)$ is the utility loss to α if he raises a dollar by forgoing consumption (and making sales) at his initial endowment. Thus, condition (CIII) states that if

$p_j \to \infty$ for all $j \in L$

and

$x^\alpha \to e^\alpha$ for all $\alpha \in H$,

then the marginal disutility of a dollar's unpaid debt > the marginal disutility of a dollar's credit (taken out of consumption) for each $\alpha \in H$.

THEOREM 4 Consider the case of general penalties. Suppose conditions CI and CIII hold. Then an M.E. exists.

Remark 7 (Neutrality of Money) If we double M and each m^α, then clearly the M.E. remain unaffected in the case of real penalties, except that all prices are also doubled. This happens in the nominal case as well, except that it is now also necessary to halve the penalty rates λ^α (unless the λ^α were

sufficiently high to have ruled out bankruptcy in the first place). These changes are analogous to a change in units, and it would be surprising indeed if a switch from dollars to cents caused rational agents to behave differently. But it is worth reiterating that even such a change in units will be non-neutral if the "courts" are not directed (in the nominal case) to suitably adjust the bankruptcy laws (i.e., the λ^α).

Remark 8 (Non-Neutrality of Money) Suppose the penalities are harsh enough to rule out bankruptcy (in accordance with theorem 3)—indeed, take $\lambda^\alpha = \infty$ for all α—and that $\sum_{\alpha \in H} m^\alpha > 0$. Then the interest rate is $\sum_{\alpha \in H} m^\alpha / M$ at any M.E. Thus, an injection of bank money (with endowments of money held fixed) will lower the interest rate, while gifts of fiat money to agents (hold the supply of bank money fixed) will raise it. In either case there will be inflation and the M.E. allocation of commodities will be affected.

When there is bankruptcy, the effect of injections of bank money or private endowments will be non-neutral, and the effects on the interest rates and inflation will tend to be in the direction specified above, but not necessarily always.

Remark 9 (Dropping the Gains-from-Trade Hypothesis) Consider an economy for which the initial endowment is not Pareto optimal. As $M \to \infty$,

$$\sum_{\alpha \in H} m^\alpha / M \to 0$$

and we reach a level M^* such that condition CI holds automatically for all $M \geq M^*$, ensuring (by theorems 1, 2, and 4) the existence of an M.E. (given, of course, condition CII or CIII for real or general penalties).

More generally, let us fix utilities and the m^α and suppose that endowments vary in some hypercube in $(\mathbb{R}^L_+)^H$. Then the "volume" of economies for which condition CI fails (and hence, possibly, M.E. also fail to exist) will go to 0 as $M \to \infty$.

Remark 10 (M.E. vs. Walrasian Equilibria) Suppose throughout there is no bankruptcy (e.g., $\lambda^\alpha = \infty$). If $\sum_{\alpha \in H} m^\alpha = 0$, then the interest rate $\theta = 0$, and it is immediate that the M.E. prices and allocations are Walrasian. If $\sum_{\alpha \in H} m^\alpha > 0$, then $\theta = \sum m^\alpha / M \to 0$ as $M \to \infty$. It follows that the wedge between buying and selling prices approaches zero, and the M.E. outcomes converge to Walrasian. (Indeed, if we specialize to the nominal case, then,

as $M \to \infty$, bankruptcies will disappear even if the penalties were not harsh enough to begin with, and Walrasian outcomes will occur in the limit.) Thus, our existence theorem for M.E. provides an alternative proof of the existence of Walras equilibrium (see Dubey and Geanakoplos 1989b).

Remark 11 (Distinction of M.E. from Walras Equilibria) We have already seen that when $\sum_{\alpha \in H} m^\alpha > 0$, the interest rate $\theta > 0$ and then the M.E. outcomes are not Walrasian.

Let $\sum_{\alpha \in H} m^\alpha = 0$. At any Walras equilibrium there is an implicit penalty rate for each agent, given by his "marginal utility of income" (with $p \cdot a \equiv$ one unit of income). Thus, if we choose $\eta(\theta,p,x) \equiv \eta(x)$ and each $\lambda^\alpha(d^\alpha, p, \eta(x))$ strictly smaller than these implicit rates, as (p,x) ranges over all Walras equilibria, no M.E. (in the case of real penalties) will be Walrasian. Thus, our model provides a new type of equilibrium, distinct from the Walrasian, in which liquidity constraints come to the fore. (Of course, under the conditions given in remark 10, we get Walras equilibrium as a special case of M.E.)

6 Determinacy of M.E. and the Value of Money

Let $\lambda^\alpha = \infty$ for $\alpha \in H$, and $\sum_{\alpha \in H} m^\alpha = 0$. As we saw, the M.E. are Walrasian; but it is equally clear that there is great indeterminacy of the commodity price levels: they can be scaled down arbitrarily (with agents hoarding increasing amounts of bank money M) without disturbing the M.E.

But the moment $\sum_{\alpha \in H} m^\alpha > 0$ we must have $\theta = \sum_{\alpha \in H} m^\alpha / M > 0$. Consequently there is no hoarding and the above indeterminacy abruptly disappears. In particular, the value of money (given by the price levels) is determinate.

We can state this intuition formally as follows. Let $\mathcal{U}$ be a finite-dimensional vector space of infinitely differentiable utilities which satisfies the property that if $c \in \mathbb{R}^L$ is small enough, and $u \in \mathcal{U}$, then $v \in \mathcal{U}$, where v is defined by $v(x) \equiv u(x) + c \cdot x$.

THEOREM 5 For an open, dense, and full measure set $\mathcal{E}$ of vectors

$$E = (M, (\lambda^\alpha, e^\alpha, m^\alpha, u^\alpha)_{\alpha \in H}) \in \mathbb{R}_{++} \times (\mathbb{R}_{++} \times \mathbb{R}^L_{++} \times \mathbb{R}_{++} \times \mathcal{U})^H,$$

the set of M.E. for the economy defined by E is finite in number, where λ^α refers to either real or nominal bankruptcy penalties.

Remark 12 This theorem applies whether or not there is bankruptcy.

7 Exogenous Debts

We can extend our model to allow for the possibility that agents have exogenous debts to the bank, on account of unpaid loans from the past (which have been rolled over into the current period). Let Δ^α denote such a debt of $\alpha \in H$. If Δ^α is positive, then this means that α owes Δ^α to the bank; if it is negative, the bank owes $-\Delta^\alpha$ to α. (In the interpretation, agents may owe each other money, with the banking sector acting like a "clearing house" through which all debts are channeled.)

Define M.E. exactly as before, with one amendment: d^α is replaced by $d^\alpha + \Delta^\alpha$ throughout, in the argument of U^α, for all $\alpha \in H$. Thus, the payment of Δ^α is called for at the end of trade, and the current debt d^α (incurred in trade) is adjusted by the exogenous debt Δ^α, to arrive at the net final debt $(d^\alpha + \Delta^\alpha)_+$ of α to the bank.

We extend the gains-from-trade hypothesis (CI) to allow for debts:

(CI*) The initial endowment is not

$$\left(\frac{\sum_{\alpha \in H} m^\alpha - \sum_{\alpha \in H} \Delta^\alpha}{M}\right)_+ \text{-Pareto optimal,}$$

i.e.,

$$\gamma^*(e) > \left(\frac{\sum_{\alpha \in H} m^\alpha - \sum_{\alpha \in H} \Delta^\alpha}{M}\right)_+ .$$

THEOREM 6 Consider the case of general penalties. Suppose conditions CI* and CIII hold, and furthermore

$$\sum_{\alpha \in H} m^\alpha - \sum_{\alpha \in H} \Delta^\alpha \geq 0.$$

Then an M.E. exists. Moreover, if penalties are nominal or real (cases II and III), then for large enough $(\lambda^\alpha)_{\alpha \in H}$ there is no bankruptcy at any M.E.

With nominal penalties, existence is guaranteed no matter how large, or what the sign, of the Δ^α.

THEOREM 7 If penalties are nominal and CI* holds, then for any $\{\Delta^\alpha\}_{\alpha \in H}$ an M.E. exists.

Theorem 7 holds without the artificial assumption of separability between consumption and penalties. We now define general nominal penalties, and then restate the theorem.

DEFINITION $U^\alpha(x^\alpha,d^\alpha,\theta,p,\omega)$ corresponds to *general nominal penalties* if, in place of conditions iva and v of general penalties, we assume that for $x^\alpha \leq \sum_{\beta \in H} e^\beta$ and any θ, p, ω,

(i)** there exists $\bar{c} > 0$ such that, for $d^\alpha + \Delta^\alpha > 0$,

$$\text{(a)} \quad -\frac{\partial U^\alpha}{\partial d^\alpha}(x^\alpha, d^\alpha + \Delta^\alpha, \theta, p, \omega) < \bar{c}$$

$$\text{(b)} \quad -\frac{\partial U^\alpha}{\partial d^\alpha}(e^\alpha, d^\alpha + \Delta^\alpha, \theta, p, \eta(\theta,p,e)) > 1/\bar{c}.$$

(ii)** there exists $\bar{D} > 0$ such that, for $d^\alpha > \bar{D}$,

$$U^\alpha(x^\alpha,d^\alpha + \Delta^\alpha,\theta,p,\omega) < U^\alpha(e^\alpha,\Delta^\alpha,\theta,p,\omega).$$

THEOREM 8 Consider the case of general *nominal* penalties. Suppose condition CI* holds. Then, for any $\{\Delta^\alpha\}_{\alpha \in H}$ an M.E. exists.

8 Proofs

Proof of Theorem 4 For any $\varepsilon > 0$, we construct an extensive game Γ^ε, with a continuum of players, that is partitioned into h types. An M.E. is then obtained as a limit of plays of type-symmetric strategic equilibria of Γ^ε, $\varepsilon \to 0$.

The set of players is the interval $[0,h)$ equipped with the Lebesgue measure, and for $t \in [\alpha - 1, \alpha)$ we have

$$e^t = e^\alpha, \quad m^t = m^\alpha, \quad U^t = U^\alpha,$$

where $1 \leq \alpha \leq h$. At the start, all $t \in [0,h)$ simultaneously send an "I.O.U. note" $\mu^t \in [\varepsilon, 1/\varepsilon]$ to the bank. The interest rate θ forms according to the rule

$$1 + \theta = \frac{\int_0^h \mu^t \, dt}{M}, \tag{1}$$

and t obtains

$$c^t = \frac{\mu^t}{1+\theta} \tag{2}$$

units of bank money. In the second stage of the game, all $t \in [0,h)$ simultaneously choose $(b^t,q^t) \in \mathbb{R}^L_+ \times \mathbb{R}^L_+$ subject to the constraints

$$\sum_{j \in L} b^t_j \leq \varepsilon + m^t + c^t \tag{3}$$

and

$$q^t_j \leq e^t_j. \tag{4}$$

Here b^t_j is the money sent by t for purchase of commodity j, and q^t_j is the quantity of j put up by him for sale. In the final stage of the game, each $t \in [0,h)$ returns all the money he has to the bank, which is optimal for him since money yields no utility. For each $j \in L$, the price p_j is formed by

$$p_j = \frac{\varepsilon + \int_0^h b^t_j \, dt}{\varepsilon + \int_0^h q^t_j \, dt} \tag{5}$$

and t obtains the final bundle $x^t \in \mathbb{R}^L_+$, and the debt d^t, as given by the right-hand sides of equations (I) and (II) in section 1 (after substituting t for α, and $\varepsilon + m^t$ for m^α, and the money t has on hand after trade for r^α); i.e.,

$$x^t_j = e^t_j - q^t_j + \frac{b^t_j}{p_j} \text{ for } j \in L, \tag{6}$$

$$d^t = \mu^t - \left(c^t + m^t + \varepsilon - \sum_{j \in L} b^t_j + \sum_{j \in L} p_j q^t_j \right). \tag{7}$$

The payoff to t is then

$$U^t(x^t,d^t,\theta,p,\omega),$$

where

$$\omega = \eta\left(\theta, p, \int_0^1 x^t \, dt, \ldots, \int_{h-1}^h x^t \, dt\right).$$

Finally, assume that at the end of stage 1 each t observes the interest rate θ. This completes the description of the extensive game Γ^e.

A strategic equilibrium (S.E.) is a choice of strategies by all the players in $[0,h)$ such that no player can improve his payoff by a unilateral change

of strategy. It is called type-symmetric (and denoted T.S.S.E.) if, for each $\alpha \in H$, all $t \in [\alpha - 1, \alpha)$ choose the same strategy.

Note that in the game Γ^ε the effect of the "ε-perturbation" is threefold:

(a) Each player has an extra endowment ε of fiat money.

(b) Each player is constrained to keep his I.O.U. note within the range $[\varepsilon, 1/\varepsilon]$.

(c) In each commodity market, an "external agency" supplies ε units of money and ε units of the commodity.

CLAIM 1 A T.S.S.E. exists in Γ^ε for any $\varepsilon > 0$.

Proof Each $t \in [0,h)$ chooses $\mu^t \leq 1/\varepsilon$, and θ forms as in (1), so we must have

$$\frac{h\varepsilon}{M} \leq 1 + \theta \leq \frac{h/\varepsilon}{M}.$$

So consider a "generalized game" in which each $t \in [\alpha - 1, \alpha)$ has the *ambient* strategy set

$$\sum^\alpha = \left\{(\mu^\alpha, b^\alpha, q^\alpha) \in \mathbb{R}_+ \times \mathbb{R}_+^L \times \mathbb{R}_+^L : \varepsilon \leq \mu^\alpha \leq \frac{1}{\varepsilon},\ q^\alpha \leq e^\alpha,\ \sum_{j \in L} b_j^\alpha \leq \varepsilon + m^\alpha + \frac{\mu^\alpha}{h\varepsilon/M}\right\},$$

which is compact and convex. For a *type-symmetric* choice

$$\sigma = (\sigma^1, \ldots, \sigma^h) \in \Sigma^1 \times \cdots \times \Sigma^h$$

by the players in $[0,h]$, where $\sigma^\alpha = (\mu^\alpha, b^\alpha, q^\alpha)$, the formulas for θ, p, etc. simplify to the following:

$$1 + \theta(\sigma) = \frac{\sum_{\alpha \in H} \mu^\alpha}{M},$$

$$p_j(\sigma) = \frac{\varepsilon + \sum_{\alpha \in H} b_j^\alpha}{\varepsilon + \sum_{\alpha \in H} q_j^\alpha}, \quad j \in L$$

$$x_j^\alpha(\sigma) = e_j^\alpha - q_j^\alpha + \frac{b_j^\alpha}{p_j(\sigma)}, \quad j \in L \text{ and } \alpha \in H$$

$$\eta(\sigma) = \eta(\theta(\sigma), p(\sigma), x^1(\sigma), \ldots, x^h(\sigma)).$$

Note that θ, p, and η are continuous functions of σ. Also, for $\sigma \in \Sigma^1 \times \cdots \times \Sigma^h$ and $\alpha \in H$, define

$$\Sigma^\alpha(\sigma) = \left\{(\mu^\alpha, b^\alpha, q^\alpha) \in \Sigma^\alpha : \sum_{j \in L} b_j^\alpha \leq \varepsilon + m^\alpha + \frac{\mu^\alpha}{1 + \theta(\sigma)}\right\}$$

to be the *feasible* strategy set of $t \in [\alpha - 1, \alpha)$ when all others choose strategies according to σ. Since, as already remarked,

$$1 + \theta(\sigma) \geq h\varepsilon/M \text{ for } \sigma \in \Sigma^1 \times \cdots \times \Sigma^h,$$

we have

$$\Sigma^\alpha(\sigma) \subset \Sigma^\alpha$$

for all such σ. It is also clear that $\Sigma^\alpha(\sigma)$ is compact and convex, and continuous in σ. Now define

$$\tilde{\Sigma}^\alpha(\sigma) = \underset{\tau \in \Sigma^\alpha(\sigma)}{\text{Arg max}}\, U^\alpha(x(\tau,\sigma), d(\tau,\sigma), \theta(\sigma), p(\sigma), \eta(\sigma)),$$

where, with $\tau \equiv (\mu, b, q)$, we mean

$$x_j(\tau,\sigma) = e_j^\alpha - q_j + \frac{b_j}{p_j(\sigma)} \quad \text{for } j \in L,$$

$$d(\tau,\sigma) = \mu - \left(\varepsilon + m^\alpha + \frac{\mu}{1 + \theta(\sigma)} - \sum_{j \in L} b_j + \sum_{j \in L} p_j(\sigma) q_j\right).$$

In other words, $\tilde{\Sigma}^\alpha(\sigma)$ is the best-response set of $t \in [\alpha - 1, \alpha)$, when others make the (type-symmetric) choice of σ, so that t now faces *fixed* $\theta(\sigma)$, $p(\sigma)$, and $\eta(\sigma)$. The map

$$\tau \mapsto U^\alpha(x(\tau,\sigma), d(\tau,\sigma), \theta(\sigma), p(\sigma), \eta(\sigma))$$

is continuous and concave by assumptions (i) and (ii) on U^α and the fact that $\tau \mapsto (x(\tau,\sigma), d(\tau,\sigma))$ is linear. We conclude that $\tilde{\Sigma}^\alpha(\sigma)$ is nonempty, compact and convex, and upper semi-continuous in σ.

Then the correspondence

$$\sigma \mapsto \tilde{\Sigma}^1(\sigma) \times \cdots \times \tilde{\Sigma}^h(\sigma),$$

satisfies all the conditions of Kakutani's fixed-point theorem. Any fixed point is easily seen to be a T.S.S.E. of our generalized game. Using these moves on the tree, and defining arbitrary but type-symmetric moves at

other positions on the tree, we then obtain a T.S.S.E. of Γ^ε. This proves claim 1. ∎

Let

$$\theta(\varepsilon), p(\varepsilon), x^1(\varepsilon), \ldots, x^h(\varepsilon), d^1(\varepsilon), \ldots, d^h(\varepsilon)$$

denote the outcome at a T.S.S.E. of Γ^ε in all steps below, and let $\mu^\alpha(\varepsilon)$, $b^\alpha(\varepsilon)$, and $q^\alpha(\varepsilon)$ denote the moves by players of type α along the T.S.S.E. play of the game.

CLAIM 2 $\theta(\varepsilon) \geq (h\varepsilon + \sum_{\alpha \in H} m^\alpha)/M > 0$ for sufficiently small ε.

Proof Choose ε small enough to ensure

$$\frac{1/\varepsilon}{M} > \frac{h\varepsilon + \sum_{\alpha \in H} m^\alpha + M}{M}.$$

If $\mu^\alpha(\varepsilon) = 1/\varepsilon$, for any α, then clearly the claim is true. So assume $\mu^\alpha(\varepsilon) < 1/\varepsilon$ for $\alpha \in H$ from now on.

Clearly at a T.S.S.E. no agent will end up with a negative debt; for if $-d^\alpha(\varepsilon) = K > 0$ then he could borrow an additional

$$0 < \frac{\Delta}{1 + \theta(\varepsilon)} < \frac{K}{1 + \theta(\varepsilon)}$$

from the bank (by raising $\mu^\alpha(\varepsilon)$ by Δ), use this money to buy any commodity he liked, and defray the loan by increasing $d^\alpha(\varepsilon)$ by Δ without incurring any default penalty. This improves his utility, a contradiction.

Since all the money in the system is returned to the bank, the total I.O.U. notes sent to the bank are at least

$$h\varepsilon + M + \sum_{\alpha \in H} m^\alpha,$$

proving Claim 2. ∎

CLAIM 3 There is an $R > 0$ such that

$$\frac{p_i(\varepsilon)}{p_j(\varepsilon)} < R$$

for any $i \in L, j \in L$, and all sufficiently small ε.

Proof By claim 2, $\theta(\varepsilon) > 0$, and at least one agent type chooses $\mu^\alpha(\varepsilon) > \varepsilon$. We claim that no agent whose I.O.U. note exceeds ε will hoard the bank money that he borrowed (i.e., not spend it on purchases). Otherwise, let him reduce his I.O.U. note by δ and spend an extra $\delta\theta/(1+\theta)$ on any good. His debt remains unchanged, but his consumption increases, so his utility goes up, a contradiction.

Clearly such agents (whose I.O.U. notes exceed ε) acquire at least $M - (h-1)\varepsilon$ units of bank money (since $\theta(\varepsilon) > 0$). Choose ε small to ensure $M - (h-1)\varepsilon > M/2$. Since they do not hoard, there is at least one commodity j^* with

$$p_{j^*}(\varepsilon) \geq \frac{M}{2\ell Q} \equiv \bar{Q} \text{ (say)},$$

where

$$Q = 1 + \max_{j \in L} \left\{ \sum_{\alpha \in H} e_j^\alpha \right\},$$

and we take $\varepsilon \leq 1$. (Q is the maximum offering of commodities, including the external agent's ε.) Let $I \subset L$ be the set of commodities that are being purchased by some agent in H, and let $J = L \backslash I$. Note that $j^* \in I$.

If $i \in I$, let α be a player-type that is purchasing i. Then we must have

$$\frac{D_i U(x^\alpha(\varepsilon), \ldots)}{D_k U(x^\alpha(\varepsilon), \ldots)} \geq \frac{p_i(\varepsilon)}{p_k(\varepsilon)}$$

for any $k \in L$. Otherwise, let $t \in [\alpha - 1, \alpha)$ spend Δ less on i and use Δ to purchase k; his gain in utility will be

$$D_k U^\alpha(x^\alpha(\varepsilon), \ldots)\frac{\Delta}{p_k} - D_i U^\alpha(x^\alpha(\varepsilon), \ldots)\frac{\Delta}{p_i} > 0,$$

a contradiction. Put

$$\tilde{Q} = \sup\left\{ \frac{D_i U^\alpha(x, \ldots)}{D_k u^\alpha(x, \ldots)} : \alpha \in H, i \in L, k \in L, x_j \leq Q \text{ for } j \in L \right\}.$$

(Note that $\tilde{Q}$ is finite on account of condition (iv) (b) on the U^α.) We conclude that if $i \in I$ and $k \in L$,

$$\frac{p_i(\varepsilon)}{p_k(\varepsilon)} \leq \tilde{Q}.$$

Now take $i \in J$. Then

$$p_i(\varepsilon) = \frac{\varepsilon}{\varepsilon + \sum_{\alpha \in H} q_i^\alpha(\varepsilon)} \leq 1.$$

So for any $k \in L$,

$$\frac{p_i(\varepsilon)}{p_k(\varepsilon)} = \frac{p_i(\varepsilon)}{p_{j^*}(\varepsilon)} \cdot \frac{p_{j^*}(\varepsilon)}{p_k(\varepsilon)} \leq \frac{1}{\underline{Q}} \cdot \tilde{Q}.$$

Take $R = \max\{\tilde{Q}, \tilde{Q}/\underline{Q}\}$. ∎

CLAIM 4 There is a $\underline{B} > 0$ such that $p_j(\varepsilon) > \underline{B}$ for $j \in L$ and sufficiently small ε.

Proof This follows from claim 3 and the fact (proved inside claim 3) that there is a $j^* \in L$ with

$$p_{j^*}(\varepsilon) \geq \frac{M}{2\ell\left(1 + \max_{j \in L}\left\{\sum_{\alpha \in H} e_j^\alpha\right\}\right)}.$$ ∎

Consier a sequence of T.S.S.E. of Γ^ε as $\varepsilon \to 0$.

CASE A There exists a subsequence and a $\overline{B} > 0$ such that

$$p_j(\varepsilon) < \overline{B}$$

for all $j \in L$ and all ε.

CLAIM 5 In case A, there is a $\tilde{B}$ such that $\theta(\varepsilon) < \tilde{B}$ and $\mu^\alpha(\varepsilon) < \tilde{B}$ for $\alpha \in H$ and sufficiently small ε.

Proof Note that $\theta(\varepsilon) \to \infty$ if and only if there exists an $\alpha \in H$ such that $\mu^\alpha(\varepsilon) \to \infty$ on a further subsequence. If $\mu^\alpha(\varepsilon) \to \infty$, then

$$d^\alpha(\varepsilon) \geq \mu^\alpha(\varepsilon) - \left(M + \sum_{\alpha \in H} m^\alpha + h\varepsilon\right) \to \infty.$$

Then, by condition (v) on the U^α, $t \in [\alpha - 1, \alpha)$ would do better by not trading at all sufficiently far down in the sequence, a contradiction. ∎

Thus, in case A we can select a subsequence so that

$$\mu^\alpha(\varepsilon), b^\alpha(\varepsilon), q^\alpha(\varepsilon) \to \mu^\alpha, b^\alpha, q^\alpha$$

for all $\alpha \in H$; moreover,

$p_j(\varepsilon) \to p_j$ for $j \in L$,

$\theta(\varepsilon) \to \theta$,

and

$x^\alpha(\varepsilon), d^\alpha(\varepsilon) \to x^\alpha, d^\alpha$ for $\alpha \in H$.

It is readily verified that the limit points yield an M.E. (Note that the constraint of $1/\varepsilon$ on μ^t is not binding since $\mu^\alpha < \tilde{B} < 1/\varepsilon$ for small enough ε. Then since $U^t \equiv U^\alpha$ is concave in μ^t, b^t, q^t for any *fixed* p, θ, ω, it follows that we could have dropped the $1/\varepsilon$ constraint altogether.)

CASE B Case A does not occur. Then, using claim 3, there is a subsequence such that $p_j(\varepsilon) \to \infty$ for all $j \in L$. But

$$p_j(\varepsilon) \leq \frac{\varepsilon + \sum_{\alpha \in H} b_j^\alpha(\varepsilon)}{\varepsilon + \sum_{\alpha \in H} q_j^\alpha(\varepsilon)} \leq \frac{\varepsilon + h\varepsilon + M + \sum_{\alpha \in H} m^\alpha}{\varepsilon + \sum_{\alpha \in H} q_j^\alpha(\varepsilon)}.$$

Hence, $\sum_{\alpha \in H} q_j^\alpha(\varepsilon) \to 0$ for $j \in L$, i.e., $q^\alpha(\varepsilon) \to 0$ for $\alpha \in H$. Then

$$x_j^\alpha(\varepsilon) = e_j^\alpha - q_j^\alpha(\varepsilon) + \frac{b_j^\alpha(\varepsilon)}{p_j(\varepsilon)} \to e_j^\alpha$$

for $\alpha \in H, j \in L$; i.e., $x^\alpha(\varepsilon) \to e^\alpha$ for $\alpha \in H$.

We break the analysis of case B into two subcases.

SUBCASE BI $\exists$ a type α and a subsequence of T.S.S.E. such that α goes bankrupt throughout the subsequence.

Then it must be that, for small enough ε,

$$\inf \left\{ \frac{\dfrac{\partial U}{\partial x}(x^\alpha(\varepsilon), d^\alpha(\varepsilon), \theta(\varepsilon), p(\varepsilon), \eta(\theta(\varepsilon), p(\varepsilon), x(\varepsilon))) \cdot a}{p(\varepsilon) \cdot a} : a \in \mathbf{R}_+^L \setminus \{0\}, a \leq e^\alpha \right\}$$
$$\geq \left| \frac{\partial U^\alpha}{\partial d^\alpha}(x^\alpha(\varepsilon), d^\alpha(\varepsilon), \theta(\varepsilon), p(\varepsilon), \eta(\theta(\varepsilon), p(\varepsilon), x(\varepsilon))) \right|.$$

Otherwise (recalling that $x^\alpha(\varepsilon) \to e^\alpha$) any $t \in [\alpha - 1, \alpha)$ could choose $\tilde{a}(\varepsilon) \in \mathbf{R}_+^L \setminus \{0\}$, such that (i) $\tilde{a}(\varepsilon) < e^\alpha$ and (ii) for all $\Delta > 0$,

$$\Delta \frac{\dfrac{\partial U}{\partial x}(x^\alpha(\varepsilon), d^\alpha(\varepsilon), \theta(\varepsilon), p(\varepsilon), \eta(\theta(\varepsilon), p(\varepsilon), x(\varepsilon))) \cdot \tilde{a}(\varepsilon)}{p(\varepsilon) \cdot \tilde{a}(\varepsilon)}$$

$$< \Delta \left| \frac{\partial U^\alpha}{\partial d^\alpha}(x^\alpha(\varepsilon), \ldots, \eta(\theta(\varepsilon), p(\varepsilon), x(\varepsilon))) \right|.$$

But then t would sell the vector $\Delta\tilde{a}(\varepsilon)/p \cdot \tilde{a}(\varepsilon)$ of commodities and raise Δ units of money for defraying his unpaid bank loan. The loss in his payoff (for small Δ) is given by the LHS and the gain by the RHS, so that he can improve his payoff, a contradiction.

Let $\varepsilon \to 0$, and recall that $x^\alpha(\varepsilon) \to e^\alpha$ for all $\alpha \in H$. Taking limits on the inequality, we contradict condition CIII. So subcase BI cannot occur. This leaves us with subcase BII:

SUBCASE BII Subcase BI does not occur.

Then no one goes bankrupt throughout the sequence. This implies that

$$1 + \theta(\varepsilon) = \frac{h\varepsilon + M + \sum_{\alpha \in H} m^\alpha}{M}$$

for all ε, and hence

$$\theta(\varepsilon) \to \frac{\sum_{\alpha \in H} m^\alpha}{M} \equiv \tilde{\gamma}.$$

Now it is easy to check that $e = (e^1, \ldots, e^h)$ is $\tilde{\gamma}$-Pareto optimal, contradicting condition CI.

Indeed, note that by claim 3 relative prices are uniformly bounded; hence, there is some $p \gg 0$ such that on a subsequence

$p_i(\varepsilon)/p_j(\varepsilon) \to p_i/p_j$ for all $i, j \in L$.

But then, if $p(\varepsilon) \cdot z \leq 0$ and $z \geq -\frac{1}{2}e^\alpha$, we must have

$u^\alpha(x^\alpha(\varepsilon) + z(\theta(\varepsilon)) \leq u^\alpha(x^\alpha(\varepsilon))$

for small ε. Passing to the limit, we see that if $p \cdot z \leq 0$ then

$u^\alpha(e^\alpha + z(\tilde{\gamma})) \leq u^\alpha(e^\alpha)$,

recalling the notation

$$z_j(\gamma) = \min\left\{z_j, \frac{z_j}{1+\gamma}\right\}.$$

By remark 3, $(e^1,\ldots,e^h)$ is $\tilde{\gamma}$-Pareto optimal.

This eliminates case B altogether, leaving us with case A as the only viable one and concluding the proof of theorem 4. ∎

Proof of Theorem 1 It is easily checked that if

$$U^\alpha(x^\alpha,d^\alpha,\theta,p,\omega) = u^\alpha(x^\alpha) - \lambda^\alpha d^\alpha_+$$

then assumptions (i), (ii), (iii), (iv), and (v) on U^α hold. Further,

$$\lambda^\alpha(p) = \lambda^\alpha.$$

Let $E = \max D_j u^\alpha(e^\alpha)$. Then, if $\sum_{j\in L} a_j = 1$,

$$\frac{Du^\alpha(e^\alpha)\cdot a}{p\cdot a} \leq \frac{E}{\min\limits_{j\in L} p_j}.$$

This implies that

$$v^\alpha(p) \leq \frac{E}{\min\limits_{j\in L} p_j}.$$

We conclude that

$$\frac{\lambda^\alpha(p)}{v^\alpha(p)} \geq \frac{\lambda^\alpha \min\limits_{j\in L} p_j}{E} \to \infty \quad \text{as } p \to \infty,$$

and so condition CIII holds. Then, by theorem 4, an M.E. exists in case II of nominal penalties. ∎

Proof of Theorem 2 Here

$$U^\alpha(x^\alpha,d^\alpha,\theta,p,\omega) = u^\alpha(x^\alpha) - \frac{\lambda^\alpha}{p\cdot a} d^\alpha_+$$

and, once again, assumptions (i)–(v) of U^α are easily seen to hold. Here

$$\lambda^\alpha(p) = \frac{\lambda^\alpha}{p\cdot a}$$

and

$$v^\alpha(p) \leq \frac{(Du^\alpha(e^\alpha))\cdot a}{p\cdot a},$$

so

$$\frac{\lambda^\alpha(p)}{v^\alpha(p)} \geq \frac{\lambda^\alpha}{(Du^\alpha(e^\alpha))\cdot a} > 1$$

by condition CII. So condition CIII holds, and by theorem 4 an M.E. exists. ∎

Proof of Theorem 3 First consider the case of real penalties, i.e.,

$$U^\alpha(x^\alpha,d^\alpha,p) \equiv u^\alpha(x^\alpha) - \frac{\lambda^\alpha d^\alpha_+}{p\cdot a}.$$

Let

$$Q^* = 1 + \max\left\{\frac{D_i u^\alpha(x)}{D_k u^\alpha(x)} : \alpha \in H, i \in L, k \in L, x \in \square\right\},$$

where

$$\square = \left\{x \in \mathbb{R}^L_+ : x_j \leq 1 + \max_{j\in L} \sum_{\alpha\in H} e^\alpha_j\right\}.$$

Put

$$\hat{\lambda}^\alpha = \ell^2 Q^* \bar{a} \max\{D_j u^\alpha(x) : j \in L, x \in \square\},$$

where

$$\bar{a} = \max_{i\in L} a_i.$$

It will be shown that if $\lambda^\alpha > \hat{\lambda}^\alpha$ for all $\alpha \in H$, there can be no bankruptcy at any M.E.

Suppose, to the contrary, that β is going bankrupt in such a situation. Let

$$L^* = \{j \in L : a_j > 0\},$$

$$J^* = \{j \in L : q^\beta_j < e^\beta_j\} \cap L^*,$$

$$K^* = L^* \setminus J^*.$$

Since, by assumption CII, $e_j^\beta > 0$ for $j \in L^*$, we see that there is positive trade in commodities in K^* (indeed, β has sold them); hence, there are buyers of K^*, and then (as was shown in the proof of claim 3)

(i)* $p_k \leq Q^* p_i$ for all $i \in L, k \in K^*$.

If $K^* = \phi$, let β sell εa (which is certainly feasible for him) and use the money to repay the bank. His loss in utility is

$$\varepsilon[Du^\beta(x^\beta) \cdot a] \leq \varepsilon \hat{\lambda}^\beta$$

and gain is

$$\frac{\varepsilon \lambda^\beta p \cdot a}{p \cdot a} = \varepsilon \lambda^\beta.$$

Since $\lambda^\beta > \hat{\lambda}^\beta$, α improves his payoff, a contradiction.

If $K^* \neq \phi$, then β has sold goods in K^* so he must have bought some good j. Let β buy $\varepsilon \ell Q^* \bar{a}$ less of good j, and sell εa_i of each good $i \in J^*$. Then the money available to β to repay the bank is

$$\text{(ii)}^* \quad \varepsilon \sum_{i \in J^*} p_i a_i + \varepsilon p_j \ell Q^* \bar{a} \geq \varepsilon \sum_{i \in J^*} p_i a_i + \varepsilon \sum_{k \in K^*} p_k a_k$$

$$= \varepsilon p \cdot a.$$

(The inequality follows from (i)*.) His loss in utility is at most

$$\varepsilon \left[\sum_{i \in J^*} D_i u^\beta(x^\beta) a_i + D_j u^\beta(x^\beta) \ell Q^* \bar{a} \right] < \varepsilon \hat{\lambda}^\beta,$$

and gain is at least (using (ii)*)

$$\frac{\varepsilon \cdot \lambda^\beta p \cdot a}{p \cdot a} = \varepsilon \lambda^\beta,$$

again a contradiction, since $\lambda^\beta > \hat{\lambda}^\beta$.

This proves theorem 3 for real penalties. The proof for nominal penalties is even simpler. Note that there is bankruptcy $\Rightarrow \theta > 0 \Rightarrow$ there is no hoarding $\Rightarrow$ all of M is spent $\Rightarrow$

$$p_{j^*} \geq \frac{M}{\max\limits_{j \in L} \sum\limits_{\alpha \in H} e_j^\alpha} \equiv C \text{ (say)}$$

for some $j^* \in L \Rightarrow C \leq p_{j^*} \leq Q^* p_j$ for all $j \in L$. (The last implication follows, as in (i)*, from the proof of claim 3.) Thus, $p_j \leq C/Q^*$ for all $j \in L$.

Suppose β is going bankrupt. Then he must be purchasing some good j. Let him reduce his purchase by ε and return ε to the bank. His loss in utility is at most

$$\frac{\varepsilon D_j u^\beta(x^\beta)}{p_j} \leq \frac{\varepsilon D_j u^\beta(x^\beta) Q^*}{C}$$

and gain is at least $\varepsilon\lambda^\beta$. So if we take each

$$\lambda^\alpha > \frac{Q^*}{C} \max\{D_j u^\alpha(x^\alpha): \alpha \in H, j \in L, x^\alpha \in \square\}$$

it is clear that no agent will go bankrupt. ∎

Proof of Theorem 5 (Sketch) At any M.E. $(\theta, p, (x^\alpha)_{\alpha \in H}) \gg 0$, the following conditions must hold. (We write out the conditions for nominal penalties.)

$$\sum_{\alpha \in H} (x^\alpha - e^\alpha) = 0; \tag{1}$$

$$\left.\begin{aligned} &\frac{\partial U^\alpha}{\partial x_j} - t_\alpha p_j(1+\theta) = 0 \text{ if } \alpha \text{ buys } x_j \\ &\frac{\partial U}{\partial x_j} - t_\alpha p_j = 0 \text{ if } \alpha \text{ sells } x_j \end{aligned}\right\} \text{ if } \alpha \text{ borrows money,} \tag{2}$$

$x_j^\alpha - e_j = 0$ if α neither borrows nor sells j;

$$p \cdot (x^\alpha - e^\alpha)_+ (1+\theta) + p \cdot (x^\alpha - e^\alpha)_- - (1+\theta) m^\alpha - d^\alpha = 0 \text{ for all } \alpha \in H; \tag{3}$$

$$d^\alpha = 0 \text{ for all } \alpha \text{ that do not go bankrupt,} \tag{4}$$

$$\frac{dU^\alpha}{\partial x_j} \frac{1}{p_j} - (1+\theta)\lambda^\alpha = 0 \text{ for all } \alpha \text{ that do go bankrupt, if } \alpha \text{ is buying } j;$$

$$\sum_{\alpha \in H} p \cdot (x^\alpha - e^\alpha)_+ - \left(M + \sum_{\alpha \in H} m^\alpha\right) = 0. \tag{5}$$

Equation (1) simply says that all commodity markets clear; (2) and (4) reflect the fact that α has maximized U^α, where $t_\alpha \equiv$ marginal utility of

money for α; (3) may be viewed as a definition of the debt d^α. If α does not borrow, then with $\theta > 0$, α spends exactly m^α, sells nothing, and

$$\frac{\partial U^\alpha}{\partial x_j} = t_\alpha p_j k$$

for goods j that α buys, where $1 \leq k \leq (1 + \theta)$. The variable k is new, but we also add a new equation $m^\alpha = p \cdot (x - e^\alpha)_+$.

The crucial equation is (5), which asserts that total expenditures are equal to the total stock of money. Note that this equality must hold when $\theta > 0$, since (as we have seen before) when $\theta > 0$ there can be no hoarding. An agent who has a dollar will always spend it, even if he has borrowed it. (Equation (1) implies that

$$\sum_{\alpha \in H} p \cdot (x^\alpha - e^\alpha)_+ = \sum_{\alpha \in H} p \cdot (x^\alpha - e^\alpha)_-,$$

that is, money spent on purchases equals money obtained from sales. Also, (3) and (5) together yield $\theta = (\sum_{\alpha \in H} m^\alpha + \sum_{\alpha \in H} d^\alpha)/M$.)

Note that in (4), if α is going bankrupt, he must be borrowing, and purchasing some good j; hence, he could contemplate a dollar less or more of borrowing, spending a dollar less or more on good j. If neither direction is to increase his utility, then the second statement in (4) must hold.

So let the map

$$f: \mathscr{E} \times \mathbb{R}^L_{++} \times \mathbb{R}_{++} \times (\mathbb{R}^L_{++})^H \times \mathbb{R}^H \times \mathbb{R}^H \to \mathbb{R}^L \times \mathbb{R}^{LH} \times \mathbb{R}^H \times \mathbb{R}^H \times \mathbb{R}$$

be given by

$$f((M, (\lambda^\alpha, e^\alpha, m^\alpha, u^\alpha)_{\alpha \in H}, (p, \theta, (x^\alpha)_{\alpha \in H}, (d^\alpha)_{\alpha \in H}, (t^\alpha)_{\alpha \in H})$$

$$\equiv \text{the LHSs of the above equalities.}$$

We can check immediately that $f \pitchfork 0$, as follows: By varying M, we perturb (5) however we like, without disturbing any other equation. By varying m^α, we can perturb (3) without changing anything except (5), which is already "controllable" (i.e., it can be compensated for by a change in M). By varying d^α or λ^α, we can perturb (4) however we like, disturbing only (3) and (5), which are controllable. By perturbing U^α, we perturb (2) however we like (presuming there is trade by every individual in every good), affecting only (4), which is under control. Finally, we can perturb x^α to control (1), and that may affect all of (2)–(5), but that is no matter, since they are all under control.

It remains only to check that if some α does not trade j, then the same argument goes through. If j is being purchased by some other agent β, then (1) can be perturbed by varying x_j^β, and (2) can be controlled by varying $\partial U^\beta/\partial x_j$, as we have seen. To perturb $x_j^\alpha - e_j$ for α (not trading j), vary x_j^α; this can be compensated by varying x_j^β, and (2) β can be compensated by varying $\partial U^\beta/\partial x_j$.

For now, let us restrict attention to the open set of endogenous variables $\hat{V}$ at which for every j there is some β with $x_j^\beta - e_j^\beta \neq 0$. We have just proved that $f: \mathscr{E} \times \hat{V} \to \mathbb{R}^{L+LH+2H+1}$ is transverse to 0. Since the dimension of $\hat{V}$ is also $L + LH + 2H + 1$, we deduce from the transversality theorem that for almost all parameters $(M, (\lambda^\alpha, e^\alpha, m^\alpha, u^\alpha)_{\alpha \in H})$ the resulting economy has a finite number of equilibria with endogenous variables in $\hat{V}$.

To finish the proof, let $\hat{L} \subset L$ be any nonempty subset of L. For all $j \notin \hat{L}$, fix $x_j^\alpha = e_j^\alpha$ for all $\alpha \in H$, defining the subeconomy on $\hat{L}$. For this subeconomy the above argument shows that, for a generic set of parameters, there are only a finite number of equilibria with endogenous variables in $V_{\hat{L}}$ (where for each $j \in \hat{L}$, $x_j^\alpha - e_j^\alpha \neq 0$ for some α).

Since $m^\alpha > 0$, each agent is buying some good at equilibrium. Hence, any equilibrium of the original economy will be an equilibrium of the subeconomy $\hat{L}$ with endogenous variables in $V_{\hat{L}}$, for some nonempty set $\hat{L} \subset L$ of traded goods. Since there are only a finite number of subsets $\phi \neq \hat{L} \subset L$, the result follows for nominal penalties.

The proof of real penalties is exactly the same, except that in (4) we substitute $\lambda^\alpha/p \cdot a$ for λ^α. ∎

Proof of Theorem 6 Define Γ_ε as before, with the following substitution:

$$d^t \to d^t + \Delta^t \text{ (where } \Delta^t \equiv \Delta^\alpha \text{ for } t \in [\alpha - 1, \alpha))$$

in the argument of U^t, for $t \in [0,h)$. Then a T.S.S.E. exists exactly as in claim 1. (We have not altered the strategy sets of the agents, or their payoff functions. Only the outcome has got adjusted as indicated above.)

Now reread the proof of claim 2 as follows. As before, no agent will hold positive amounts of fiat money at the end of trade, so all the money in the system,

$$h\varepsilon + \sum_{\alpha \in H} m^\alpha + M,$$

is returned to the bank. Now, however, agents may well repay more than their I.O.U. notes to the bank (in an effort to clear their exogenous debts),

but clearly (by the same argument as in the proof of claim 2) no agent t of type α will repay more than his total debt,

$$\mu^t(\varepsilon) + \Delta^t \equiv \mu^\alpha(\varepsilon) + \Delta^\alpha,$$

at a T.S.S.E. Hence,

$$\sum_{\alpha \in H} \mu^\alpha(\varepsilon) + \sum_{\alpha \in H} \Delta^\alpha \geq h\varepsilon + \sum_{\alpha \in H} m^\alpha + M,$$

which implies that

$$\sum_{\alpha \in H} \mu^\alpha(\varepsilon) \geq h\varepsilon + \sum_{\alpha \in H} m^\alpha - \sum_{\alpha \in H} \Delta^\alpha + M.$$

Since

$$\sum_{\alpha \in H} m^\alpha - \sum_{\alpha \in H} \Delta^\alpha \geq 0$$

by assumption, we have

$$\sum_{\alpha \in H} \mu^\alpha(\varepsilon) \geq h\varepsilon + M > M,$$

and hence

$$\theta(\varepsilon) = \frac{\sum_{\alpha \in H} \mu^\alpha(\varepsilon)}{M} - 1 > 0.$$

The rest of the proof of theorem 6 goes through like the proof of theorem 4, replacing d^α by $d^\alpha + \Delta^\alpha$ throughout (in the argument of U^α). Similarly, the proof of theorem 3 goes through as well. ∎

Proof of Theorem 8 (and thus also theorem 7) Define Γ_ε as in the proof of theorem 6, and note that a T.S.S.E. exists.

CLAIM 6 $\exists\, \underline{B}^*$ such that, for sufficiently small ε,

$$p_j(\varepsilon) > \underline{B}^*$$

for $1 \leq j \leq l$.

Proof Suppose $\theta(\varepsilon) \to \infty$. Then the total debt of all agents is at least

$$\theta(\varepsilon)M + \sum_{\alpha \in H} \Delta^\alpha,$$

and so agents of some type β have debt at least

$$\frac{\theta(\varepsilon)M + \sum_{\alpha \in H} \Delta^\alpha}{h} \to \infty$$

and hence, by (ii)**,

$$U^\beta(x^\beta(\varepsilon), d^\beta + \Delta^\beta, p(\varepsilon), \theta(\varepsilon), \eta((\theta(\varepsilon), p(\varepsilon), x(\varepsilon)))$$
$$< U^\beta(e^\beta, \Delta^\beta, p(\varepsilon), \theta(\varepsilon), \eta(\theta(\varepsilon), p(\varepsilon), x(\varepsilon)))$$

for small enough ε. Then any agent of type β would do better not trading at all, a contradiction.

We conclude that $\theta(\varepsilon)$ is bounded from above, hence so is each $\mu^\alpha(\varepsilon)$, and therefore the bound of $1/\varepsilon$ on the I.O.U. notes is not binding for small enough ε.

Suppose $p_j(\varepsilon) \to 0$ for some $j \in L$. Let any agent of type α increase his I.O.U. note by δ, obtain

$$\frac{\delta}{1 + \theta(\varepsilon)}$$

units of bank money, and purchase

$$\frac{\delta}{(1 + \theta(\varepsilon))p_j(\varepsilon)}$$

of j. His gain in utility (for small ε) is at least

$$\frac{\partial U^\alpha}{\partial x_j}(x^\alpha(\varepsilon), d^\alpha(\varepsilon), p(\varepsilon), \theta(\varepsilon), \eta(\theta(\varepsilon), p(\varepsilon), x(\varepsilon)))\frac{\delta}{(1 + \theta(\varepsilon))p_j(\varepsilon)}$$

and loss is at most (using (i)**)

$$\bar{c}\delta.$$

Since $\theta(\varepsilon)$ is bounded from above, and $\partial U^\alpha/\partial x_j$ is bounded away from zero, we see that he can improve his payoff, a contradiction. This establishes claim 6.

Since condition CIII is obviously implied by hypothesis (b), the rest of the proof of theorem 7 proceeds exactly as the proof of theorem 4 (after claim 4). ∎

Remark 13 (on a variation of condition CII) We can drop the requirement that $a_j > 0$ only if $e_j^\alpha > 0$ for all $\alpha \in H$ by making the λ^α harsher. More precisely, let

$$\bar{Q} = \frac{M}{2\ell \max\limits_{j \in L} \sum\limits_{\alpha \in H} e^{\alpha}},$$

$$\hat{Q}^* = 1 + \max\left\{\frac{D_i u^{\alpha}(e^{\alpha})}{D_k u^{\alpha}(e^{\alpha})} : \alpha \in H, i \in L, k \in L\right\},$$

$$R^* = \max\left\{\hat{Q}^*, \frac{\hat{Q}^*}{\bar{Q}}\right\}.$$

Assume

(CII)* $\lambda^{\alpha} > \ell^2 R^* \bar{a} \max\{D_j u^{\alpha}(e^{\alpha}): j \in L\}$ for $\alpha \in H$,

where (recall)

$$\bar{a} = \max_{j \in L} a_j.$$

Then our results hold with (CII)* in place of (CII). To see this, reread the proof of theorem 3, remembering that as $p(\varepsilon) \to \infty$

$$x^{\alpha}(\varepsilon) \to e^{\alpha},$$

and obtain the same contradiction. (Note that, as in claim 3, $p_i(\varepsilon)/p_k(\varepsilon) < R^*$, so we can substitute R^* for Q^* in rereading the proof of theorem 3.)

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