Chapter 35

OVERLAPPING GENERATIONS

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0. Introduction

Competitive equilibria in economies of overlapping generations are different from competitive equilibria in economies that extend over finitely many periods, finite economies for short. These differences concern the properties of competitive equilibria, such as existence, optimality and determinacy or local uniqueness; and the phenomena compatible with competitive equilibria, such as net aggregate debt or fiat money with a positive price.

Ever since the introduction of the model of overlapping generations by Allais (1947) and Samuelson (1958), economic theorists have striven to isolate the reasons for the differences between this model and the definitive model of a finite economy elaborated by Arrow (1951), Debreu (1951, 1970) and Arrow and Debreu (1954). This is also our focus.

In the original model of Allais and Samuelson, time evolves over discrete periods. In each period, a new generation is born and lives for two or three periods. Later extensions of the model permit generations to live longer and even be immortal, include many commodities in each period and introduce uncertainty; the latter is referred to as the stochastic overlapping generations model.

All the variants of the model of overlapping generations allow for an infinity of time periods and hence for infinitely many commodities. This infinity of time periods and commodities is not a mathematical curiosity, but rather is central to the economic significance of the model. Consider a pay-as-you-go system of social security. Each period, the young generation makes payments that are given directly to the old. The young cooperate in this enterprise because they expect to receive payments when they grow old. Were it believed that there would be no next generation, the system would surely break down immediately. In economic models in which the rationality of individuals is presumed to be unlimited, the fact that no new generation would appear at some point in the very distant future would also lead to the immediate break down of the social security system. Individuals would reason that no payment would be made at the period in which no young generation would appear, the generation just before would refuse to make its payment, therefore the generation just before that would also refuse, etc. For exactly the same reason, fiat money can have positive value in an economy of overlapping generations, whereas in a finite economy, as Hahn (1965) has pointed out, it must have zero value in the last period and hence, by backward induction, it must have zero value in every period.

In an economy of overlapping generations, the temporal and demographic structure is explicit, which evidently enriches the study of problems such as the transfer of value over time. It also allows the claim that the inability of individuals to trade directly with individuals whose consumption and endowment spans commence after they have perished is the distinguishing feature of economies of overlapping generations. This, we argue, is not essential. It is possible to make the demographic structure explicit in a finite economy as well and to suppose that the consumption of any single individual extends over only few periods. Yet, equilibria in this economy share the qualitative properties of equilibria in abstract Arrow–Debreu economies provided the asset market is complete. A complete asset market, which is essential in the construction of Arrow and Debreu, implies that trades occur as if all individuals, irrespective of the period of their biological birth, could participate in an initial exchange of contracts for the dated and possibly contingent delivery of commodities or revenue. Furthermore, there is no issue of bequest motives. This way of modelling exchange at an Archimedean point, without reference either to the biological lifespan of individuals or of the mediation of transactions, is evidently metaphorical. But the metaphor is no more strained for overlapping generation than for Arrow–Debreu economies. We maintain the hypothesis of a complete asset market and hence of a unique budget constraint throughout this survey.

What distinguishes economies of overlapping generations is the countable infinity of individuals and commodities. This has two consequences, one straightforward, and the other depending on two further assumptions. First, recall that in a finite economy, prices determine the value of arbitrary commodity bundles; technically, they define a linear functional on the commodity space [Debreu (1954)]. In an economy of overlapping generations, the finite valuation of arbitrary commodity bundles need not be possible. In particular, even if individual consumption bundles have finite value, the corresponding aggregate consumption bundle need not. Thus, Walras' law need not hold for economies of overlapping generations. For a large class of economies of overlapping generations, we can always find allocations in which aggregate consumption equals aggregate supply for every commodity, and yet at the same time every individual is spending less on his consumption than the value of his endowment. Second, with countably infinite individuals and commodities, competitive equilibria are limits of sequences of allocations for finite economies. These allocations are competitive, except that some markets are allowed not to clear. Since the periods at which markets fail to clear tend to infinity, the model of overlapping generations has been interpreted as "lack of market clearing at infinity" [Geanakoplos (1987)]. This approximation by finite economies is possible because of two properties that further distinguish economies of overlapping generations, the continuity of individual preference relations or utility functions, which can be interpreted as impatience, and the hypothesis that only a finite number of individuals desire and essentially own any commodity.

The failure of finite valuation accounts for the features of competitive equilibria in economies of overlapping generations. Market clearing may require that the value of the consumption bundle of an individual exceed the value of his initial endowment. Prices may fail to convey the aggregate scarcity of commodities and competitive equilibrium allocations may fail to be Pareto optimal. When more than one commodity market fails to clear at infinity, as is the case in economies with more than one commodity per period, more degrees of freedom are operative at a competitive equilibrium than the one required to account for the budget constraint and competitive equilibrium allocations may be indeterminate. Net aggregate transfers revenue may be compatible with a competitive equilibrium; and this is closely linked with the possibility that fiat money which provides no utility or liquidity services maintains a positive price at equilibrium.

The interpretation of economies of overlapping generations as a lack of market clearing at infinity also explains the properties of equilibria in these models, of which the auxiliary hypotheses of impatience and finite ownership and desire for any commodity hold. Since dropping the requirement that all markets clear only eases the existence problem, economies of overlapping generations satisfying the auxiliary hypotheses and a further hypothesis on individual endowments have equilibria. Moreover, there may be many dimensions of equilibria, since not all the markets clear. The dimension of potential indeterminacy is related to the number of markets that do not clear, which in turn depends on the number of commodities per period and the lifetime of the agents. Furthermore, since markets do not necessarily clear, Pareto optimality cannot be expected, unless the value of the commodities in the markets that do not clear is essentially zero. For the same reason, fiat money may have positive value in equilibrium.

It is very interesting that many of the properties of finite economies are restored in infinite horizon economies if there is some finite set of individuals whose aggregate endowment is a nonnegligible fraction of the aggregate endowment of the whole economy. Since individuals are characterized by impatience, market clearing requires that each agent have finite wealth. It follows that when a finite number of agents control a nonnegligible fraction of the resources of the whole economy, the aggregate endowment has finite value and Walras' law is restored. Moreover, the value of the commodities in the markets at infinity that do not clear is essentially zero. This argument rests entirely on the endowment stream of the nonnegligible agents, and not at all on whether they consume in every period, or "live" forever.

The infinite time horizon allows the demographic structure, the preferences and endowments of individual consumers or the production possibilities of firms to display simple recursive patterns, which also distinguishes economies of overlapping generations. Competitive equilibria in which endogenous variables, prices and quantities, inherit these recursive patterns are referred to as steady states. Alternatively, endogenous variables at competitive equilibria may display recursive patterns more complex than the simple pattern displayed by the exogenous structure of the economy; such equilibria are referred to as endogenous business cycles. Chaos is observed when endogenous variables display no pattern whatsoever.

Production introduces no essential difference as long as the production spans of firms are finite. We do not consider production economies in this survey.

The countably infinite index of commodities need not refer to calendar time. Location or any other characteristic suffice to give rise to economies analytically equivalent to economies of overlapping generations.

Uncertainty alters the argument significantly if the asset market is incomplete. As in the case of a finite economy, a complete asset market reduces analytically an economy under uncertainty to an economy under certainty [Arrow (1953), Debreu (1959b)]. When the asset market is incomplete, beyond the existence, optimality and determinacy of competitive equilibrium allocations which are problematic, as they are in a finite economy [Cass (1985), Hart (1975)], novel considerations arise; among them, whether endogenous variables retain the stochastic properties of exogenous variables, such as serial dependence or memory [Duffie, Geanakoplos, Mas-Colell and McLennan (1989), Dutta and Polemarchakis (1990)].

1. The economy

We denote commodities by

$$l \in \mathbf{L},$$

and individuals by

$$h \in \mathbf{H}.$$

Assumption 1. The set of commodities is non-empty and countable, $\mathbf{L} = \{1, \ldots\}$; also, the set of individuals, $\mathbf{H} = \{1, \ldots\}$.

Either the set of commodities or the set of individuals may be finite; nevertheless, most interestingly, they are both countably infinite.

We denote commodity bundles by

$$x = (\ldots, x_l, \ldots) \in \Lambda,$$

where the commodity space, Λ, is the Euclidean space of dimension equal to the cardinality of the set of commodities. The context should clarify whether x_k refers to the kth component of a commodity bundle or the kth term in a sequence of commodity bundles $(x_n\colon n = 1, \ldots)$.

The non-negative orthant of the commodity space is $\Lambda_+ = \{x \in \Lambda: x \geq 0\}$ and the projection on the lth coordinate is Λ_l. The vector of units is $\tilde{1}$, and $\tilde{1}_l$ is the vector with a unit in the lth coordinate and zero otherwise.

An individual is characterized by a triple

$$(\mathbf{X}^h, u^h, w^h)\,, \quad \text{for } h \in \mathbf{H}\,,$$

of a consumption set, a utility function and an initial endowment.

Assumption 2. For $h \in \mathbf{H}$, the consumption set, $\mathbf{X}^h \subseteq \Lambda$, is convex, bounded from below, for simplicity $\mathbf{X}^h \subseteq \Lambda_+$, and allows for free disposal, $\mathbf{X}^h + \Lambda_+ \subseteq \mathbf{X}^h$. The utility function, $u^h: \mathbf{X}^h \to \mathbf{R}$, is quasi-concave, $u^h(x') > u^h(x) \Rightarrow u^h(\lambda x' + (1-\lambda)x) > u^h(x)$, for $0 < \lambda \leq 1$, and weakly monotonically increasing, $x' \gg x \Rightarrow u^h(x') > u^h(x)$, for $x', x \in \mathbf{X}^h$. The initial endowment, $0 < w^h \in \mathbf{X}^h$, is a positive consumption bundle such that $(1-\varepsilon^h)w^h \in \mathbf{X}^h$, for some $0 < \varepsilon^h < 1$.

The individual does not desire commodity l at $x \in \mathbf{X}^h$ if and only if $(x - x_l\tilde{1}_l) \in \mathbf{X}^h$ and $u^h(x) = u^h(x - x_l\tilde{1}_l)$; the individual does not desire the commodity if and only if he does not desire the commodity anywhere in the consumption set. The set of commodities desired by the individual, somewhere in his consumption set, desired for short, is $\mathbf{L}^h \subseteq \mathbf{L}$. The individual is endowed with commodity l if and only if $w_l^h > 0$.

In examples we suppose, for simplicity, that the consumption sets of individuals coincide with the non-negative cone of the commodity space, $\mathbf{X}^h = \Lambda_+$, unless we explicitly state otherwise.

The weak monotonicity of the utility function complements the free disposal assumption on the consumption sets.

The assumptions on individual characteristics are standard [Arrow and Debreau (1954); also Debreu (1959a), Arrow and Hahn (1971)], and require no comment.

The only necessary novelty here is the choice of a topology for the commodity space and the assumptions that in this topology individual consumption sets are closed and utility functions are continuous.

The commodity space is a topological vector space with the product topology. A base for the product topology consists of cylinder sets of the form $\Pi_{l \in L_F} \mathbf{O}_l \times \Pi_{l \in L/L_F} \Lambda_l$, where $\mathbf{O}_l \subseteq \Lambda_l$ are open and $\mathbf{L}_F \subset \mathbf{L}$ is finite. It is the weakest topology for which the projection maps $\pi_l: \Lambda \to \Lambda_l$ are continuous. By the Tychonoff theorem, a product set $\mathbf{C} = \Pi_{l \in L} \mathbf{C}_l$ is compact if the sets $\mathbf{C}_l \subset \Lambda_l$ are compact. A sequence $(x_n: n = 1, \ldots)$ converges, $\lim_{n\to\infty} x_n = x$, if and only if every component converges, $\lim_{n\to\infty} x_{l,n} = x_l$, for $l \in \mathbf{L}$. A set, $\mathbf{C}$, is compact if and only if, for any sequence, $(x_n: x_n \in \mathbf{C},\ n = 1, \ldots)$, there exists a convergent subsequence $(x_{n_k}: k = 1, \ldots)$, and $x \in \mathbf{C}$ such that $\lim_{k\to\infty} x_{n_k} = x$.

Assumption 3. For $h \in \mathbf{H}$, the consumption set, $\mathbf{X}^h$, is closed, and the utility function, u^h, is continuous in the product topology.

If a commodity bundle does not permit an individual to survive, $x \not\in \mathbf{X}^h$, the assumption that the consumption set is closed, and hence its complement is open, implies that there exists a neighborhood, $\mathrm{V}(x)$, such that $\mathrm{V}(x) \cap \mathbf{X}^h = \emptyset$. Since $\mathrm{V}(x)$ is open in the product topology, there exists a finite set $\mathbf{L}_F \subset \mathbf{L}$ and $\varepsilon > 0$ such that $x' \not\in \mathbf{X}^h$, where $|x'_l - x_l| < \varepsilon$, for $l \in \mathbf{L}_F$, no matter what the value of x'_l, for $l \not\in \mathbf{L}_F$. Sufficiently distant modifications, even if unbounded, do not suffice to permit survival. Similarly, if a commodity bundle, x', yields higher utility than another, x, $u^h(x') > u^h(x)$, continuity of the utility function implies that there exists a neighborhood, $\mathrm{V}(x')$, such that $u^h(x'') > u^h(x)$, for $x'' \in \mathrm{V}(x)$. Since $\mathrm{V}(x')$ is open in the product topology, there exists a finite set $\mathbf{L}_F \subset \mathbf{L}$ and $\varepsilon > 0$ such that the commodity bundle $x'' \in \mathrm{V}(x')$, if $|x''_l - x'_l| < \varepsilon$ for $l \in \mathbf{L}_F$, no matter what the value of x''_l, for $l \not\in \mathbf{L}_F$, and hence $u^h(x'') > u^h(x)$. Sufficiently distant modifications, even if unbounded, do not reverse the order of preference.

In the above comparison, we could take $\mathbf{L}_F = \{l \in \mathbf{L}\colon l \leq \bar{l}, \text{ for some } \bar{l}\}$, the set of all commodities with index less than some $\bar{l}$. Thus, continuity in the product topology implies that, in a sense, individuals "agree" that commodities with a lower index, l, are more important than commodities with a higher index. As we point out later, when the index of commodities involves calendar time that extends infinitely into the future but not the past, the assumption of continuity in the product topology suggests that individuals are "impatient". There is no presumption of uniform impatience, however; it may well be the case that individual h desires only commodity $l = h$, for $h = 1, \ldots$. As we see later, the failure of uniform impatience is related to the possible inefficiency of competitive equilibria.

The study of economies with infinitely many commodities, typically a continuum, and finitely many individuals has often restricted attention to bounded commodity bundles, $x \in \Lambda_\infty = \{x \in \Lambda\colon \|x\|_\infty < \infty\}$, where $\|x\|_\infty = \sup\{|x_l|\colon l \in \mathbf{L}\}$, and has imposed continuity of individual utility functions in the Mackey topology [Bewley (1972)]. The Mackey topology is the strongest topology under which the topological dual of the commodity space is the space of summable prices, and is evidently weaker than the topology defined by the supremum norm, $\|\ \|_\infty$. The Mackey topology is stronger than the weak topology, the weakest topology in which all linear functionals are continuous. The weak topology is stronger than the product topology. The sequence of bounded commodity bundles $(x_n\colon n = 1, \ldots)$, where $x_{n,l} = 1$, for $l \leq n$ and $x_{n,l} = n$, for $l > n$, for $n = 1, \ldots$, converges to the commodity bundle $\bar{x} = \tilde{1}$ in the product topology but not in the weak topology and hence not in the Mackey topology. Thus, our assumption of continuity in the product topology is strong. On the other hand, a sequence of bounded commodity bundles converges in the weak topology if and only if it converges in the Mackey topology; that is, these topologies induce the same concept of convergence of sequences on bounded commodity bundles [Hildenbrand (1990)]. Moreover, the convergence of a uniformly bounded sequence of commodity bundles is the same in the Mackey and the product topologies.

Since the commodity space endowed with the product topology is separable and the individual consumption sets are convex and hence connected, the existence of a continuous utility function is no stronger than the assumption that the underlying preference relation be complete and continuous [Debreu (1954b)]. It is straightforward to handle incomplete preference relations [Wilson (1981), Burke (1988)].

Assumption 4. The aggregate endowment of each commodity is finite and strictly positive,

$$0 < w_l = \sum_{h \in \mathbf{H}} w_l^h < \infty\,, \quad \text{for } l \in \mathbf{L}\,,$$

and thus

$$0 \ll w \in \Lambda\,.$$

Since, for any commodity, l, and any $1 > \varepsilon > 0$, there exists a finite set of individuals, who jointly are endowed with $(1-\varepsilon)w_l$, each commodity is essentially owned by finitely many individuals.

The aggregate endowment need not be uniformly bounded. By rescaling the units of measurement of different commodities we could always suppose this to be the case. Continuity in the product topology is not affected by such rescaling. For the purposes of equilibrium theory, we need not contemplate individual consumptions that exceed the aggregate endowment, though this may be contrary to the spirit of a competitive equilibrium, since individuals might, indeed, contemplate unbounded consumption bundles. Pursuing this direction, nevertheless, we could suppose that $w \in \Lambda_{\infty,+} = \{x \in \Lambda_\infty\colon x \geq 0\}$, and also $\mathbf{X}^h \subseteq \mathbf{X}_{2w} = \{x \in \Lambda_+\colon \|x\|_\infty < 2w\}$, for $h \in \mathbf{H}$. Continuity with respect to the product topology in $\mathbf{X}_{2w}$, which as we have noted is equivalent to Mackey continuity, is weaker than the assumption of continuity in the product topology in Λ; from the economic point of view, it amounts, roughly, to the assumption that the impatience of individuals exceeds the growth rate of the aggregate endowment. Our stronger assumption has the advantage of keeping separate the restrictions on preferences and the restrictions on endowments. The weaker assumption would mix the two by imposing continuity of the utility function on a domain that is dependent on the aggregate endowment through the rescaling of the units of measurement of commodities.

The set of individuals who desire commodity l is

$$\mathbf{D}_l = \{h \in \mathbf{H}: l \in \mathbf{L}^h\}\,, \quad \text{for } l \in \mathbf{L}\,.$$

Assumption 5. For $l \in L$, $\mathbf{D}_l$ is finite, at most finitely many individuals desire each commodity.

An allocation is an array of individual consumption bundles,

$$x^{\mathbf{H}} = \{x^h \in \mathbf{X}^h: h \in \mathbf{H}\}\,.$$

An allocation is feasible if and only if $\Sigma_{h\in\mathbf{H}}\, x^h = w$. Since the individual consumption sets allow for free disposal, while utility functions are weakly monotonic, there is no loss of generality in stating the feasibility condition with equality.

Lemma 1. *Let* $\{\mathbf{D}'_l: \mathbf{D}'_l \subset \mathbf{H}, l \in \mathbf{L}\}$ *be a collection of finite sets of individuals. Suppose* $(w_n: w_n \geq 0,\ n = 1, \ldots)$ *is a convergent sequence of commodity bundles,* $w = \lim_{n\to\infty} w_n$, *and let* $(x_n^{\mathbf{H}}: \Sigma_{h\in\mathbf{H}}\, x_n^h = w_n,\ x_{n,l}^h = 0 \text{ if } h \not\in \mathbf{D}'_l,\ l \in \mathbf{L},\ n = 1, \ldots)$ *be an associated sequence of allocations. There exists a convergent subsequence,* $(x_{n_k}^{\mathbf{H}}: k = 1, \ldots)$, *with* $x^{\mathbf{H}} = \lim_{k\to\infty} x_{n_k}^{\mathbf{H}}$ *a feasible allocation,* $\Sigma_{h\in\mathbf{H}}\, x^h = w$.

Proof. Let $\bar{w} \in \Lambda$ be such that $w_n \leq \bar{w}$, for $n = 1, \ldots$. Let $\mathbf{X}_{\bar{w}}^h = \{x \in \mathbf{X}^h: x \leq \bar{w}\}$, for $h \in \mathbf{H}$, be the set of consumption bundles for individual h bounded above by $\bar{w}$. Since individual consumption sets are closed and bounded below, the set $\mathbf{X}_{\bar{w}}^h$ is closed and bounded. Since the set $\Pi_{l\in\mathbf{L}}\, [0, \bar{w}_l]$ is compact, $\mathbf{X}_{\bar{w}}^h$ is a compact subset of $\mathbf{X}^h$. It follows that the set $\mathbf{X}_{\bar{w}}^{\mathbf{H}} = \Pi_{h\in\mathbf{H}}\, \mathbf{X}_{\bar{w}}^h$ is compact. Hence, a convergent subsequence exists, $(x_{n_k}^{\mathbf{H}}: k = 1, \ldots)$, with $x^{\mathbf{H}} = \lim_{k\to\infty} x_{n_k}^h$. Since individual consumption sets are closed, $x^{\mathbf{H}}$ is an allocation. It remains to show that $x^{\mathbf{H}}$ is feasible or, equivalently, that $\Sigma_{h\in\mathbf{H}}\, x^h = w$. But this is evident, since $\Sigma_{h\in\mathbf{H}}\, x_l^h = \Sigma_{h\in\mathbf{D}'_l}\, x_l^h = \Sigma_{h\in\mathbf{D}'_l} \lim_{k\to\infty} x_{n_k}^h = \lim_{k\to\infty} \Sigma_{h\in\mathbf{D}'_l}\, x_{n_k,l}^h = w_l$, for $l \in \mathbf{L}$. The transposition of the summation and the limit is possible since $\mathbf{D}'_l$ is a finite set. □

If, for each commodity, individual consumption bundles are not restricted to vanish for all but finitely many individuals, the argument fails. If the feasibility constraint is imposed as an inequality, this does not occur, but the same point arises in the argument for the existence of competitive equilibria [Burke (1988)].

Associated with a feasible allocation there is an allocation of utilities,

$$u^{\mathbf{H}}(x^{\mathbf{H}}) = \{u^h(x^h): h \in \mathbf{H}\}\,.$$

Definition 1. An allocation, $x^{\mathbf{H}}$, Pareto dominates another, $x'^{\mathbf{H}}$, if and only if

$$u^h(x'^h) \geq u^h(x^h), \quad \text{for } h \in \mathbf{H},$$

with some strict inequality. An allocation is Pareto optimal if and only if it is feasible and no feasible allocation Pareto dominates it.

We denote commodity prices by

$$p = (\ldots, p_l, \ldots) \in \mathbf{P},$$

an element of the positive price domain, $\mathbf{P} = \Lambda_+ / \{0\}$. Again, the context should clarify whether p_k refers to the kth component of a price vector or to the kth term in a sequence of prices $(p_n: n = 1, \ldots)$.

At commodity prices p, the value of a consumption bundle, x, is

$$px = \sum_{l \in \mathbf{L}} p_l x_l,$$

which may be infinite. Note that prices do not define a linear functional on the commodity space. They do define a linear functional on $\Lambda_F = \{x \in \Lambda: x_l = 0$, if $l \not\in \mathbf{L}_F$ for a finite set $\mathbf{L}_F \subseteq \mathbf{L}\}$. Moreover, any non-trivial, weakly monotonically increasing linear functional of Λ_F is described by a $p \in \mathbf{P}$.

At commodities prices p, the individual optimization problem is

$$\max u^h(x) \text{ s.t. } px \leq pw^h, \quad \text{for } h \in \mathbf{H}.$$

Definition 2. A competitive equilibrium is a pair of prices and a feasible allocation, $(p^*, x^{*\mathbf{H}})$, such that, the commodity bundle x^{*h} solves the individual optimization problem at p^*,

$$u^h(x) > u^h(x^{*h}) \Rightarrow p^*x > p^*w^h, \quad \text{for } x \in \mathbf{X}^h,$$

and

$$p^*x^{*h} = p^*w^h, \quad \text{for } h \in \mathbf{H}.$$

It follows from the weak monotonicity of individual utility functions that at a competitive equilibrium

$$p^*w^h < \infty, \quad \text{for } h \in \mathbf{H}.$$

Note that competitive equilibrium defines a single budget constraint for each individual. Thus, implicitly, the asset market is complete. Each individual chooses a consumption bundle that he pays for at once. We can thus imagine that trade occurs ex ante, under a veil of ignorance, before any uncertainty is resolved. Many applications of the model of overlapping generations study economies in which the special demographic structure suggests multiple budget constraints. The resulting competitive equilibria under multiple budget constraints sometimes can be reduced to the equilibria described above, but often they cannot be. We have chosen not to survey the vast literature on competitive equilibria with multiple effective budget constraints in economies of overlapping generations.

Definition 3. An economy is individually finite if and only if, for $h \in \mathbf{H}$, the set of commodities desired by the individual, L^h, is finite.

A uniform bound on the number of commodities desired by each individual is not important. Each commodity can be replaced by a finite, yet unbounded over the set of commodities, number of perfect substitutes, thus destroying the uniform upper bound yet yielding an essentially equivalent economy, in particular preserving continuity in the product topology.

This concludes the description of an abstract exchange economy,

$$\mathbf{E} = \{\mathbf{L}, \mathbf{H}, (\mathbf{X}^h, u^h, w^h): h \in \mathbf{H}\}\ .$$

Note that we have incorporated the hypotheses that there is a countable infinity of commodities and a countable infinity of individuals, each commodity is desired and is essentially owned by finitely many individuals and that the utility functions of individuals display impatience. It remains to represent the recursive possibilities inherent in economies of overlapping generations.

Definition 4. An abstract exchange economy, $\mathbf{E} = \{\mathbf{L}, \mathbf{H}, (\mathbf{X}^h, u^h, w^h): h \in \mathbf{H}\}$, reduces to another, $\mathbf{E}' = \{\mathbf{L}', \mathbf{H}', (\mathbf{X}^{h'}, u^{h'}, w^{h'}): h' \in \mathbf{H}'\}$,

$$\mathbf{E} \rightarrow \mathbf{E}'\ ,$$

if and only if there exists a bijective map, $g: \mathbf{H} \rightarrow \mathbf{H}'$, and a linear, continuous, weakly monotonically increasing, surjective map, $\varphi: \Lambda \rightarrow \Lambda'$, such that, for $h \in \mathbf{H}$, $\mathbf{X}^{g(h)} = \varphi(\mathbf{X}^h)$, $w^{g(h)} = \varphi(w^h)$, and $u^{g(h)} = u^h \circ \varphi^{-1}$.

Economies are equivalent,

$$\mathbf{E} \sim \mathbf{E}'\ ,$$

if and only if there exist bijective, equivalence maps, $f: \mathbf{L} \rightarrow \mathbf{L}'$ and $g: \mathbf{H} \rightarrow \mathbf{H}'$, between the sets of commodities and individuals, respectively, in the two economies such that for $h \in \mathbf{H}$, $\mathbf{X}^{g(h)} = \varphi(\mathbf{X}^h)$, $w^{g(h)} = \varphi(w^h)$ and $u^{g(h)} =$

$u^h \circ \varphi^{-1}$, where $\varphi : \Lambda \to \Lambda'$ is defined componentwise by $\varphi_{l'}(x) = x_{f^{-1}(l')}$, for $l' \in \mathbf{L}'$.

Evidently, if the economies $\mathbf{E}$ and $\mathbf{E}'$ are equivalent under the equivalence maps $f: \mathbf{L} \to \mathbf{L}'$ and $g: \mathbf{H} \to \mathbf{H}$, $(p^*, x^{*\mathbf{H}})$ is a competitive equilibrium for the economy $\mathbf{E}$ if and only if $(p'^*, x'^{*\mathbf{H}})$ is a competitive equilibrium for the economy $\mathbf{E}'$, where $p'^*_{l'} = p^*_{f^{-1}(l')}$, for $l' \in \mathbf{L}'$, and $x'^{*g(h)}_{l'} = x^{*h}_{f^{-1}(l')}$, for $l \in \mathbf{L}$ and $h \in \mathbf{H}$.

If an economy, $\mathbf{E}$, is obtained from an economy, $\mathbf{E}'$, by replacing a commodity by a finite number of perfect substitutes, the economy $\mathbf{E}$ reduces to the economy $\mathbf{E}'$.

Economies are equivalent if they differ only in the indexation of commodities and individuals.

Evidently, the identity maps $f = i_{\mathbf{L}}$ and $g = i_{\mathbf{H}}$ establish the equivalence of an abstract exchange economy of overlapping generations, $\mathbf{E}$, with itself. But there may be other such equivalence maps.

Definition 5. The group, $\mathbf{G}$, of symmetrics of an abstract exchange economy, $\mathbf{E}$, is the group of equivalence maps of $\mathbf{E}$, where $(f_1, g_1) \circ (f_2, g_2) = (f_1 \circ f_2, g_1 \circ g_2)$ defines the composition, and the identity maps $(i_{\mathbf{L}}, i_{\mathbf{H}})$ define the unit element.

It is often of great interest whether or not there exist competitive equilibrium allocations, or other special allocations, $x^{\mathbf{H}}$, that are invariant under all symmetries, that is, for every symmetry, $(f, g) \in \mathbf{G}$, $x^{g(h)} = \varphi(x^h)$, for $h \in \mathbf{H}$. Another class of interesting allocations are those that are invariant to a subgroup, $\mathbf{G}' \subsetneq \mathbf{G}$, of symmetries of $\mathbf{E}$. When the temporal structure of the economy is explicit, symmetries can be interpreted as time invariance.

For a positive integer, n, $\mathbf{G}^n$ is the nth power of the group $\mathbf{G}$, a subgroup.

Definition 6. An allocation for the exchange economy $\mathbf{E}$ is a cycle of order n if and only if it is invariate under the subgroup $\mathbf{G}^n \subseteq \mathbf{G}$ of the group of symmetrics of $\mathbf{E}$, but is not invariant under any subgroup $\mathbf{G}^m$, for $m < n$.

1.1. The temporal and demographic structure

Economies of overlapping generations received their name from the special pattern of preferences and endowments that Allais and Samuelson used in their first examples in which individuals lived for two or three overlapping periods consuming and exchanging only in their lifetimes. We present now an explicit demographic and temporal structure underlying the abstract economy that we introduced above. This formulation includes as special cases the first examples as well as the so-called stochastic overlapping generations models as long as the set of histories of realizations of uncertainty up to any date is at most countable. The essential reason why all these models can be understood simply as instances of an abstract exchange economy is that in all of them individuals optimize under a single budget constraint.

We denote time periods by

$$t \in \mathbf{T}_{\underline{t},\bar{t}} = \{\underline{t}, \ldots, \bar{t}\}\ .$$

Thus, time extends infinitely into the future but not into the past if $\mathbf{T}_{\underline{t},\bar{t}} = \mathbf{T}_{1,\infty}$, and it extends infinitely into the future as well as into the past if $\mathbf{T}_{\underline{t},\bar{t}} = \mathbf{T}_{-\infty,\infty}$. Time is finite if both $\underline{t}$ and $\bar{t}$ are finite; they may coincide.

Under uncertainty, we denote date-events by

$$(s, t) \in \mathbf{S}\ ,$$

where $\mathbf{S} = \bigcup_{t \in \mathbf{T}_{\underline{t},\bar{t}}} \mathbf{S}_t$ is a non-empty, countable set, and

$$\mathbf{S}_t = \{1, \ldots, \mathbf{S}_t\} \times \{t\}\ , \quad \text{for } t \in \mathbf{T}_{\underline{t},\bar{t}}\ .$$

If $(s, t) \in \mathbf{S}_t$, we interpret s as an event at t or a history of realizations of uncertainty up to t. For $\underline{t} \leqslant t < \bar{t}$, there is associated with every date-event $(s, t+1) \in \mathbf{S}_{t+1}$ a unique date-event $\psi(s, t+1) = (s', t) \in \mathbf{S}_t$, its immediate predecessor. Thus $(\mathbf{S}, \psi)$ is a tree. A date-event (s', t') precedes a date-event (s, t) if and only if $(s', t') = \psi^n(s, t)$, for some $1 \leqslant n < \infty$; this we denote by $(s', t') < (s, t)$. For a date-event $(s, t) \in \mathbf{S}$, the branch that contains (s, t) is $\mathbf{S}_{(s,t)} = \{(s', t') \in \mathbf{S} : (s', t') = (s, t) \text{ or } (s', t') < (s, t) \text{ or } (s, t) < (s', t')\}$.

A temporal structure is thus a triple

$$(\mathbf{T}_{\underline{t},\bar{t}}, \mathbf{S}, \psi)\ .$$

In the special case of certainty, the temporal structure is simply $\mathbf{T}_{\underline{t},\bar{t}}$.

When time extends infinitely into the future but not into the past under uncertainty, one of countably many states of nature $\theta_t \in \Theta_t$, for $t \in \mathbf{T}_{1,\infty}$, may be realized each period; in particular, a date-event is $(s, t) = (\theta_1, \ldots, \theta_t, t) \in \mathbf{S}_t = \Pi_{t'=1}^{t} \Theta_{t'} \times \{t\}$. When time extends infinitely into the past as well as into the future, it is important to preserve the countability of the set of events at any finite time. This can only be maintained if there is no uncertainty before some finite time. For simplicity, we require that one of countably many states of nature, $\theta_{-\infty} \in \Theta_{-\infty}$, is realized at "$t = -\infty$" with no further uncertainty until $t = 0$, while one of countably many states of nature, $\theta_t \in \Theta_t$, may be realized each period $t \geqslant 1$; in particular, a date-event is $(s, t) = (\theta_{-\infty}, t)$, for $t < 1$, or $(\theta_{-\infty}, \theta_1, \ldots, \theta_t, t)$, for $t \geqslant 1$.

Definition 7. An exchange economy of overlapping generations is a triple,

$$(\mathbf{E}, (\mathbf{T}_{\underline{t},\bar{t}}, \mathbf{S}, \psi), (\sigma_{\mathrm{L}}, \sigma_{\mathrm{H}}))\,,$$

of an abstract exchange economy, $\mathbf{E}$, a temporal structure, $(\mathbf{T}_{\underline{t},\bar{t}}, \mathbf{S}, \psi)$, and a pair $(\sigma_{\mathrm{L}}, \sigma_{\mathrm{H}})$, where $\sigma_{\mathrm{L}} : \mathbf{L} \to \mathbf{S}$ is a function, while $\sigma_{\mathrm{H}} : \mathbf{H} \to \mathbf{S}$ is a correspondence such that (i) for $(s, t) \in \sigma_{\mathrm{H}}(h)$, either $w_l^h > 0$ or $l \in \mathbf{L}^h$ for some $l \in \sigma_{\mathrm{L}}^{-1}(s, t)$, (ii) if $w_l^h > 0$ or $l \in \mathbf{L}^h$ and $\sigma_{\mathrm{L}}(l) \not\in \sigma_{\mathrm{H}}(h)$, $(s', t') < \sigma_{\mathrm{L}}(l)$ for some $(s', t') \in \sigma_{\mathrm{H}}(h)$, and (iii) for $(s', t') \in \mathbf{S}$, there exists $(s, t) \in S$ such that $(s', t') < (s, t)$ and $\sigma_{\mathrm{H}}^{-1}(s, t) \neq \emptyset$.

From (iii), time extends infinitely into the future. Commodity l is available at $\sigma_{\mathrm{L}}(l)$. Individual h, conditional on (s, t), is "born" at $\underline{t}(h; (s, t)) = \sup\{t' \in \mathbf{T}_{\underline{t},\infty} :$ if $t'' < t'$ and $\sigma_{\mathrm{L}}(l) = (s'', t'') \in \mathbf{S}_{(s,t)}$, $w_l^h = 0$ and $l \not\in \mathbf{L}^h\}$; individual h is born at $\underline{t}(h) = \sup\{t \in \mathbf{T}_{\underline{t},\infty} :$ if $t' < t$ and $\sigma_{\mathrm{L}}(l) = (s, t)$, $w_l^h = 0$ and $l \not\in \mathbf{L}^h\} = \inf\{t' \in \mathbf{T}_{\underline{t},\infty} : t' = \underline{t}(h; (s, t))$ for some $(s, t) \in \mathbf{S}\}$. From (i) and (ii), $\underline{t}(h; (s, t)) = \inf\{t' \in \mathbf{T}_{\underline{t},\infty} : (s', t') \in \sigma_{\mathrm{H}}(h) \cap \mathbf{S}_{(s,t)}\}$ and $\underline{t}(h) = \inf\{t' \in \mathbf{T}_{\underline{t},\infty} : (s, t') \in \sigma_{\mathrm{H}}(h)\}$. From (iii), further, individuals do not cease to be born. The periods of consumption or endowment of the individual indeed begin at $\underline{t}(h)$ or $\underline{t}(h; (s, t))$, conditional on (s, t), and not earlier. The individual, conditional on (s, t), "dies" at $\bar{t}(h; (s, t)) = \inf\{t' \in \mathbf{T}_{\underline{t},\infty} :$ if $t'' > t'$ and $\sigma_{\mathrm{L}}(l) = (s'', t'') \in \mathbf{S}_{(s,t)}$, $w_l^h = 0$ and $l \not\in \mathbf{L}^h\}$; the individual dies at $\bar{t}(h) = \inf\{t' \in \mathbf{T}_{\underline{t},\infty} :$ if $t' > t$ and $\sigma_{\mathrm{L}}(l) = (s, t')$, $w_l^h = 0$ and $l \not\in \mathbf{L}^h\} = \sup\{t' \in \mathbf{T}_{\underline{t},\infty} : t' = \bar{t}(h; (s, t))$, for some $(s, t) \in \mathbf{S}_{(s,t)}\}$.

The date of birth of an individual may not be finite, $\underline{t}(h) = -\infty$. An individual may be "immortal", $\bar{t}(h) = \infty$.

As a special case, we obtain a model in which the life spans of individuals are stochastic [Yaari (1985)]. The state of nature at t, θ_t, may determine the individuals who are born as well as the individuals who die at t. In this case, the number of individuals alive at any date-event may be non-stochastic, while an individual may be alive at any date following his birth and is thus immortal.

We refer to σ_{H} as the demographic structure. Note that, since σ_{H} is a correspondence and not necessarily a function, we may not need to consider individuals as distinct according to the date-event at their birth.

Under certainty, we write $(\tau_{\mathrm{L}}, \tau_{\mathrm{H}})$ for $(\sigma_{\mathrm{L}}, \sigma_{\mathrm{H}})$.

A change of notation is convenient when the temporal structure is explicit. We denote commodities available at (s, t) by

$$(l, (s, t) \in \mathbf{L}_{(s,t)} = \{1, \ldots, L_{(s,t)}\} \times \{(s, t)\}\,, \quad \text{for } (s, t) \in \mathbf{S}\,,$$

the set of commodities at t is $\mathbf{L}_t = \bigcup_{(s,t)\in \mathbf{S}_t} \mathbf{L}_{(s,t)}$, and the set of commodities is $\mathbf{L} = \bigcup_{t\in \mathbf{T}_{\underline{t},\bar{t}}} \mathbf{L}_t$.

A commodity bundle is

$$x = (\ldots, x_{(s,t)}, \ldots) \in \Lambda ,$$

where $\Lambda = \Pi_{(s,t)\in \mathbf{S}} \Lambda_{(s,t)}$ and $x_{(s,t)} = (\ldots, x_{(l,(s,t))}, \ldots) \in \Lambda_{(s,t)}$ is a commodity bundle at (s, t); also, $\Lambda_t = \Pi_{(s,t)\in \mathbf{S}_t} \Lambda_{(s,t)}$, and $x_t \in \Lambda_t$ is a commodity bundle at t.

Individuals who are born at t, and thus form a generation, we denote by

$$(h, t) \in \mathbf{H}_t = \{1, \ldots, H_t\} \times \{t\} , \quad \text{for } t \in \mathbf{T} ,$$

and the set of all individuals is $\mathbf{H} = \bigcup_{t\in\mathbf{T}} \mathbf{H}_t$.

The periods of consumption of an individual are $\mathbf{T}^{(h,t)} = \{\underline{t}^{(h,t)}, \ldots, \bar{t}^{(h,t)}\} \subseteq \mathbf{T}_{\underline{t},\bar{t}}$, such that $t' \not\in \mathbf{T}^{(h,t)} \Rightarrow \mathbf{L}_{t'} \cap \mathbf{L}^{(h,t)} = \emptyset$, and his consumption span is $1 \leq A^{(h,t)} \leq \bar{t}^{(h,t)} - \underline{t}^{(h,t)} + 1$. The periods of endowment of an individual are $\mathbf{T}_w^{(h,t)} = \{\underline{t}_w^{(h,t)}, \ldots, \bar{t}_w^{(h,t)}\} \subseteq \mathbf{T}_{\underline{t},\bar{t}}$, such that $t' \not\in \mathbf{T}_w^{(h,t)} \Rightarrow w_{t'}^{(h,t)} = 0$, and his endowment span is $1 \leq A_w^{(h,t)} \leq \bar{t}_w^{(h,t)} - \underline{t}_w^{(h,t)} + 1$. Thus, $t = \min\{\underline{t}^{(h,t)}, \underline{t}_w^{(h,t)}\}$. The periods of consumption and endowment of an individual need not coincide. In particular, the endowment span of an individual may be infinite even though his consumption span is finite.

As a special case we obtain individually finite economies in which the consumption, and possibly the endowment span as well, of each individual is finite, which have been traditionally referred to as economies of overlapping generations.

Exchange economies of overlapping generations are equivalent,

$$(\mathbf{E}, (\mathbf{T}_{\underline{t},\bar{t}}, \mathbf{S}, \psi), (\sigma_{\mathbf{L}}, \sigma_{\mathbf{H}})) \sim (\mathbf{E}', (\mathbf{T}_{\underline{t}',\bar{t}'}, \mathbf{S}', \psi'), (\sigma'_{\mathbf{L}}, \sigma'_{\mathbf{H}}))$$

if and only if the abstract economies, $\mathbf{E}$ and $\mathbf{E}'$, are equivalent, $\mathbf{E} \sim \mathbf{E}'$.

Definition 8. The temporal structure is simple if and only if time extends infinitely into the future but not into the past, under certainty. The simple temporal structure is thus $\mathbf{T}_{1,\infty}$.

Lemma 2. *Every exchange economy of overlapping generations is equivalent to an economy with a simple temporal structure.*

Proof. Let $(\mathbf{E}, (\mathbf{T}_{\underline{t},\bar{t}}, \mathbf{S}, \psi), (\sigma_{\mathbf{L}}, \sigma_{\mathbf{H}}))$ be an exchange economy of overlapping generations. Let $\mathbf{T}_{1,\infty}$ be the simple temporal structure. Consider the map $\tau_{\mathbf{L}} : \mathbf{L} \rightarrow \mathbf{T}_{1,\infty}$ defined by $\tau_{\mathbf{L}}(l) = t$, if $t \geq 1$, and $\tau_{\mathbf{L}}(l) = 1 - t$, if $t \leq 0$, where $(t, s) = \sigma_{\mathbf{L}}(l)$. Consider the map $\tau_{\mathbf{H}} : \mathbf{H} \rightarrow \mathbf{T}_{1,\infty}$ defined by $\tau_{\mathbf{H}}(h) = t$, if $(t, s) \in \sigma_{\mathbf{H}}(h)$ and $t \geq 1$, $\tau_{\mathbf{H}}(h) = 1 - \bar{t}(h)$, if $\bar{t}(h) \leq 0$, $(s, t) \in \sigma_{\mathbf{H}}(h)$ and $t \leq 0$, and $\tau_{\mathbf{H}}(h) = 1$ if $\bar{t}(h) \geq 1$, $(s, t) \in \sigma_{\mathbf{H}}(h)$ and $t \leq 0$. Evidently, $(\mathbf{E}, \mathbf{T}_{1,\infty}, (\tau_L, \tau_{\mathbf{H}}))$ is an exchange economy of overlapping generations. □

This equivalence reduces economies in which time extends infinitely into the future as well as into the past, under certainty as well as under uncertainty, to economies under certainty in which time extends infinitely into the future but not into the past.

When time extends infinitely into the future but not into the past, continuity of the utility function in the product topology can be interpreted as impatience. The impatience implied by continuity in the product topology is stronger than that implied by continuity in the weak and hence the Mackey or norm topologies. Nevertheless, any continuous, intertemporally separable utility function $u^{(h,t)} = \Sigma_{t' \in \mathbf{T}^{(h,t)}} u_{t'}^{(h,t)}$ that are well defined everywhere on the consumption set with $\mathbf{X}^{(h,t)} = \Pi_{t' \in \mathbf{T}^{(h,t)}} \mathbf{X}_{t'}^{(h,t)}$ and $u_{t'}^{(h,t)} : \mathbf{X}_{t'}^{(h,t)} \to \mathbf{R}$ a continuous function, for $t' \in \mathbf{T}^{(h,t)}$, does satisfy continuity in the product topology. An important example is the function $u^{(h,t)} = \Sigma_{t' \in \mathbf{T}^{(h,t)}} \beta^{t'} v^{(h,t)}$, where $\mathbf{L}_{t'} = \bar{\mathbf{L}}$ for $t' \in \mathbf{T}^{(h,t)}$, $v^{(h,t)} : \bar{\mathbf{L}} \to \mathbf{R}$ is a continuous, bounded function, and $0 < \beta < 1$.

In examples, we suppose, without loss of generality, that the temporal structure is simple, unless we explicitly mention otherwise.

Example 1. Economic activity extends infinitely into the past as well as into the future under certainty. One commodity is available each period, $\mathbf{L}_t = \{(1, t)\}$, and one individual is born, $\mathbf{H}_t = \{(1, t)\}$. The economy is equivalent to an economy in which time extends infinitely into the future but not into the past, $\mathbf{T}_{t',\bar{t}'} = \mathbf{T}_{1,\infty}$, two commodities are available each period, $\mathbf{L}'_{t'} = \{(1, t), (2, t)\}$, and two individuals are born each period, $\mathbf{H}'_{t'} = \{(1, t'), (2, t')\}$. It suffices to identify periods $t = t' \geq 1$ and $t = 1 - t' \leq 0$ with period $t' \geq 1$, individual $(1, t)$ with individual $(1, t')$, for $t = t' \geq 1$, and individual $(1, t)$ with individual $(2, t')$ for $t = 1 - t' \leq 0$, and similarly commodity $(1, t)$ with commodity $(1, t')$, for $t = t' \geq 1$, and commodity $(1, t)$ with commodity $(2, t')$, for $t = 1 - t' \leq 0$.

Definition 9. The demographic structure is simple if and only if the temporal structure is simple and, in addition, the consumption span of each individual is two. This does not restrict the endowment spans of individuals.

Lemma 3 [Balakso, Cass and Shell (1980)]. *Every exchange economy of overlapping generations that is individually finite is equivalent to an economy with a simple demographic structure.*

Proof. The argument is constructive. Let $\mathbf{E}$ be an abstract economy that is individually finite.

Let L_0 be any finite set of commodities, and let $\mathbf{L}_1 = \{1\} \cup \mathbf{L}_0$ and $\mathbf{H}_1 = \{1\} \cup \{h \in \mathbf{H}: \mathbf{L}_1 \cap \mathbf{L}^h \neq \emptyset$ or $w_l^h > 0$, for $l \in \mathbf{L}_1\}$. Define inductively $\mathbf{L}_t = \{t\} \bigcup_{h \in \mathbf{H}_{t-1}} \mathbf{L}^h / \bigcup_{t' \leq t-1} \mathbf{L}_{t'}$, and $\mathbf{H}_t = \{t\} \cup \{h \in \mathbf{H}: \mathbf{L}_t \cap \mathbf{L}^h \neq \emptyset$ or $w_l^h > 0$, for $l \in \mathbf{L}_t\} / \bigcup_{t' \leq t-1} \mathbf{H}_{t'}$. Evidently, $\{\mathbf{L}_t: t \in \mathbf{T}_{1,\infty}\}$ is a partition of the set of commodities, $\mathbf{L}$, and $\{\mathbf{H}_t: t \in \mathbf{T}_{1,\infty}\}$ is a partition of the set of individuals, $\mathbf{H}$.

Consider the economy $(\mathbf{E}, \mathbf{T}_{1,\infty}, (\tau_{\mathbf{L}}, \tau_{\mathbf{H}}))$, where the maps $\tau_{\mathbf{L}}: \mathbf{L} \to \mathbf{T}_{1,\infty}$ and $\tau_{\mathbf{H}}: \mathbf{H} \to \mathbf{T}_{1,\infty}$ are defined, respectively, by $\tau_{\mathbf{L}}(l) = t$ such that $l \in \mathbf{L}_t$ and $\tau_{\mathbf{H}}(h) = t$ such that $h \in \mathbf{H}_t$. By construction, $\mathbf{L}^{(h,t)} \subseteq \mathbf{L}_t \cup \mathbf{L}_{t+1}$ and thus, without loss of generality, $\mathbf{T}^{(h,t)} = \{t, t+1\}$ or $A^{(h,t)} = 2$. □

Example 2. Economic activity extends infinitely into the future but not into the past, under certainty. One commodity is available each period, $\mathbf{L}_t = \{(1, t)\}$ and one individual is born, $\mathbf{H}_t = \{(1, t)\}$. The life span of an individual is three, $\mathbf{T}^{(h,t)} = \{t, t+1, t+2\}$. The economy is equivalent to an economy in which time extends infinitely into the future but not into the past, $\mathbf{T}' = \{1, \ldots\}$, two commodities are available each period, $\mathbf{L}'_{t'} = \{(1, t'), (2, t')\}$, and two individuals are born, $\mathbf{H}'_{t'} = \{(1, t'), (2, t')\}$. It suffices to identify period t with period $t' = [t/2]$, commodity $(1, t)$ with commodity $(1, t')$, for $t' = [t/2]$, if t is odd and $(2, t')$ if t is even, and similarly individual $(1, t)$ with individual $(1, t')$ for $t' = [t/2]$ if t is odd and $(2, t')$ if t is even, where $[k]$ is the smallest integer greater than or equal to k.

2. The existence of competitive equilibria

We consider first examples that isolate the reasons for the failure of existence of competitive equilibria. In some sense, assumptions 1–5 are necessary in order to ensure the existence of competitive equilibria, but they are not sufficient. Even after introducing assumptions 6 and 7, we only guarantee the existence of "compensated equilibria". The existence of competitive equilibria requires yet another condition.

Competitive equilibria may fail to exist when consumption bundles are uniformly bounded, but not continuous in the product topology, if the aggregate initial endowment is not bounded.

Example 3 [Wilson (1981)]. One commodity is available each period, $(1, t) = t$, and one individual is born, $(1, t) = t$. The utility function of individual t is $u^t = x_t + \alpha x_{t+1}$ with $\alpha < \frac{1}{2}$, and his initial endowment is $w^t = (\ldots, 0, w^t_t = 2^t, w^t_{t+1} = 2^{t+1}, 0, \ldots)$. In addition, a second individual is born in the first period, $(2, 1) = 0$. The utility function of individual 0 is $u^0 = \Sigma_{t=1}^{\infty} \beta^t x_t$, with $\frac{1}{2} < \beta < 1$, and his initial endowment is $w^0 = (w^0_1 = 2, w^0_2 = 0, \ldots)$. Observe that the utility function of individual 0 is indeed Mackey continuous on $\Lambda_{\infty,+}$. However, the aggregate initial endowment is $w = (4, 8, 16, \ldots)$ and hence not bounded, $w \not\in \Lambda_{\infty,+}$. In order to show that no competitive equilibria exist, we argue by contradiction. Suppose p^* are competitive equilibrium prices. Note first that $p^*_{t+1} \leq \frac{1}{2} p^*_t$; otherwise, from the optimization of individual t it follows that at the associated allocation $x^{*t}_{t+1} = 0$ and hence, from the budget constraint, $p^*_t x^{*t}_t = p^*_t 2^t + p^*_{t+1} 2^{t+1}$ which implies, if $p^*_{t+1} > \frac{1}{2} p^*_t$, that $x^{*t}_t > 2^{t+1} =$

w_t, a contradiction. But $p^*_{t+1} \leq \frac{1}{2} p^*_t$ implies, since $\beta > \frac{1}{2}$, that there is no solution to the optimization problem of individual 0.

Competitive equilibria may fail to exist if infinitely many individuals desire some commodity.

Example 4 [Burke (1988)]. One commodity, $(1, t) = t$, is available in periods following the first, $t = 2, \ldots$, while two commodities, $(1, 1) = 0$ and $(2, 1) = 1$, are available in the first period. In the first period, a countable infinity of individuals, $(h, 1) = h$ for $h \in \mathbf{H}_1 = \{1, \ldots\}$, are born and they are the only individuals in the economy. Individual h has utility function $u^h = x_0 + x_h$, and his initial endowment is $w^h = (0, \ldots, 0, w^h_{h-1} = 1, w^h_h = 1, 0, \ldots)$. Note that all individuals desire commodity 0. In order to show that no competitive equilibria exist, we argue by contradiction. Suppose (p^*, x^{*H}) is a competitive equilibrium. Evidently, $p^* \gg 0$. Utility maximization implies that $x^{*h}_t = 0$, for $t \neq 0, h$, and the market clearing conditions reduce to $\Sigma^{\infty}_{h=1} x^{*h}_0 = 1$, while $x^{*h}_h = w^h_h + w^{h+1}_h = 2$, for $h = 1, \ldots$. From the budget constraint of individual h it follows then that $p^* w^h = p^*_h + p^*_{h-1} \geq 2p^*_h$ or $\cdots \leq p^*_h \leq p^*_{h-1} \leq \cdots \leq p^*_0$. Indeed, $\cdots = p^*_h = p^*_{h-1} = \cdots = p^*_0$; for if $p^*_h < p^*_{h-1}$ for some h, $p^*_h < p^*_0$, $x^{*h}_0 = 0$, and then, from the linearity of the utility function of individual h and again from his budget constraint, $p^*_h = p^*_{h-1}$, a contradiction. But the constancy of prices leads to a contradiction since it implies from the individual budget constraints and market clearing that $x^{*h}_0 = 0$, for $h = 1, \ldots$.

With a particular commodity desired by infinitely many individuals, the limit of competitive equilibrium allocations for the finite economies obtained by restricting attention to individuals $h \leq n$ and commodities $h \leq n$ may not be a competitive equilibrium for the full economy. For a particular commodity, the aggregate feasibility constraint may be satisfied with equality all along the sequence of competitive equilibrium allocations for the truncated economies but not at the limit, even though the price of the commodity remains positive. The limit operation need not commute with aggregation across individuals when the latter involves an infinite sum and thus the set of feasible allocations is not compact; equivalently, the infinite sum of upper-semi-continuous correspondences, the individual excess demands, need not be upper-semi-continuous.

Indeed, Assumption 5 requires that at most finitely many individuals desire each commodity.

Definition 10. An abstract, exchange economy is irreducible if and only if, at any feasible allocation, $x^{\mathbf{H}}$, and for any non-trivial partition, $\{\mathbf{H}^1, \mathbf{H}^2\}$, of the set of individuals, there exists an individual, $h_2 \in \mathbf{H}^2$, such that

$$u^{h_2}\left(x^{h_2} + \sum_{h \in \mathbf{H}^1} w^h\right) > u^{h_2}(x^{h_2}) .$$

This does not allow for a reallocation of commodities and thus strengthens the analogous condition for finite economies [Nikaido (1956), McKenzie (1959); also, Debreu (1962), Arrow and Hahn (1971)].

Assumption 6. The abstract exchange economy is irreducible.

If the economy is not irreducible, competitive equilibra may not exist; this is the case in a finite economy as well.

Example 5 [Arrow (1951)]. Consider an abstract finite exchange economy. There are two commodities, $l = 1, 2$, and two individuals, $h = 1, 2$. Individual $h = 1$ has utility function $u^1 = x_1$ and initial endowment $w^1 = (1, 1)$. Individual 2 has utility function $u^2 = x_2$ and endowment $w^2 = (0, 1)$. Evidently, the economy is not irreducible. For the partition $\mathbf{H}^1 = \{2\}$ and $\mathbf{H}^2 = \{1\}$, no individual in $\mathbf{H}^2$ benefits by receiving the aggregate endowment of individuals in $\mathbf{H}^1$. In order to show that competitive equilibria do not exist we argue by contradiction. Suppose $p^* = (p_1^*, p_2^*)$ are competitive equilibrium prices. If $p_2^* > 0$, commodity 2 is in excess supply since individual 1 supplies the commodity inelastically while individual 2 is not endowed with commodity 1 to offer in exchange. If $p_2^* = 0$, there is no solution to the optimization problem of individual 2.

2.1. *Truncations*

Let

$$\mathbf{E} = \{\mathbf{L}, \mathbf{H}, (\mathbf{X}^h, u^h, w^h)\colon h \in \mathbf{H}\}$$

be an abstract exchange economy.

The argument for the existence of competitive equilibria proceeds by considering a sequence of finite or "truncated" economies that tend to the "full" economy, at the limit.

Consider a finite set of commodities, $\mathbf{L}^n \subset \mathbf{L}$, and a finite set of individuals, $\mathbf{H}^n \subset \mathbf{H}$.

Consider the commodity space Λ^n, Euclidean space of dimension $\mathbf{L}^n$, the cardinality of $\mathbf{L}^n$ and commodity bundles $x^n \in \Lambda^n$; for a commodity bundle $x \in \Lambda$, $x^n = \mathrm{proj}_{\Lambda^n} x$; $\Lambda_+^n = \{x^n \in \Lambda^n\colon x^n \geq 0\}$.

For a commodity bundle $x^n \in \Lambda^n$ and for an individual $h \in \mathbf{H}$, we write $\hat{x}^{n,h} \in \Lambda$ for the commodity bundle defined by $\hat{x}_l^{n,h}$, for $l \in \mathbf{L}^n$, and $\hat{x}_l^{n,h} = w_l^h$, for $l \not\in \mathbf{L}^n$. For prices $p^n \in \mathbf{P}^n = \Lambda_+^n / \{0\}$, we write $\hat{p}^n \in \mathbf{P}$ for the prices defined by $\hat{p}_l^n$, for $l \in \mathbf{L}^n$, and $\hat{p}_l^n = 0$, for $l \not\in \mathbf{L}^n$. For commodity bundles $x \in \Lambda$ and $x' \in \Lambda$, we write $(x \wedge_n x') \in \Lambda$ for the commodity bundle defined by $(x \wedge_n x')_l = x_l$, for $l \in \mathbf{L}^n$, and $(x \wedge_n x')_l = x'_l$, for $l \not\in \mathbf{L}^n$. The vector of units in

Λ^n is $\tilde{1}^n$. The context should make clear whether x^k refers to a consumption bundle for the kth individual in the full economy or to a commodity bundle in the kth truncation.

A finite economy

$$\mathbf{E}^n = \{\mathbf{L}^n, \mathbf{H}^n, (\mathbf{X}^{n,h}, u^{n,h}, w^{n,h}): h \in \mathbf{H}^n\}$$

is obtained by considering commodities in $\mathbf{L}^n$ and individuals in $\mathbf{H}^n$ and restricting the characteristics of individuals to $\Lambda^n \subset \Lambda$. The consumption set of an individual is $\mathbf{X}^{n,h} = \{x^n: \hat{x}^{n,h} \in \mathbf{X}^h\}$, his utility function is $u^{n,h}: \mathbf{X}^{n,h} \to \mathbf{R}$ defined by $u^{n,h}(x^n) = u^h(\hat{x}^{n,h})$, and his initial endowment is $w^{n,h}$. An allocation is an array $x^{n,\mathbf{H}^n} = \{x^{n,h} \in \mathbf{X}^{n,h}: h \in \mathbf{H}^n\}$. An allocation, $x^{n,\mathbf{H}^n}$, of commodities in the truncated economy is unambiguously associated with an allocation $\hat{x}^{n,\mathbf{H}}$ in the full economy, where $\hat{x}^{n,h} = w^h$, for $h \not\subseteq \mathbf{H}^n$. The desirability of commodities, Pareto dominance, irreducibility and competitive equilibrium are defined by analogy to the definitions in the full economy or, equivalently, in a standard finite economy.

An increasing sequence of sets of commodities, $(\mathbf{L}^n : \mathbf{L}^n \subseteq \mathbf{L}^{n+1}, n = 1, \ldots)$, converges to the set of commodities, $\mathbf{L}$, if and only if $\bigcup_{n=1}^{\infty} \mathbf{L}^n = \mathbf{L}$; similarly, an increasing sequence of sets of individuals, $(\mathbf{H}^n: \mathbf{H}^n \subseteq \mathbf{H}^{n+1}, n = 1, \ldots)$, converges to the set of individuals, $\mathbf{H}$, if and only if $\bigcup_{n=1}^{\infty} \mathbf{H}^n = \mathbf{H}$. The sequence of finite truncated economies, $(\mathbf{E}^n: n = 1, \ldots)$, associated with convergent increasing sequences of finite sets of individuals and commodities we say converges to the economy $\mathbf{E}$.

The economy $\mathbf{E}$ is sequentially irreducible if and only if it is the limit of a sequence of finite irreducible economies.

Perhaps surprisingly, an economy may be sequentially irreducible even though it fails to be irreducible and competitive equilibria fail to exist.

Example 6. Two commodities are available each period, (l, t), for $l = 1, 2$. An individual, $(2, 1) = 2$, has utility function $u^2 = x_{(2,1)}$ and initial endowment $w^2 = (w_1^2 = (\frac{1}{2}, 0), \ldots, w_t^2 = (\frac{1}{2}, 0), \ldots)$. In addition, each period, an individual is born, $(1, t)$, and has utility function

$$u^{(1,t)} = \begin{cases} x_{(1,t)} + x_{(2,t+1)} - 1\,, & \text{for } 0 \leq x_{(2,t+1)}, x_{(1,t)} \leq 1 \text{ and } x_{(2,t+1)} + x_{(1,t)} \leq 1\,, \\ \dfrac{2x_{(2,t+1)} + 2x_{(1,t)} - 2}{1 + x_{(1,t)}}\,, & \text{for } 0 \leq x_{(2,t+1)} \leq 2,\ 0 \leq x_{(1,t)} \leq 1 \text{ and } 1 \leq x_{(2,t+1)} + x_{(1,t)}\,, \\ x_{(2,t+1)}\,, & \text{for } x_{(2,t+1)} \geq 2 \text{ or } x_{(1,t)} \geq 1\,, \end{cases}$$

and initial endowment $w^{(1,t)} = (\ldots 0, w_t^{(1,t)} = (0, 2), 0, \ldots)$. Observe that individual $(1, t)$ desires commodity $(1, t)$ only as long as $x_{(2,t+1)} \leq 2$. Consider first the truncated economy $\mathbf{E}_{\bar{t}}$, associated with $\mathbf{L}^{\bar{t}} = \bigcup_{t=1}^{\bar{t}} \mathbf{L}_t$ and individuals

$\mathbf{H}^{\bar{t}} = \bigcup_{t=1}^{\bar{t}} \mathbf{H}_t$. In order to show that it is irreducible, suppose irreducibility fails for some non-trivial partition $\{\bar{\mathbf{H}}^{\bar{t},1}, \bar{\mathbf{H}}^{\bar{t},2}\}$. It follows from the preferences and endowments of individuals that if $(1,t) \in \bar{\mathbf{H}}^{\bar{t},1}$ also $(1, t-1) \in \bar{\mathbf{H}}^{\bar{t},1}$ and thus $(1,1) \in \bar{\mathbf{H}}^{\bar{t},1}$. Suppose $2 \in \bar{\mathbf{H}}^{\bar{t},2}$. Then $(1,1) \in \bar{\mathbf{H}}^{\bar{t},2}$ and hence $\bar{\mathbf{H}}^{\bar{t},2} = \mathbf{H}$, a contradiction. Conversely, suppose $2 \in \mathbf{H}^{\bar{t},1}$. Thus, since at a feasible allocation for the truncated economy individual $(1,\bar{t})$ indeed desires commodity $(1,\bar{t})$ with which individual 2 is endowed, $(1,\bar{t}) \in \bar{\mathbf{H}}^{\bar{t},1}$ and hence $\bar{\mathbf{H}}^{\bar{t},1} = \mathbf{H}^{\bar{t}}$, which is a contradiction. Competitive equilibria prices for the truncated economy are $p^{*\bar{t}} = (p_1^{*\bar{t}} = (0,1), p_2^{*\bar{t}} = (0,1), \ldots, p_{\bar{t}-1}^{*\bar{t}} = (0,1), p_{\bar{t}}^{*\bar{t}} = (4,1))$ and the associated allocation is described by $x^{*\bar{t},2} = (x_1^{*\bar{t},2} = (0,2), 0, \ldots)$, $x^{*\bar{t},(1,t)} = (\ldots 0, x_{t+1}^{*\bar{t},(1,t)} = (0,2), 0, \ldots)$, for $t = 1, \ldots, \bar{t}-1$, and $x^{*\bar{t},(1,\bar{t})} = (\ldots, 0, x_{\bar{t}}^{*\bar{t},(1,\bar{t})} = (\frac{1}{2}, 0))$. Alternatively consider the full economy, $\mathbf{E}$, and the feasible allocation $\bar{x}^* = \lim_{\bar{t}\to\infty} \hat{x}^{*,\bar{t}}$. At this allocation, commodities $(1,t)$, with which individual 2 is endowed, are not desired by any other individual, and hence the utility of no individual in $\bar{\mathbf{H}}^2 = \{(1,1), \ldots\}$ would increase if he were to receive the aggregate endowment of individuals in $\bar{\mathbf{H}}^1 = \{2\}$. It follows that the full economy, $\mathbf{E}$, is not irreducible. In order to show that indeed no competitive equilibria exist for the full economy we argue by contradiction. Suppose p^* are competitive equilibrium prices. From the strict monotonicity of the utility functions of individuals 2 and $(1,t)$ in commodities $(2,1)$ and $(2,t+1)$, respectively $p^*_{(2,t)} > 0$; since $(2,1)$ is not desired by any individual other than 2, and since this individual is endowed only with commodity $(1,t)$, equilibrium requires that $p^*_{(1,t)} > 0$ for some t. But this is a contradiction since commodity $(2,t)$ is desired only by individual $(1, t-1)$ and hence at the associated competitive equilibrium allocation $x^{*(1,t)}_{(2,t+1)} = 2$, which in turn implies that commodity $(1,t)$ is not desired by any individual. Observe that as the point of truncation tends to infinity, the competitive equilibrium prices tend to $\lim_{\bar{t}\to\infty} \hat{p}^{*\bar{t}} = p^* = (p_1^* = (0,1), \ldots, p_t^* = (0,1), \ldots)$ at which the revenue of individual 2 vanishes.

Competitive equilibria may fail to exist due to the structure of an individual consumption set; in particular if a commodity bundle that coincides with a consumption bundle up to time t and with the initial endowment of the individual after time t, fails to be in the consumption set.

Example 7 [Burke (1988)]. One commodity is available each period, $(1,t) = t$. In the first period, two individuals are born, $(1,1) = i$ and $(2,1) = j$, and they are the only individuals in the economy. Individuals i has consumption set $\mathbf{X}^i = \{x\colon x \geq 0 \text{ and } x_t + 2^{t-3}(x_2 - 1) \geq 0 \text{ for } t = 3, \ldots\}$, his utility function is $u^i = x_1$, and his initial endowment is $w^i = (\ldots, 2, \ldots)$. Note that a consumption bundle such as $x = (1, \frac{1}{2}, \frac{1}{2}, 1, 2, 4, 8, \ldots)$ does not yield a consumption bundle if x_t is replaced by w_t, for t large. Individual j has a consumption set $\mathbf{X}^j = \{x\colon x \geq 0\}$, utility function $u^j = x_1 + \Sigma_{t=2}^{\infty} 4^{-(t+1)} \min\{x_t, 5\}$ and initial endowment $w^j = (\dots, 2, \dots)$. In order to show that no competitive equilibria exist we argue by contradiction. Suppose $(p^*, x^{*\mathbf{H}})$ be a competitive equilibrium. By the strict monotonicity of the utility function of individual j, $p^* \gg 0$. Market clearing, the boundedness from below of $\mathbf{X}^i$, the requirement that $x^{*i} \in \mathbf{X}^i$ and in particular the structure of $\mathbf{X}^i$ imply $x_2^{*i} \geq 1$. It follows that $x^{*i} = (x_1^{*i}, 1, 0, \dots)$ while $x^{*j} = (4 - x_1^{*i}, 3, 4, \dots)$. Utility maximization by individual j then implies that $p^* = (p_1^*, p_2^*, \dots, 2^{-2(t-2)} p_2^*, \dots)$. But this is a contradiction since at prices p^* individual i does not maximize utility subject to the budget constraint at x^{*i}; the alternative bundle $x' = x(_1^{*i} + p_2^*/2p_1^*, 0, \dots, 2^{t-3}, \dots)$ is a preferred point in the consumption set and it is affordable, $p^* x' = p_1^* x_1^{*i} + \frac{1}{2} p_2^* + \Sigma_{t=3}^{\infty} (2^{t-3})(2^{-2(t-2)}) p_2^* = p_1^* x_1^{*i} + \Sigma_{t=2}^{\infty} (\frac{1}{2})^{t-1} p_2^* = p_1^* x_1^{*i} + p_2^* = p^* x^{*i}$.

Definition 11. Autarky is everywhere eventually individually feasible, if and only if, for $h \in \mathbf{H}$, for any increasing, convergent sequence of finite sets of commodities, $(\mathbf{L}^n\colon n = 1, \dots)$ and for any consumption bundle $x^h \in \mathbf{X}^h$ there exists $\bar{n}$ such that $\hat{x}^{n,h} \in \mathbf{X}^h$, for $n = \bar{n}, \dots$.

This restricts the structure of individual consumption sets; yet, it is weaker than the assumption that $\mathbf{X}^h = \Lambda_+$ [Wilson (1981), Burke (1988)] and is necessarily satisfied in individually finite economies.

Assumption 7. In the abstract exchange economy, autarky is everywhere individually feasible.

Competitive equilibria may fail to exist in the presence of individuals whose consumption-endowment spans are infinite yet their initial endowment is an eventually negligible proportion of the aggregate endowment.

Example 8 [Wilson (1981)]. One commodity is available each period, $(1, t) = t$. In the first period, two individuals are born; the first, $(1, 1) = 1$, has a consumption span of two, while the second, $(2, 1) = i$, has an infinite consumption span. In periods following the first, one individual is born, $(1, t) = t$, whose consumption span is two. The utility function of an individual with a two-period consumption span is $u^t = x_t + 3x_{t+1}$, and this initial endowment is $w^t = (\dots, 0, w_t^t = 1, w_{t+1}^t = 1, 0, \dots)$. The utility function of the individual with infinite consumption is $u^i = x_1$, and his initial endowment is $w^i = (\dots, w_t^i = (\frac{1}{2})^t, \dots)$. Note that the endowment of individual i is eventually a negligible proportion of the aggregate endowment; that individual i derives utility from consumption only in the first period simplifies the argument but is not essential. In order to show that no competitive equilibria exist, we argue by contradiction. Let p^* be competitive equilibrium prices. Since every commodity enters the utility function of at least one individual strictly monotonically, $p^* \gg 0$. Let $\mathbf{T}_{1,\infty}^* = \{t > 1\colon p_{t+1}^* < 3p_t^*\} \subset \mathbf{T}$. Observe first that $\mathbf{T}_{1,\infty}^*$ is an infinite set; otherwise, $p^* w^i = \infty$, which is incompatible with equilibrium, in particular with market clearing in the first period. Further, for $t \in \mathbf{T}^*_{1,\infty}$, $x^{*t}_t = 0$ and hence $x^{*t-1}_t = w^{t-1}_t + w^t_t + w^i_t > 2$; from the budget constraint of individual $(t-1)$ then, $2p^*_t < p^*_t x^{*t-1}_t = p^*_{t-1} + p^*_t$ and hence $p^*_{t-1} > p^*_t$. Since $\mathbf{T}^*_{1,\infty}$ is an infinite set, $p^*_1 > p^*_2 > \cdots > p^*_{t-1} > p^*_t > \cdots$. It follows that $p^* w^i = \Sigma^{\infty}_{t=1} 2^{-t} p^*_t < p^*_1$. From the budget constraint of individual i, $x^{*i}_1 < 1$, while $x^{*1}_1 = 0$ since $p^*_2 < p^*_1$. But this is a contradiction since $w^i_1 + w^1_1 = \frac{3}{2} > 1 > x^{*i}_1 + x^{*1}_1$, while $x^{*t}_1 = 0$ for $t > 1$.

In the same economy, it is remarkable that by permitting some individuals to overspend their budget, without requiring any individuals to spend less than their budget, market clearing can be restored. Moreover, the income transfer can be restricted to individuals with infinite endowment spans. The argument is constructive. Let $\bar{p}^* = (\ldots, \bar{p}^*_t, \ldots)$, where $\bar{p}^*_1 = 1$ and $\bar{p}^*_{t+1} = (1 + 2^{-t})^{-1} p^*_t$, for $t = 2, \ldots$. Consider the allocation $\bar{x}^{*\mathbf{H}} = \{\bar{x}^{*i}, \bar{x}^{*t} : t = 1, \ldots\}$ where $\bar{x}^{*i} = (\frac{3}{2}, 0, \ldots)$, while $\bar{x}^{*t} = (\ldots, 0, \bar{x}^{*t}_{t+1} = 2 + 2^{-(t+1)}, 0, \ldots)$. Since $\bar{p}^* w^i < \bar{p}^*_1$ while $\bar{p}^* x^{*i} = (\frac{3}{2})\bar{p}^*_1$, a net transfer of revenue $\tau^{*i} = \bar{p}^* x^{*i} > (\frac{1}{2})\bar{p}^*_1 > 0$ implements $(\bar{p}^*, \bar{x}^{*\mathbf{H}})$ as a competitive equilibrium with a positive net transfer of revenue or a compensated equilibrium.

Definition 12. A compensated equilibrium is a pair, $(\bar{p}^*, \bar{x}^{*\mathbf{H}})$, of prices and a feasible allocation such that

$$u^h(x) > u^h(\bar{x}^{*h}) \Rightarrow \bar{p}^* x > \bar{p}^* \bar{x}^{*h}, \quad \text{for } x \in \mathbf{X}^h$$

and

$$\bar{p}^* \bar{x}^{*h} \geq \bar{p}^* w^h, \quad \text{for } h \in \mathbf{H},$$

with equality whenever an individual is endowed with finitely many components.

From the weak monotonicity of the individual utility functions, it follows that at a competitive equilibrium

$$\bar{p}^* w^h \leq \bar{p}^* \bar{x}^{*h} < \infty, \quad \text{for } h \in \mathbf{H}.$$

A compensated equilibrium is a competitive equilibrium if and only if

$$\bar{p}^* \bar{x}^{*h} = \bar{p}^* w^h, \quad \text{for } h \in \mathbf{H}.$$

From the weak monotonicity of the utility functions, it follows that at a compensated equilibrium

$$u^h(x) \geq u^h(\bar{x}^{*h}) \Rightarrow \bar{p}^* x \geq \bar{p}^* \bar{x}^{*h}, \quad \text{for } x \in \mathbf{X}^h, h \in \mathbf{H}.$$

Theorem 1 [Wilson (1981), Burke (1988); also, Balasko, Cass and Shell (1980), Okuno and Zilcha (1982)]. *In an abstract exchange economy, under Assumptions 1–7, compensated equilibria exist.*

We consider a convergent sequence of prices and a convergent sequence of feasible allocations for the full economy obtained from the competitive equilibria for a sequence of finite modified truncated economies.

Lemma 4. *If* $(p_n \in \mathbf{P}: n = 1, \ldots)$ *and* $(x_n \in \Lambda_+: n = 1, \ldots)$ *are convergent sequences of prices and consumption bundles.*

$$(\lim_{n\to\infty} p_n)(\lim_{n\to\infty} x_n) \leq \liminf_{n\to\infty}(p_n x_n)\,.$$

Proof. Let $p = \lim_{n\to\infty} p_n$ and $x = \lim_{n\to\infty} x_n$. For any k, it follows from positivity that $(p_n \wedge_k 0)(x_n \wedge_k 0) \leq p_n x_n$. Letting $n \to \infty$, we obtain that $(p \wedge_k 0)(x \wedge_k 0) \leq \liminf_{n\to\infty}(p_n x_n)$. Since $px = \lim_{k\to\infty}(p \wedge_k 0)(x \wedge_k 0)$, $px \leq \liminf_{n\to\infty}(p_n x_n)$. □

It is worth remarking that the reverse inequality in Lemma 4 need not hold. Take $p_n = x_n = \tilde{1}_n$, $n = 1, \ldots$; evidently, $\lim_{n\to\infty} p_n = \lim_{n\to\infty} x_n = 0$. Then $(\lim_{n\to\infty} p_n)(\lim_{n\to\infty} x_n) = 0 < 1 = \liminf_{n\to\infty} p_n x_n$.

A difficulty in proving the existence of compensated equilibria arises from the possible failure of the economy to be sequentially irreducible. In order to obtain competitive equilibria for a convergent sequence of finite truncated economies, it is necessary to perturb the structure of each economy with the perturbation vanishing at the limit. If the perturbation involves only the endowment of each individual, it may not be limited to finitely many commodities. It is then necessary, and rather involved, to show that at the limit the value of the perturbation vanishes [Burke (1988)]. In our argument we perturb the utility functions as well as the initial endowments. The perturbation of the utility functions allow us to perturb the endowment of each individual in only one commodity and simplifies the limiting argument. The perturbation of the initial endowments allow us to perturb the marginal utility of each individual in only one commodity, different for each individual, thus preserving the property, essential for the limiting argument, that finitely many individuals desire each commodity.

Proof of Theorem 1. Consider the convergent increasing sequence of sets of commodities, $(\mathbf{L}^n: \mathbf{L}^n \subset \mathbf{L}^{n+1}, n = 1, \ldots)$, where $\mathbf{L}^n = \{1, \ldots, n\}$, and the convergent increasing sequence of finite sets of individuals, $(\mathbf{H}^n: \mathbf{H}^n \subset \mathbf{H}^{n+1}, n = 1, \ldots)$, where $\mathbf{H}^n = \{1, \ldots, n\} \subset \mathbf{H}$.

Recall that Λ^n is the Euclidean space of dimension L^n, and, for $x \in \Lambda$, $x^n = \mathrm{proj}_{\Lambda^n} x$ while, for $x^n \in \Lambda^n$ and $h \in \mathbf{H}$, $\hat{x}^{n,h}$ is the vector defined by $\hat{x}^{n,h}_l = x^n_l$, for $l \in \mathbf{L}^n$, and $\hat{x}^{n,h}_l = w^h_l$, for $l \not\in \mathbf{L}^n$. Prices are $p^n \in \mathbf{P}^n = \Lambda^n_+/\{0\}$, and $\hat{p}^n$ is defined by $\hat{p}^n_l = p_l$, for $l \in \mathbf{L}^n$ and $\hat{p}^n_l = 0$, for $l \not\in \mathbf{L}_n$. The vector of units in Λ^n is $\tilde{1}^n$.

For $n = 1, \ldots$ the modified truncated economy $\mathbf{E}'^n$ is obtained by first perturbing the utility functions and the initial endowments of individuals in the full economy, and then truncating, according to

$$u'^{n,h}(x^n) = u^h\left(\hat{x}^{n,h} + \frac{1}{n} x_h \tilde{1}\right), \quad \text{for } h \in \mathbf{H}^n ,$$

and

$$w'^{n,h} = \left(w^h + \frac{1}{n} \tilde{1}_{h+1}\right)^n , \quad \text{for } h \in \mathbf{H}^n/\{n\} ,$$

while

$$w'^{n,n} = \left(w^n + \frac{1}{n} \tilde{1}_1\right)^n .$$

The truncated, perturbed utility function of individual h, $u'^{n,h}$, is strictly increasing in commodity h, since the utility function, u^h, is weakly monotonic; also, $u'^{n,h}$ is quasi-concave. The modified, truncated initial endowment of individual $h < n$, $w'^{n,h}$, is positive in commodity $h + 1$, and of individual $h = n$ in commodity 1.

The set, $\mathbf{D}'^n_l$, of individuals who desire commodity l in the modified truncated economy is contained in $\mathbf{D}_l \cup \{l\}$, a finite set independent of n. The aggregate endowment is $(\Sigma_{h \in \mathbf{H}_n} w^h + (1/n)\tilde{1})^n \gg 0$.

Let $(p^{*n}, x^{*n,\mathbf{H}^n})$ be a competitive equilibrium for the modified truncated economy $\mathbf{E}'^n$. That a competitive equilibrium exists follows by the standard argument for finite economies, since the economy is irreducible [McKenzie (1959), Debreu (1962), Arrow and Hahn (1971)]. Furthermore, the income of each individual at a competitive equilibrium is positive, $p^{*n} w'^{n,h} > 0$ and $p^{*n} \gg 0$. Without loss of generality we suppose that $p^{*n} w'^{n,1} = 1$. Also, since individuals do not display satiation while consumption sets allow for free disposal we have that $x^{*n,h}_l = 0$ for $h \not\in \mathbf{D}_l \cup \{l\}$.

Let $w'^n = w + (1/n)(\tilde{1} \wedge_n 0) \in \Lambda$. Note that, for $h \in \mathbf{H}$, $\lim_{n\to\infty} \hat{w}'^{n,h} = w^h$ and also $\lim_{n\to\infty} w'^n = w$.

Let $\hat{x}^{*n,\mathbf{H}}$ be the allocation for the full economy associated with the equilibrium allocation for the modified truncated economy, defined by $\hat{x}^{*n,h} = w^h$, for $h \not\in \mathbf{H}^n$. Observe that by construction $\Sigma_{h \in \mathbf{H}} \hat{x}^{*n,h} = w'^m$. By passing to a subsequence, by Lemma 1 we obtain that

$$\bar{x}^{*\mathbf{H}} = \lim_{n\to\infty} \hat{x}^{*n,\mathbf{H}}$$

is a feasible allocation.

Claim 1. *For $h \in \mathbf{H}$, there exist finite, positive scalars, $0 < \underline{c}^h \leq \bar{c}^h < \infty$, such that*

$$0 < \underline{c}^h \leq \liminf_{n\to\infty} p^{*n} w^{n,h} \leq \limsup_{n\to\infty} p^{*n} w'^{n,h} \leq \bar{c}^h < \infty .$$

We argue by contradiction. Let $\mathbf{H}^1 = \{h \in \mathbf{H}: \limsup_{n\to\infty} p^{*n} w'^{n,h} = \infty\}$ and let $\mathbf{H}^2 = \{h \in \mathbf{H}: \limsup_{n\to\infty} p^{*n} w'^{n,h} < \infty\}$. Suppose $\mathbf{H}^1 \neq \emptyset$; since $1 \in \mathbf{H}^2$, the partition $\{\mathbf{H}^1, \mathbf{H}^2\}$ is non-trivial. By the irreducibility of the full economy, there exists an individual, $h_1 \in \mathbf{H}^1$, such that $u^{h_1}(\bar{x}^{*h_1} + \Sigma_{h\in\mathbf{H}^2} w^h) > u^{h_1}(\bar{x}^{*h_1})$. By continuity, there exists $\bar{n}$ such that $u^{h_1}(\bar{x}^{*h_1} + \Sigma_{h\in\mathbf{H}^2}(w^h \wedge_{\bar{n}} 0)) > u^{h_1}(\bar{x}^{*h_1})$. By the convergence of the competitive equilibrium allocations for the modified truncated economies, the finiteness of the aggregate endowment, $w_l < \infty$, for $l = 1, \ldots, \bar{n}$, the continuity of the utility functions in the product topology and the structure of the consumption sets, there exist scalars, $0 < \varepsilon < 1$ and $0 < \delta < 1$, $\bar{\bar{n}} > \bar{n}$ and a finite set $\mathbf{H}^2_F \subseteq \mathbf{H}^2$, such that $u'^{n,h_1}(\delta x^{*n,h_1} + (1-\delta) \times (1-\varepsilon) w'^{n,h_1} + \Sigma_{h\in\mathbf{H}^2_F}(w^h \wedge_{\bar{n}} 0)_n) > u'^{n,h_1}(x^{*n,h_1})$, for $n = \bar{\bar{n}}, \ldots$. From individual optimization in the truncated economies, $p^{*n} \Sigma_{h\in\mathbf{H}^2_F}(w^h \wedge_{\bar{n}} 0)^n > (1-\delta)\varepsilon p^{*n} w'^{n,h_1}$, for $n = \bar{\bar{n}}, \ldots$. But this is a contradiction, since the left side is bounded as $n \to \infty$, while the right side is not. It follows that $\mathbf{H}^1 = \emptyset$ or equivalently, $p^{*n} w'^{n,h} \leq \bar{c}^h < \infty$.

To argue the lower bound, let $\mathbf{H}^1 = \{h \in \mathbf{H}: \liminf_{n\to\infty} p^{*n} w^{n,h} > 0\}$ and $\mathbf{H}^2 = \{h \in \mathbf{H}: \liminf_{n\to\infty} p^{*n} w^{n,h} = 0\}$. Proceed exactly as above, getting an inequality in which the higher side goes to zero and the lower stays bounded away from zero, a contradiction.

For $l \in \mathbf{L}$, take $h \in \mathbf{H}$ with $w^h_l > 0$, this is possible by assumption 4. By claim 1, there exists $\bar{n}_l$ such that $0 \leq \hat{p}^{*n}_l \leq \bar{c}^h / w^h_l$, for $n = \bar{n}_l, \ldots$. Thus, by passing to a subsequence, we obtain $\bar{p}^* \in \Lambda_+$ with $\bar{p}^*_l = \lim_{n\to\infty} \hat{p}^{*n}_l$, for $l \in \mathbf{L}$.

Claim 2. *For $h \in \mathbf{H}$,*

$$0 < \bar{p}^* w^h < \infty .$$

In particular

$$\bar{p}^* > 0$$

and hence $\bar{p}^*$ are prices for the full economy, $\bar{p}^* \in \mathbf{P}$.

By Lemma 4, $\bar{p}^* w^h \leq \liminf_{n\to\infty}(\bar{p}^{*n} \hat{w}'^{n,h}) \leq \bar{c}^h < \infty$; half the claim follows at once. To argue the other half, let $\mathbf{H}^2 = \{h \in \mathbf{H}: \bar{p}^* w^h = 0\}$; note that for any $\bar{n}$ and any $h \in \mathbf{H}^2$, $\lim_{n\to\infty} p^{*n}(w^h \wedge_{\bar{n}} 0) = 0$. Let $\mathbf{H}^1 = \{h \in \mathbf{H}: \bar{p}^* w^h > 0\}$. For $h \in \mathbf{H}^1$, $\liminf_{n\to\infty} p^{*n} w'^{n,h} > 0$. If $\mathbf{H}^2 \neq \emptyset$ and $\mathbf{H}^1 \neq \emptyset$, we get a contradiction as in the proof of Claim 1. If $\mathbf{H}^1 = \emptyset$ and $\mathbf{H}^2 = \mathbf{H}$, note that, by weak monotonicity, $u^1(\bar{x}^{*1} + \Sigma_{h\in\mathbf{H}^2} w^h) = u^1(\bar{x}^{*1} + w) > u^1(\bar{x}^{*1})$. Again by proceeding as in the proof of Claim 1 we get a contradiction.

Claim 3. *For* $h \in \mathbf{H}$,

$$\bar{p}^* \bar{x}^{*h} < \infty .$$

To see this note again that, by Lemma 4, $\bar{p}^* \bar{x}^{*h} \leq \liminf_{n\to\infty} \hat{p}^{*n} \hat{x}^{*n,h} = \liminf_{n\to\infty} p^{*n} x^{*n,h} = \liminf_{n\to\infty} p^{*n} w'^{n,h} \leq \bar{c}^h < \infty$.

Claim 4. *For* $h \in \mathbf{H}$,

$$\bar{p}^* \bar{x}^{*h} \geq \bar{p}^* w^h .$$

We argue by contradiction. Suppose that for some individual $\bar{p}^* \bar{x}^{*h} < \bar{p}^* w^h < \infty$. By weak monotonicity, there is $\bar{n}$ such that $u^h(\bar{x}^h + (\tilde{1} \wedge_{\bar{n}} 0)) > u^h(\bar{x}^{*h})$. By quasi-concavity, for $1 > \varepsilon > 0$, $u^h(x_\varepsilon) > u^h(\bar{x}^{*h})$, where $x_\varepsilon = \bar{x}^{*h} + \varepsilon(\tilde{1} \wedge_{\bar{n}} 0)$. But for small ε, $\bar{p}^* x_\varepsilon < \bar{p}^* w^h$. Since the two series converge, the tails of both infinite sums in the last inequality must be negligible. Hence, there is $\bar{\bar{n}}$ such that $\bar{p}^*(x_\varepsilon \wedge_{\bar{\bar{n}}} w^h) < \bar{p}^{*n} w^h$, for $n = \bar{\bar{n}}, \ldots$. But $u^h(x_\varepsilon \wedge_{\bar{\bar{n}}} w^h) > u^h(\bar{x}^{*h} + (1/n)\mathbf{1})$ for n large. Finally, for large n, $p^{*n}(x_\varepsilon \wedge_{\bar{\bar{n}}} w^h)^n < p^{*n} w^{n,h} < p^{*n} w'^{n,h}$, contradicting individual optimization in the modified truncated economy.

Claim 5. *For* $h \in \mathbf{H}$,

$$u^h(x) > u^h(\bar{x}^{*h}) \ \Rightarrow\ \bar{p}^* x > \overline{px}^{*h} .$$

We argue by contradiction. Suppose that for some individual, h, and for some $x \in \mathbf{X}^h$, $u^h(x) > u^h(\bar{x}^{*h})$, while $\bar{p}^{*h} \leq \bar{p}^* \bar{x}^{*h}$. Since $\bar{p}^* \bar{x}^{*h} \geq \bar{p}^* w^h > 0$, it follows from the structure of the consumption set that there exists $x' = \delta x + (1-\delta)(1-\varepsilon) w^h \in \mathbf{X}^h$ such that $u^h(x') > u^h(\bar{x}^{*h})$, while $\bar{p}^* x' < \bar{p}^* \bar{x}^{*h}$. But from continuity, convergence and the structure of the consumption sets, there exists $\bar{n}$ such that this contradicts the optimization of the individual in the truncated economies for $n = \bar{n}, \ldots$.

In order to complete the argument that $(\bar{p}^*, \bar{x}^{*\mathbf{H}})$ is a compensated equilibrium, it remains to show that $\bar{p}^* \bar{x}^{*h} = \bar{p}^* w^h$ if w^h vanished in all but finitely many components. But $\lim_{n\to\infty}(p^{*n} w'^{n,h} - p^{*n} w^{n,h}) = 0$, since $w'^{n,h}$ and $w^{n,h}$ differ only in the component l_{h+1}, for $n \geq h+2$. Thus, $\bar{p}^* \bar{x}^{*h} \leq \lim_{n\to\infty} p^{*n} x^{*n,h} = \lim_{n\to\infty} p^{*n} w'^{n,h} = \lim_{n\to\infty} p^{*n} w^{n,h} = p^* w^h$, where the first equality follows from the budget constraint in the truncated economies, while the last equality follows from the convergence, by construction, of the modified endowments to w^h and by the fact that the latter vanishes in all but finitely many components.

Thus $(\bar{p}^*, \bar{x}^{*\mathbf{H}})$ is a compensated equilibrium. □

Corollary 1. *In an abstract exchange economy, under Assumptions* 1–7, *if every individual is endowed with at most finitely many commodities, competitive equilibria exist.*

Definition 13. A finite set of non-negligible individuals exists everywhere, if and only if for any feasible allocation, $x^{\mathbf{H}}$, there exists a finite set of individuals, $\mathbf{H}_F \subseteq \mathbf{H}$, a commodity bundle that vanishes in all but finitely many components, w_F, a scalar $k > 0$ and an allocation $x'^{\mathbf{H}}$ such that

$$\sum_{h \in \mathbf{H}} x'^h = k \sum_{h \in \mathbf{H}_F} w^h + w_F$$

and

$$u^h(x'^h) \geq u^h(x^h), \quad \text{for } h \in \mathbf{H}.$$

The definition, evidently, generalizes the condition that a finite set of individuals own a non-negligible fraction of the aggregate endowment of all but finitely many commodities [Wilson (1981); also Burke (1988)]. It allows the sets of individuals, $\mathbf{H}_F$, and the commodity bundle w_F to vary with the allocation $x_{\mathbf{H}}$. More importantly, it is based on a utility comparison and not on a commodity by commodity comparison and thus it is invariant to inessential changes in the indexation of commodities.

Theorem 2. *In an abstract exchange economy, if a finite set of non-negligible individuals exists everywhere, under Assumptions* 1 *and* 2, *in particular if individual utility functions are weakly monotonically increasing, every compensated equilibrium,* $(\bar{p}^*, \bar{x}^{*\mathbf{H}})$, *is a competitive equilibrium,*

$$\bar{p}^* \bar{x}^{*h} = \bar{p}^* w^h, \quad \textit{for } h \in \mathbf{H},$$

and the value of the aggregate endowment is finite,

$$\bar{p}^* w < \infty.$$

In particular, under Assumptions 1–7, if a finite set of non-negligible individuals exists everywhere, competitive equilibria exist.

Proof. Consider a compensated equilibrium, $(\bar{p}^*, \bar{x}^{*\mathbf{H}})$. Let the set of individuals $\mathbf{H}_F$, the commodity bundle w_F, the scalar k and the allocation $x'^{\mathbf{H}}$ be as in the definition of a finite set of non-negligible individuals.

It follows from the definition of a compensated equilibrium that, since individual utility functions are weakly monotonic, $\bar{p}^* \bar{x}^h \leq \bar{p}^* x'^h$, and hence $\bar{p}^* w \leq \bar{p}^*(w_F + k \Sigma_{h \in \mathbf{H}_F} w^h) < \infty$. Since $\bar{p}^* \bar{x}^h \geq \bar{p}^* w^h$ while $\Sigma_{h \in \mathbf{H}} \bar{x}^h = w$, $\bar{p}^* \bar{x}^h = \bar{p}^* w^h$, for $h \in \mathbf{H}$ □

3. The optimality of equilibrium allocations

We first consider examples that illustrate the failure of Pareto optimality of competitive equilibrium allocations.

Competitive equilibrium allocations may fail to be Pareto optimal if the value of the aggregate endowment at the equilibrium prices is infinite.

Example 9. One commodity is available each period, $(1, t) = t$ and one individual is born, $(1, t) = t$. The utility function of an individual is $u^t = \ln x_t + \frac{1}{2} \ln x_{t+1}$, and his initial endowment is $w^t = (\ldots, 0, w_t^t = 5, w_{t+1}^t = 1, 0, \ldots)$. In addition , an individual is born in period 1, $(2, 1) = 0$, with utility function $u^0 = x_1$ and initial endowment $w^0 = (w_1^0 > 0, 0, \ldots)$. Prices $p^* = (1, \ldots, p_{t+1}^* = \frac{5}{2} p_t^*, \ldots)$ are competitive equilibrium prices and the associated allocation coincides with the initial endowment. The feasible allocation x'^H described by $x'^t = (\ldots, 0, x_t'^t = 4, x_{t+1}'^t = 2, 0, \ldots)$ and $x'^0 = (x_1'^0 = w_1^0 + 1, 0, \ldots)$ Pareto dominates the initial endowment.

Alternatively, suppose that individual 0 has utility function $u^0 = \ln x_1 + \Sigma_{t=1}^{\infty} (\frac{1}{2})^{t-1} \ln(\max\{8, x_t\})$ and initial endowment $w^0 = (\ldots, w_t^0 = (\frac{1}{5})^{t-1}, \ldots)$. Prices p^* again suppose the initial endowment as a competitive equilibrium allocation which is suboptimal.

Observe that in both cases the value of the aggregate endowment at the equilibrium prices fails to be finite. This is of interest in the second case in particular, since there is an individual, 0, whose consumption span as well as his endowment span are infinite. Nevertheless, the individual fails to be non-negligible and thus fails to impose a finite value on the aggregate endowment.

Competitive equilibrium allocations may fail to be Pareto optimal, even if the value of the aggregate endowment at the equilibrium prices is finite, though individual utility functions fail to be weakly monotonic even if they are locally nonsatiated, for $x \in \mathbf{X}^h$ and $\mathbf{V}(x)$ a neighbourhood of x, there exists $x' \in \mathbf{V}(x)$ such that $u^h(x') > u^h(x)$.

Example 10. Consider an abstract exchange economy with commodities $\mathbf{L} = \{1, \ldots\}$ and individuals $\mathbf{H} = \{1, 2\}$. Individual 1 has utility function $u^1 = x_1$ and initial endowment $w^1 = (1, 0, \ldots)$. Individual 2 has utility function $u^2 = \inf\{x_l : l \in \mathbf{L}\}$ and initial endowment $w^2 = (\ldots, w_l^2 = 2^{-l}, \ldots)$; evidently, the utility function of individual 2 is not weakly monotonic, if $\Delta x = (\ldots, \Delta x_l, \ldots) \gg 0$, but $\lim_{l \to \infty} \Delta x_l = 0$ and $x' = x + \Delta x$, $u^2(x') = u^2(x) = 0$ even though $x' \gg x$. Prices $p^* = (1, \ldots)$ are autarky competitive equilibrium prices, the associated allocation coincides with the initial endowment. At p^*, w^1 evidently solves the optimization problem of individual 1 and so does w^2 for individual 2 since no consumption bundle, whose value does not exceed $p^* w^2 = 1$ at p^*, yields greater utility to the individual. Note also that $p^*(w^1 + w^2) < \infty$. On the other hand, the allocation described by $x'^1 = (\frac{3}{2}, 0, \ldots)$ and $x'^2 = (0, \frac{1}{4}, \ldots, x_l' = 2^{-l}, \ldots)$ Pareto dominates the initial endowment allocation.

In a finite economy, local non-satiation implies that at any prices, p, for any consumption bundle, $x \in \mathbf{X}^h$, and any $x > 0$, there exists a consumption bundle $x' \in \mathbf{X}^h$ such that $p(x' - x) < \varepsilon$ and $u^h(x') > u^h(x)$, at least when continuity of the utility functions fails. This is not the case in an economy with a countable infinity of commodities. Note that the continuity of the utility functions, which also fails in the above example, is not employed in the argument for the Pareto optimality of competitive allocations in a finite economy.

Theorem 3. *In an abstract exchange economy, under Assumptions* 1 *and* 2, *in particular if the individual utility function is weakly monotonically increasing, a competitive equilibrium allocation,* $x^{*\mathbf{H}}$ *is Pareto optimal if at* p^*, *the associated competitive equilibrium prices,*

$$p^* w < \infty .$$

The proof is essentially as in the case of a finite economy.

Proof. In order to show that the allocation $x^{*\mathbf{H}}$ is Pareto optimal, we argue by contradiction. Suppose the allocation $x'^{\mathbf{H}}$ is feasible and dominates the competitive allocation $x^{*\mathbf{H}}$.

Note first that

$$u^h(x'h) \geq u^h(x^{*h}) \Rightarrow p^* x'^h \geq p^* w^h , \quad \text{for } h \in \mathbf{H} .$$

This follows from the weak monotonicity of the utility function. If $p^* x'^h < p^* w^h$, the commodity bundle Δx^h defined by $\Delta x_l'^h = (p^* w^h - p^* x'^h)(2^l p_l)^{-1}$, for $l \in \mathbf{L}$ is strictly positive, $\Delta x'^h \gg 0$, and hence $u^h(x'^h + \Delta x'^h) > u^h(x'^h) \geq u^h(x^{*h})$, while $p^*(x'^h + \Delta x'^h) \leq p^* w^h$, which contradicts the optimization of individual h at prices p^*.

Also,

$$u^h(x'^h) > u^h(x^{*h}) \Rightarrow p^* x'^h > p^* w^h , \quad \text{for } h \in \mathbf{H} .$$

This follows immediately from the optimization of individual h at prices p^*.

Since the allocation $x'^{\mathbf{H}}$ dominates the competitive allocation $x^{*\mathbf{H}}$, by definition $u^h(x'^h) \geq u^h(x^{*h})$, for $h \in \mathbf{H}$, with some strict inequality. Since $p^* w = \Sigma_{h \in \mathbf{H}} p^* w^h < \infty$, it follows that $\Sigma_{h \in \mathbf{H}} p^* x'^h > \Sigma_{h \in \mathbf{H}} p^* w^h$. But this contradicts the feasibility condition $\Sigma_{h \in \mathbf{H}} x'^h = \Sigma_{h \in \mathbf{H}} w^h$, since $p^* > 0$. □

Corollary 2. *In an abstract exchange economy in which a finite set of non-negligible individuals exists everywhere, under Assumptions* 1 *and* 2, *in particular if the individual utility functions are weakly monotonically increasing, a competitive equilibrium allocation,* $x^{*\mathbf{H}}$, *is Pareto optimal.*

Proof. It suffices to observe that at the associated competitive equilibrium prices, p^*, the value of the aggregate endowment is finite, $p^* w < \infty$.

This accounts for the Pareto optimality of competitive allocations in economies with "land", interpreted as a commodity bundle that renders its owners always non-negligible. Evidently, whether the consumptions span of the owners is infinite or not is of no consequence.

The interest of Corollary 2 lies most importantly in that it gives conditions for the Pareto optimality of competitive equilibrium allocations that refer only to the exogenous structure of the economy, the utility functions and initial endowments of individuals, and not to the competitive equilibrium prices themselves. It is thus the analogue of the first welfare theorem for finite economies [Arrow (1951), Debreu (1951)].

The optimality of competitive equilibrium allocations is complemented by the characterization of conditions under which a Pareto optimal allocation is indeed a competitive equilibrium allocation for some redistribution of initial endowments.

Theorem 4. *Let $\bar{x}^{\mathbf{H}}$ be a feasible Pareto optimal allocation: if the abstract exchange economy with initial endowments $w^h = \bar{x}^h$, for $h \in \mathbf{H}$, satisfies Assumptions* 1–7, *there exists prices p^* such that $(p^*, \bar{x}^{\mathbf{H}})$ is a competitive equilibrium.*

Even if an abstract exchange economy satisfies Assumptions 1–7, the economy obtained by substituting for the initial endowment of each individual by his consumption bundle at a Pareto optimal allocation need not satisfy the same assumptions. In particular, irreducibility may fail and competitive equilibria may fail to exist. This is the case in a finite economy as well. Thus, Theorem 4 is the analogue of the second welfare theorem for finite economics [Arrow (1951), Debreu (1951)].

Proof. Since the economy with initial endowment allocation $\bar{x}^{\mathbf{H}}$ satisfies Assumptions 1–7, it follows from Theorem 1 that a compensated equilibrium, $(p^*, x^{*\mathbf{H}})$, exists.

From the definition of a compensated equilibrium, it follows that $u^h(x^{*h}) \geqslant u^h(\bar{x}^h)$, for $h \in \mathbf{H}$. Since, by hypothesis, the allocation $\bar{x}^{\mathbf{H}}$ is Pareto optimal, $u^h(x^{*h}) = u^h(\bar{x}^h)$. But then, $(p^*, \bar{x}^{\mathbf{H}})$ satisfies the definition of a compensated equilibrium and a fortiori of a competitive equilibrium. □

It remains to characterize conditions under which competitive equilibrium allocations without a finite non-negligible set of individuals are Pareto optimal.

We restrict our attention to economics with a simple demographic structure; from Lemma 2, this is without loss of generality for economics that are individually finite.

Theorem 5 [Brown and Geanakoplos (1982)]. *In an exchange economy of overlapping generations with a simple demographic structure, under Assumptions* 1 *and* 2, *in particular if the individual utility functions are weakly montonically increasing, the allocation at a competitive equilibrium* $(p^*, x^{*\mathbf{H}})$ *is Pareto optimal if*

$$\liminf_{t\to\infty} p_t^* w_t^t = 0 ,$$

where $w_t^t = \Sigma_{(h,t)\in\mathbf{H}_t} w_t^{(h,t)}$.

Proof. Without loss of generality, suppose $\mathbf{H}_1 \neq \emptyset$. For $\bar{t}\in\mathbf{T}_{1,\infty}$, let $\Delta x^{\bar{t}}\in\Lambda^{\bar{t}}$ and consider the optimization problem

$$\max \varphi^{\bar{t}} = p_1^* x_1^{(1,1)} + p_2^* x_2^{(1,1)} \text{ s.t.}$$

$$u^{(h,t)}(x^{(h,t)}) \geqslant u^{(h,t)}(x^{*(h,t)}) , \quad \text{for } (h,t)\in\mathbf{H}_t \text{ and } t = 1,\ldots,\bar{t}-1 ,$$

$$\sum_{(h,1)\in\mathbf{H}_1} x_1^{(h,1)} \leqslant w_1 ,$$

$$\sum_{(h,t)\in\mathbf{H}_t} x_{t+1}^{(h,t)} + \sum_{(h,t+1)\in\mathbf{H}_t} x_{t+1}^{(h,t+1)} \leqslant w_{t+1} , \quad \text{for } t = 1,\ldots,\bar{t}-2 ,$$

$$\sum_{(h,\bar{t}-1)\in\mathbf{H}_{\bar{t}-1}} x_{\bar{t}}^{(h,\bar{t}-1)} \leqslant \sum_{(h,\bar{t}-1)\in\mathbf{H}_{t-1}} x_{\bar{t}}^{*(h,\bar{t}-1)} + \Delta x^{\bar{t}} .$$

Let $\varphi^{\bar{t}}(\Delta x^{\bar{t}})$ be the value of the objective function at a solution. From the weak monotonicity of the individual utility function, it follows that $\varphi^{\bar{t}}(0) = p_1^* x_1^{*(1,1)} + p_2^* x_2^{*(1,1)}$, while $\varphi^{\bar{t}}(\Delta x^{\bar{t}}) \leqslant \varphi^{\bar{t}}(0) + p^{*\bar{t}}\Delta x^{\bar{t}}$.

Suppose a feasible allocation, $x'^{\mathbf{H}}$, Pareto dominates the competitive allocation, $x^{*\mathbf{H}}$; without loss of generality, $u^{(1,1)}(x'^{(1,1)}) > u^{(1,1)}(x^{*(1,1)})$ and hence $p_1^* x_1'^{(1,1)} + p_1^* x_2'^{(1,1)}$. Since the allocation is feasible, it satisfies the constraints of the above optimization problem for $\Delta x^{\bar{t}} = w_{\bar{t}} - \Sigma_{(h,\bar{t}-1)} x_{\bar{t}}^{*(h,\bar{t}-1)}$, for $\bar{t}\in\mathbf{T}_{1,\infty}$. But this is a contradiction since $\varphi^{\bar{t}}(\Delta x^{\bar{t}}) \geqslant p_1^* x_1'^{(1,1)} + p_2^* x_2'^{(1,1)} > \varphi^{\bar{t}}(0)$, independently of $\bar{t}\in\mathbf{T}_{1,\infty}$, while $\varphi^{\bar{t}}(\Delta x^{\bar{t}}) - \varphi^{\bar{t}}(0) \leqslant p_{\bar{t}}^*(w_{\bar{t}} - \Sigma_{(h,\bar{t}-1)} x_{\bar{t}}^{*(h,\bar{t}-1)}) = p_{\bar{t}}^* w_{\bar{t}}^{\bar{t}}$ and, by hypothesis, $\liminf_{\bar{t}\to\infty} p_{\bar{t}}^* w_{\bar{t}}^{\bar{t}} = 0$. □

If $\liminf_{t\to\infty} p_t^* w_t^t > 0$, the competitive equilibrium allocation may still be Pareto optimal. Nevertheless, there exists an alternative exchange economy of overlapping generations with a simple demographic structure which differs from the original economy only in the utility functions and for which $(p^*, x^{*\mathbf{H}})$ is a competitive equilibrium while the allocation $x^{*\mathbf{H}}$ fails to be Pareto optimal. In particular, the economy in which the utility function of individual (h, t) is $u^{(h,t)} = p_t^* x_t + p_{t+1}^* x_{t+1}$.

Competitive equilibrium allocations may fail to be Pareto optimal in economies in which the marginal rate of substitution of commodities, the slope of a supporting hyperplane to the indifference curve of an individual at the equilibrium consumption bundle is invariant to small changes in the relative consumption of the individual in the two periods in his consumption span.

Example 11. One commodity is available each period, $(1, t) = t$, and one individual is born, $(1, t) = t$. The utility function of the individual is $u^t = x_t + x_{t+1}$, and his initial endowment is $w^t = (\ldots, 0, w_t^t = 1, w_{t+1}^t = 1, 0, \ldots)$. In addition, an individual $(2, 1) = 0$ is born in the first period, whose utility function is $u^0 = x_1$, and his initial endowment is $w^0 = (w_1^0 = 1, 0, \ldots)$. The initial endowment is indeed a competitive equilibrium allocation supported by prices $p^* = (\ldots, 1, \ldots)$. The allocation $x'^{\mathbf{H}}$ described by $x_1'^0 = 1 + \varepsilon$ and $x_t'^t = 1 - \varepsilon - \cdots - \varepsilon^t$, $x_{t+1}'^t = 1 + \varepsilon + \cdots + \varepsilon^t + \varepsilon^{t+1}$, for $t = 1, \ldots$ is well defined for $\varepsilon < \frac{1}{3}$ and Pareto dominates the initial endowment.

Let $(p^*, x^{*\mathbf{H}})$ be a competitive equilibrium. Consider the individual expenditure minimization problems

$$\min z_{t+1} = \frac{p_{t+1}^*}{\|p_{t+1}^*\|} (x_{t+1} - x_{t+1}^{*(t+1)}) \text{ s.t.}$$

$$u^{(h,t)}(x) \geq u^{(h,t)}(x^*) ,$$

$$\frac{p_t^*}{\|p_t^*\|} (x_t - x_t^{*(h,t)}) = z_t ,$$

$$x \in \mathbf{X}^{(h,t)} , \quad \text{for } (h, t) \in \mathbf{H} ,$$

where $x = (\ldots, 0, x_t^{*(h,t)} + z_t, x_{t+1}^{*(h,t)} + z_{t+1}, 0, \ldots)$.

The per capita endowment of a commodity is

$$\frac{1}{H_t} w_{(l,t)} , \quad \text{for } (l, t) \in \mathbf{L} .$$

From the solution of the individual expenditure minimization problem, we obtain the expenditure function $f^{(h,t)}$, for $(h, t) \in \mathbf{H}$. If, for some z_t, a solution to the minimization problem yields $f^{(h,t)}(z_t) = \infty$.

The function $f^{(h,t)}$ is, up to price normalization, the expenditure in the second period of consumption required for the individual to attain the level of utility associated with this consumption at the competitive equilibrium if the expenditure at the first period of his life is exogenously modified. In the special case of one commodity per period, the expenditure function coincides with the indifference curve through the equilibrium consumption of the individual, whenever the utility function is monotonically increasing in x_{t+1}.

Definition 13a. The competitive equilibrium, $(p^*, x^{*\mathbf{H}})$, in an exchange economy of overlapping generations with a simple demographic structure satisfies the non-vanishing Gaussian curvature condition if and only if there exists $\underline{t} \in \mathbf{T}_{1,\infty}$ and scalars $\bar{\beta}_t > 0$ and $\bar{d}_t > 0$ for $t = \underline{t}, \ldots$ such that

(i) for $t = \underline{t}, \ldots, \|p_t^*\| > 0$, $H_t > 0$,

(ii) for $(h, t) \in \mathbf{H}_t$ and $t = \underline{t}, \ldots,$

$$f^{(h,t)}(z_t) \geq f_t(z_t) = \begin{cases} -\dfrac{\|p_t^*\|}{\|p_{t+1}^*\|} z_t + \bar{\beta}_t z_t^2 , & \text{for } \|z_t\| \leq \bar{d}_t , \\ -\left(\dfrac{\|p_t^*\|}{\|p_{t+1}^*\|} - 2\bar{\beta}_t \bar{d}_t\right) z_t - \beta_t \bar{d}_t^2 , & \text{for } z_t > \bar{d}_t , \\ -\left(\dfrac{\|p_t^*\|}{\|p_{t+1}^*\|} + 2\bar{\beta}_t \bar{d}_t\right) z_t - \beta_t \bar{d}_t^2 , & \text{for } z_t < -\bar{d}_t , \end{cases}$$

(iii) for some $\bar{\rho} > 0$ and $t = \bar{t}, \ldots, \bar{\beta}_t(\|p_{t+1}^*\| / \|p_{t+1}^*\|) = \rho_t \geq \bar{\rho}$, and

(iv) for some $d > 0$ and $t = \bar{t}, \ldots, \bar{d}_t \geq d$.

In order to interpret the non-vanishing Gaussian curvature condition, consider the special case of one commodity per period, in which function, $f^{(h,t)}$, coincides with the indifference curve through the equilibrium consumption point. The function f_t coincides with $f^{(h,t)}$ at $z_t^* = 0$ or equivalently, at the equilibrium consumption point and does not lie anywhere below it. The curve f_t is linear quadratic with strictly positive Gaussian curvature at $z_t^* = 0$. If the indifference curve is smooth, which, nevertheless, we do not require, and $f^{(h,t)}$ and f_t are tangent at $z_t^* = 0$, the requirement that $f^{(h,t)} \geq f_t$ amounts to the Gaussian curvature of the indifference curve not vanishing at $z_t^* = 0$.

Theorem 5a [Cass (1972); also Benveniste and Gale (1975), Balasko and Shell (1981a)]. *Consider an exchange economy of overlapping generations with a simple demographic structure such that the per capita endowment of each commodity is bounded,*

$$\frac{1}{H_t} w_{(l,t)} < \frac{1}{L_t} k , \quad \textit{for } (l, t) \in \mathbf{L} \textit{ and some } k > 0 .$$

Under Assumptions 1 *and* 2, *a competitive equilibrium* $(p^*, x^{*\mathbf{H}})$ *in which the non-vanishing Gaussian curvature condition is satisfied, the allocations* $x^{*\mathbf{H}}$ *is Pareto optimal if*

$$\lim_{T\to\infty} \sum_{t=\bar{t}}^{T} \frac{1}{H_t \|p_t^*\|} = \infty .$$

The uniform bound on the per capita endowment of all commodities is normalized by the cardinality of the set of commodities available in each period. This is necessary for the restriction to be meaningful; otherwise it is possible to replace each commodity by a possibly large, yet finite, number of perfect substitutes in order to satisfy any positive upper-bound on the per capita endowment of each commodity. The non-vanishing Gaussian curvature condition fails in Example 11.

If population grows at a constant rate, $\#\mathbf{H}_t = (1+n)^t$, while the real rate of interest is constant, $\|p_t^*\| = (1+r^*)\|p_{t+1}^*\|$, the divergence condition takes the familiar form

$$r^* \geq n\,.$$

If population grows at a constant rate, and the rate of interest is constant yet time extends infinitely into the future as well as into the past, the divergence condition takes the form [Samuelson (1958)]

$$r^* = n\,.$$

This follows simply by applying the construction of Lemma 2 which establishes the equivalence between economies that extend infinitely into the future as well as into the past with economies in which time extends infinitely only into the future.

The Euclidean norm is the divergence condition can be replaced by any norm.

Proof. Let $(p^*, x^{*\mathbf{H}})$ be a competitive equilibrium that satisfies the curvature condition. In order to show that with the per capita endowment of commodities uniformly bounded, the allocation $x^{*\mathbf{H}}$ is Pareto optimal if $\lim_{t\to\infty} \Sigma_{t=\bar{t}}^{T} 1/H_t \|p_t^*\| = \infty$; we argue by contradiction.

Suppose the allocation $x'^{\mathbf{H}}$ is feasible and dominates the equilibrium allocation $x^{*\mathbf{H}}$.

Let $\bar{\bar{t}} = \min\{t \geq \bar{t}\text{: } u^{(h,t)}(x'^{(h,t)}) > u^{(h,t)}(x^{*(h,t)})$, for some $(h,t) \in \mathbf{H}_t\}$. Without loss of generality, $\bar{\bar{t}} = \bar{t} = 1$.

Since p^* are competitive equilibrium prices, $\Sigma_{(h,1)\in\mathbf{H}_1} p_2^*(x_2'^{(h,1)} - x_2^{*(h,1)}) > 0$. This is the case since, from the feasibility of the allocation x', $\Sigma_{(h,1)\in\mathbf{H}_1}(x_1'^{(h,1)} - x_1^{*(h,1)}) = 0$, while, from the optimization of individuals $\mathbf{H}_1$, $\Sigma_{(h,1)\in\mathbf{H}_1} p_1^*(x_1'^{(h,1)} - x_1^{*(h,1)}) + \Sigma_{(h,1)\in\mathbf{H}_1} p_2^*(x_2'^{(h,1)} - x_2^{*(h,1)}) > 0$ since the individual utility functions are weakly monotonic. It follows that, for $t = 2, \ldots,$ $\Sigma_{(h,t)\in\mathbf{H}_t} p_t^*(x_t'^{(h,t)} - x_t^{*(h,t)}) < 0$, while $\Sigma_{(h,t)\in\mathbf{H}_t} p_{t+1}^*(x_{t+1}'^{(h,t)} - x_{t+1}^{*(h,t)}) > 0$.

Let $z_t'^{(h,t)} = (p_t^*/\|p_{t+1}^*\|)(x_t'^{(h,t)} - x_t^{*(h,t)})$ and $z_{t+1}'^{(h,t)} = (p_{t+1}^*/\|p_{t+1}^*\|) \times (x_{t+1}'^{(h,t)} - x_{t+1}^{*(h,t)})$, for $(h,t) \in \mathbf{H}$. Let $z_t' = (1/\mathbf{H}_t)\Sigma_{(h,t)\in\mathbf{H}_t} z_t^{(h,t)}$ and $z_{t+1}'' = (1/\mathbf{H}_t)\Sigma_{(h,t)\in\mathbf{H}_t} z_{t+1}'^{(h,t)}$. From the monotonicity of the function f_t^* it follows that

$f_t^*(z'') \geqslant 0$. From the uniform upper bound on the per capita endowment of all commodities and the definition of z_t'' it follows that $\|z_t''\| \leqslant k$. From the quasi-concavity of individual utility functions and consumption sets, we may suppose that $\|z_t''\| \leqslant \bar{d} \leqslant \bar{d}_t$. Thus,

$$\| p_t^* \| z_t'' + \| p_{t+1}^* \| z_{t+1}'' - \bar{\beta}_t \| p_{t+1}^* \| (z_t'')^2 \geqslant 0 \,, \quad \text{for } t = 1, \ldots .$$

Setting $\varepsilon_t = z_{t+1}'' > 0$, observing that $-H_t z_{t+1}'' = H_{t+1} z_{t+1}''^{t+1}$ and substituting, we obtain

$$-\| p_{t+1}^* \| \frac{H_t}{H_{t+1}} \varepsilon_t + \| p_{t+2}^* \| \varepsilon_{t+1} - \bar{\beta}_{t+1} \| p_{t+2}^* \| \left(\frac{H_t}{H_{t+1}} \right)^2 \varepsilon_t^2 \geqslant 0 \,, \quad \text{for } t = 1, \ldots .$$

Rearranging terms, multiplying both sides of the inequality by H_{t+1}, for $t = 1, \ldots$, and taking reciprocals, we obtain

$$\begin{aligned}
\frac{1}{\| p_{t+1}^* \| H_{t+1} \varepsilon_{t+1}} &\leqslant \frac{1}{\| p_{t+1}^* \| H_t \varepsilon_t + \bar{\beta}_{t+1} \| p_{t+2}^* \| \dfrac{(H_t)^2}{H_{t+1}} \varepsilon_t^2} \\
&= \frac{1}{\| p_{t+1}^* \| H_t \varepsilon_t} \left(\frac{\| p_{t+1}^* \| H_t \varepsilon_t}{\| p_{t+1}^* \| H_t \varepsilon_t + \bar{\beta}_{t+1} \| p_{t+2}^* \| \dfrac{(H_t)^2}{H_{t+1}} \varepsilon_t^2} \right) \\
&= \frac{1}{\| p_{t+1}^* \| H_t \varepsilon_t} \left(1 - \frac{\bar{\beta}_{t+1} \| p_{t+2}^* \| \dfrac{(H_t)^2}{H_{t+1}} \varepsilon_t^2}{\| p_{t+1}^* \| H_t \varepsilon_t + \bar{\beta}_{t+1} \| p_{t+2}^* \| \dfrac{(H_t)^2}{H_{t+1}} \varepsilon_t^2} \right) \\
&= \frac{1}{\| p_{t+1}^* \| H_t \varepsilon_t} - \frac{\bar{\beta}_{t+1} \dfrac{\| p_{t+2}^* \|}{\| p_{t+1}^* \|}}{\| p_{t+1}^* \| H_{t+1} \left(1 + \bar{\beta}_{t+1} \dfrac{\| p_{t+2}^* \|}{\| p_{t+1}^* \|} \dfrac{H_t}{H_{t+1}} \varepsilon_t \right)} \,, \\
&\quad \text{for } t = 1, \ldots .
\end{aligned}$$

Observe now that the expression

$$\frac{\bar{\beta}_{t+1} \dfrac{\| p_{t+2}^* \|}{\| p_{t+1}^* \|}}{1 + \bar{\beta}_{t+1} \dfrac{\| p_{t+2}^* \|}{\| p_{t+1}^* \|} \dfrac{H_t}{H_{t+1}} \varepsilon_t}$$

is monotonically increasing in

$$\bar{\beta}_{t+1}\frac{\|p^*_{t+2}\|}{\|p^*_{t+1}\|}$$

and monotonically decreasing in

$$\frac{H_t}{H_{t+1}}\varepsilon_t\,,\quad \text{for } t=1,\ldots .$$

By assumption,

$$\bar{\beta}_{t+1}\frac{\|p^*_{t+2}\|}{\|p^*_{t+1}\|}\geq\bar{\rho}_{t+1}\,,\quad \text{while } \frac{H_t}{H_{t+1}}\varepsilon_t=\frac{H_t}{H_{t+1}}z'^{t}_{t+1}=-z'^{t+1}_{t+1}\leq k\,,$$

for $t=1,\ldots$.

Substituting, we obtain

$$\frac{1}{\|p^*_{t+2}\|H_{t+1}\varepsilon_{t+1}}\leq\frac{1}{\|p^*_{t+1}\|H_t\varepsilon_t}-\frac{1}{\|p^*_{t+1}\|H_{t+1}}\frac{\bar{\rho}_{t+1}}{1+\bar{\rho}_{t+1}k}\,,\quad \text{for } t=1,\ldots .$$

Summing over $t=1,\ldots$, and cancelling terms, since $\bar{\rho}_{t+1}\geq\bar{\rho}>0$,

$$\sum_{t=1}^{T}\frac{1}{\|p^*_{t+1}\|H_{t+1}}<\frac{1}{\|p^*_2\|H_1\varepsilon_1}\frac{1+\bar{\rho}k}{\bar{\rho}}\,,$$

It follows that

$$\lim_{T\to\infty}\sum_{t=1}^{T}\frac{1}{H_t\|p^*_t\|}<\infty\,.$$

But this is a contradiction and hence the competitive allocation is Pareto optimal. □

Definition 13b. The competitive equilibrium $(p^*, x^{*\mathbf{H}})$ in an economy with a simple demographic structure satisfies the bounded curvature condition if and only if there exists $\bar{t}\in\mathbf{T}$ and sets of individuals $\mathbf{K}_t=\{(h,t)_1,\ldots,(h,t)_{K_t}\}\subseteq\mathbf{H}_t$, commodity bundles Δx_t, and scalars $\underline{\beta}_t\geq 0$ and $\underline{d}_t>0$, for $t=\bar{t},\ldots$, such that

(i) for $t=\bar{t},\ldots$, $p^*_t\,\Delta x_t>0$, $K_t>1$;

(ii) for $(h,t)\in\mathbf{K}_t$ and $t=\bar{t},\ldots$ the consumption bundles $x'^{(h,t)}$ defined by

$$x'^{(h,t)}_{t'}=\begin{cases}0\,, & \text{for } t'\neq t, t+1\,,\\ x^{*(h,t)}_{t'}+\dfrac{\|p^*_{t'}\|\,z'_{t'}}{p^*_{t'}\,\Delta x_{t'}}\Delta x_{t'}\,, & \text{for } \|z'_{t'}\|\leq\underline{d}_t\dfrac{K_{t'-1}}{K'_t} \text{ and } t'=t, t+1\,,\end{cases}$$

where by convention $K_{\bar{t}-1}=0$, satisfy $u^{(h,t)}(x'^{(h,t)}) > u^{(h,t)}(x^{*(h,t)})$ so long as $z_t' < 0$ and $z_{t+1}' \geq -(p_{t+1}^*/\|p_{t+1}^*\|)z_t + \beta_t z_t^2$,
(iii) for some $\underline{\rho} > 0$ and $t = \bar{t}, \ldots, \beta_t(\|p_{t+1}^*\|/\|p_t^*\|) = \underline{\rho}_t < \bar{\rho}$, and
(iv) for some $\underline{d} > 0$ and $t = \bar{t}, \ldots, \underline{d}_t \geq \underline{d}$.

In the special case of one commodity per period, the bounded curvature condition requires that, locally at $z_t^* = 0$, the indifference curve not be anywhere below a quadratic function with finite Gaussian curvature with which it coincides at $z_t^* = 0$. In particular, this excludes the case in which, for $t = \bar{t}+1, \ldots$, the utility functions of individuals are not strictly monotonically increasing in x_{t+1}.

Theorem 5b [Cass (1972), Benveniste and Gale (1975)]. *Consider an exchange economy of overlapping generations with a simple demographic structure, under Assumptions* 1 *and* 2. *The allocation at a competitive equilibrium,* $(p^*, x^{*\mathbf{H}})$, *satisfying the bounded curvature condition, is not Pareto optimal if*

$$\lim_{T\to\infty} \sum_{t=\bar{t}}^{T} \frac{1}{K_t \|p_t^*\|} < \infty .$$

Proof. The argument is constructive. Suppose, without loss of generality that $\bar{t} = 1$ and let

$$\lim_{T\to\infty} \sum_{t=1}^{T} \frac{1}{K_t \|p_t^*\|} = s < \infty .$$

Consider the allocation $x'^{\mathbf{H}}$ defined as follows. For $(h,t) \notin \mathbf{K}_t$, $x'^{(h,t)} = x^{*(h,t)}$. For $(h,t) \in \mathbf{K}_t$ if suffices to specify $z_t'^t$ and $z_{t+1}'^t$. Requiring that $-K_t z_{t+1}'^t = K_{t+1} z_{t+1}'^{t+1}$, if suffices to specify $\varepsilon_t = z_{t+1}'^t$, for $t = \bar{t}, \ldots$ and also $z_{\bar{t}}'^{\bar{t}} = 0$.

Let $z_{\bar{t}}'^{\bar{t}} = 0$ and choose $\varepsilon_{\bar{t}} > 0$. Observe that $u^{(h,\bar{t})}(x'^{(h,\bar{t})}) > u^{(h,\bar{t})}(x^{*(h,\bar{t})})$ for $(h,\bar{t}) \in \mathbf{K}_{\bar{t}}$. Define inductively ε_{t+1} by

$$\frac{1}{\|p_{t+1}^*\| \varepsilon_{t+1} K_{t+1}} = \frac{1}{\|p_{t+1}^*\| \varepsilon_t K_t} - \frac{1}{\|p_{t+1}^*\| K_{t+1}} \frac{\bar{\rho}_{t+1}}{1 + \bar{\rho}_{t+1}\left(\dfrac{K_t}{K_{t+1}}\right)\varepsilon_t} ,$$

for $t = 1, \ldots$.

Note that ε_{t+1} is well defined since $\|p_{t+1}^*\| > 0$ and $K_{t+1} \geq 1$. To complete the argument, it remains to show that $\varepsilon_t < \underline{d}$, for $t = 1, \ldots$, since then by construction, $u^{*(h,t)}(x'^{(h,t)}) \geq u^{*(h,t)}(x^{*(h,t)})$ for $(h,t) \in \mathbf{K}_t$. Summing over $t = 1, \ldots, T$ and cancelling terms we obtain

$$\frac{1}{\|p_T^*\| \varepsilon_T K_T} > \frac{1}{\|p_2^*\| \varepsilon_1 K_1} - s\underline{\rho} .$$

Since $\lim_{T\to\infty} \|p_T^*\| K_T = 0$, $\varepsilon_t < \underline{d}$, for $t = 1, \ldots$, if initially ε_1 is chosen sufficiently small. □

The optimality properties of competitive allocations may extend beyond Pareto optimality.

A coalition $\mathbf{K} \subset \mathbf{H}$ blocks an allocation, $x^{\mathbf{H}}$, in an abstract exchange economy if and only if there exists an allocation for $\mathbf{K}$, $x'^{\mathbf{K}} = \{x'^h \in \mathbf{X}^h: h \in \mathbf{K}\}$, which is feasible for $\mathbf{K}$, $\Sigma_{h\in\mathbf{K}} x'^h = \Sigma_{h\in\mathbf{K}} w^h$, and Pareto dominates $x^{\mathbf{H}}$ for $\mathbf{K}$, $u^h(x'^h) \geq u^h(x^h)$, for $h \in \mathbf{K}$, with some strict inequality. An allocation, $x^{\mathbf{H}}$, is in the core of an abstract exchange economy if and only if it is feasible and it is not blocked by any coalition. Evidently, a feasible allocation that is not Pareto optimal cannot be in the core, since it is blocked by the coalition $\mathbf{K} = \mathbf{H}$.

In finite economies, under weak monotonicity, competitive equilibrium allocations are in the core. Under stronger, convexity assumptions, the set of core and competitive equilibrium allocations coincide in particular for large economies obtained by replicating a given economy. Evidently, replication does not augment the number of commodities in the economy, which is finite [Debreu and Scarf (1963)].

It is a straightforward extension of Corollary 1 that, in an abstract exchange economy in which a finite set of non-negligible individuals exists everywhere, competitive equilibrium allocations are in the core. This follows from the finite value of the aggregate endowment and hence of the endowment of any coalition.

By a similar argument, a coalition of finitely many individuals can never block a competitive equilibrium allocation.

In the absence of a finite set of non-negligible individuals, even if the demographic structure is simple, the non-vanishing Gaussian curvature condition is satisfied and the divergence condition guarantees Pareto optimality, competitive equilibrium allocations may fail to be in the core.

Evidently, in an economy of overlapping generations, competitive equilibrium allocations need not be in the core [Gale (1971)].

4. Aggregate revenue at equilibrium

In finite economies, it is impossible for markets to clear if the expenditures of each individual is at least as high as the value of his initial endowment while for some it is strictly higher; neither if the expenditure of each individual is at most as high, while for some it is strictly lower. For economies of overlapping generations this is not the case.

Revenue permits the value of the consumption bundle of an individual at equilibrium to differ from the value of the initial endowment.

The budget constraint in the individual optimization problem with revenue is

$$px \leqslant pw^h + \tau^h \,, \quad \text{for } h \in \mathbf{H} \,,$$

where τ^h is the revenue of the individual. Revenue may be positive or negative.

An allocation of the revenue is an array

$$\tau^{\mathbf{H}} = \{\tau^h : h \in \mathbf{H}\} \,,$$

such that $\tau^h = 0$, for $h \not\in \mathbf{H}_F$, where $\mathbf{H}_F \subseteq \mathbf{H}$ is a finite set. That revenue vanishes for all but finitely many individuals, is only for simplicity. An allocation of revenue is negative if $\tau^{(h,t)} \leqslant 0$ with some strict inequality, it is positive if $\tau^h \geqslant 0$ with some strict inequality and it vanishes if $\tau^h = 0$, for $h \in \mathbf{H}$.

At an allocation of revenue $\tau^{\mathbf{H}}$, aggregate revenue is

$$\tau = \sum_{h \in \mathbf{H}} \tau^h \,.$$

A redistribution is an allocation of revenue at which aggregate revenue vanishes.

A competitive equilibrium with revenue is a triple $(p^*, x^{*\mathbf{H}}, \tau^{*\mathbf{H}})$ of prices, an allocation of commodities and an allocation of revenue such that the commodity bundle x^{*h} solves the individual optimization problem at prices p^* and revenue τ^{*h} or, equivalently,

$$u^h(x) > u^h(x^{*h}) \;\Rightarrow\; p^*x > p^*w^h + \tau^{*h} \,, \quad \text{for } x \in \mathbf{X}^h \,,$$

and

$$p^*x^{*h} = p^*w^h + \tau^{*h} \,, \quad \text{for } h \in \mathbf{H} \,.$$

Theorem 6. *In an abstract exchange economy, under Assumption* 1, *if at* $(p^*, x^{*\mathbf{H}}, \tau^{*\mathbf{H}})$, *a competitive equilibrium with revenue,*

$$p^*w < \infty \,,$$

the allocation of revenue $\tau^{*\mathbf{H}}$ *is a redistribution.*

Proof. Since $p^*x^{*h} = p^*w^h < \infty$ for $h \in \mathbf{H}/\mathbf{H}_F$ and $p^*x^{*h} = p^*w^h + \tau^{*h}$ for $h \in \mathbf{H}_F$, while $p^*w < \infty$, $p^*x^* = p^*w + \tau < \infty$. Feasibility, however, implies that $p^*x^* = p^*w$ and hence $\tau = 0$. □

This generalizes a well-known argument for finite economies.

Theorem 7. *In an abstract exchange economy, under Assumptions* 1 *and* 2, *in* *particular if individual utilty functions are weakly monotonic, if the allocation of initial endowments, $w^{\mathbf{H}} = \{w^h: h \in \mathbf{H}\}$ is Pareto optimal, and $(p^*, x^{*\mathbf{H}}, \tau^{*\mathbf{H}})$ is a competitive equilibrium with revenue, the allocation of revenue $\tau^{*\mathbf{H}}$ is not positive.*

Proof. We argue by contradiction. Suppose the allocation $\tau^{*\mathbf{H}}$ is positive.

From the individual optimization problem it follows that $u^h(x^{*h}) \geq u^h(w^h)$, with some strict inequality; the latter follows from the weak monotonicity of the utility functions of the individual(s) with $\tau^{*h} > 0$. But this contradicts the Pareto optimality of the initial endowment allocation.

Theorem 8 [Burke (1988)]. *In abstract exchange economy, under Assumptions 1–7, if every individual is endowed with at most finitely many commodities, for any array of scalars $\theta^H = \{(\theta^h): 0 \leq \theta^h < 1$ for $h \in \mathbf{H}\}$ there exists $(p^*, x^{*\mathbf{H}}, \tau^{*\mathbf{H}})$, a competitive equilibrium with revenue, such that*

$$\tau^{*h} = -\theta^h p^* w^h \leq 0\,, \quad \textit{for}\ (h, t) \in \mathbf{H}\,.$$

Proof. Consider the increasing, convergent sequence of sets of commodities, $(\mathbf{L}^n: \mathbf{L}^n \subset \mathbf{L}^{n+1},\ n = 1, \ldots)$, where $\mathbf{L}^n = \{1, \ldots, n\}$, and the convergent, increasing sequence of finite sets of individuals, $(\mathbf{H}^n: \mathbf{H}^n \subset \mathbf{H}^{n+1},\ n = 1, \ldots)$, where $\mathbf{H}^n = \{1, \ldots, n\}$.

For $n = 1, \ldots,$ the modified truncated economy $\mathbf{E}''^n$ is obtained from the modified, truncated economy $\mathbf{E}'^n$ defined in the proof of Theorem 1 by further perturbing the endowments of individuals according to

$$w''^{n,h} = (1 - \theta^h) w'^{n,h}\,, \quad \text{for } h \in \mathbf{H}^n / \{n\}$$

and

$$w''^{n,n} = w'^{n,n} + \sum_{h \in \mathbf{H}^n / \{n\}} \theta^h w'^{n,h}\,.$$

As in the proof of Theorem 1, the sequence of prices and allocations $((\hat{p}^{*n}, \hat{x}^{*\mathbf{H}^n}): n = 1, \ldots)$ associated with the sequence of competitive equilibria for the modified truncated economies converges to a pair of prices and an allocation, $(\bar{p}^*, \bar{x}^{*\mathbf{H}})$, for the economy $\mathbf{E}$.

As in Corollary 1, since w^h vanishes in all but finitely many components, $\bar{p}^* \bar{x}^{*h} = (1 - \theta^h) \bar{p}^* w^h$. Thus $(\bar{p}^*, \bar{x}^{*\mathbf{H}}, \bar{\tau}^*)$ is a competitive equilibrium with revenue, where, for $h \in \mathbf{H}$, $\bar{\tau}^{*h} = -\theta^h \bar{p}^* w^h \leq 0$. □

Examples of economies in which competitive equilibria with positive allocations of revenue exist are well known. Revenue can then be interpreted as fiat money that maintains a positive price at equilibrium.

Note that a competitive equilibrium with positive revenue in an economy **E** can be interpreted as a competitive equilibrium in an economy **E**′ in which the set of commodities is $\mathbf{L}' = \mathbf{L} \cup \{0\}$, no individual desires commodity $l = 0$, the endowment of individual h in commodity $l = 0$ is $w^h = \tau^h \geqslant 0$ and the price of commodity $l = 0$ at equilibrium is $p_0^* = 1$. The characterization of the conditions under which competitive equilibrium allocations are optimal in the economy **E**′, such as the divergence condition in Theorems 5a and 5b then carry over to the economy **E**.

No unambiguous link can be established between the positive price of fiat money at equilibrium and the optimality of competitive equilibrium allocations [Cass, Okuno and Zilcha (1979)].

The situation is different when the competitive equilibrium allocation is autarkic. In that case if it is also Pareto optimal there cannot be any monetary equilibria. Conversely, if it is not Pareto optimal, then under fairly general circumstances, there is almost surely a monetary equilibrium [Brown and Geanakoplos (1985)].

5. Stationary economies and cycles

In order to study whether recursive patterns, symmetries, in the exogenous structure of exchange economies of overlapping generations are inherited by competitive equilibrium allocations, or other allocations of interest, it is convenient, and possibly necessary, to consider economies with an elementary temporal and demographic structure.

Definition 14. In an elementary exchange economy of overlapping generations, time extends infinitely into the future as well as into the past under certainty, $\mathbf{T}_{-\infty,\infty}$, one good is available each period, $\mathbf{L}_t = \{t\}$ and the consumption as well as the endowment span of each individual is two, $\mathbf{T}^{(h,t)} = \mathbf{T}_w^{(h,t)} = \{t, t+1\}$.

This is evidently very restrictive. It is important to note, however, that, by Lemma 3, an elementary exchange economy of overlapping generations is equivalent to an economy with a simple demographic structure, in which time extends infinitely into the future but not into the past.

At strictly positive commodity prices, $p \gg 0$, relative prices are denoted by $q = (\ldots, q_t, \ldots)$, where

$$q_t = \frac{p_{t+1}}{p_t}, \quad \text{for } t \in \mathbf{T}_{-\infty,\infty}.$$

The domain of relative prices is $\mathbf{Q} = \{q\colon q \gg 0\}$. Associated with relative prices are real rates of interest $r = (\ldots, r_t, \ldots)$, where

$$r_t = \frac{1}{q_t} - 1 \,, \quad \text{for } t \in \mathbf{T}_{-\infty,\infty} \,.$$

The periods of consumption and endowment of individuals allow us to write an individual optimization problem as

$$\max \tilde{u}^{(h,t)}(z_t, z_{t+1}) \text{ s.t.}$$
$$z_t + q_t z_{t+1} = 0 \,, \quad \text{for } (h, t) \in \mathbf{H} \,,$$

where $\tilde{u}^{(h,t)}(z_t, z_{t+1}) = u^{(h,t)}(\ldots, 0, w_t^{(h,t)} + z_t, w_{t+1}^{(h,t)} + z_{t+1}, 0, \ldots)$. The excess demand correspondence of an individual is, $z^{(h,t)} = (z_t^{(h,t)}, z_t^{(h,t)}) : \mathbf{Q}_t \to \mathbf{Z}^{(h,t)}$, where $\mathbf{Z}^{(h,t)} = \{(z, z_2): (\ldots, 0, w_t^{(h,t)} + z_t, w_{t+1}^{(h,t)} + z_{t+1}, 0, \ldots) \in \mathbf{X}^{(h,t)}$ and $\mathbf{Q}_t = \{q_t: q_t > 0\}$. The aggregate excess demand correspondence of a generation is $z^t = \Sigma_{(h,t)\in \mathbf{H}_t} z^{(h,t)} : \mathbf{Q}_t \to \mathbf{Z}^t$, where $\mathbf{Z}^t = \Sigma_{(h,t)\in \mathbf{H}_t} \mathbf{Z}^{(h,t)}$.

Competitive equilibrium relative prices, q^*, are such that

$$0 \in z_t^{t-1}(q_{t-1}^*) + z_t^t(q_t^*) \,, \quad \text{for } t \in \mathbf{T}_{-\infty,\infty} \,.$$

Associated with competitive equilibrium relative prices, there is a competitive equilibrium allocation, $x^{*\mathbf{H}}$, such that, for $(h, t) \in \mathbf{H}$, $z^{*(h,t)} = (z_t^{*(h,t)}, z_{t+1}^{*(h,t)})$ is a solution to the optimization problem of individual (h, t) at q_t^*.

Definition 15. An exchange economy of overlapping generations under certainty $(\mathbf{E}, \mathbf{T}_{\underline{t},\bar{t}}, (\tau_{\mathrm{L}}, \tau_{\mathrm{H}}))$, is stationary if and only if time extends infinitely into the future as well as into the past, $\mathbf{T}_{\underline{t},\bar{t}} = \mathbf{T}_{-\infty,\infty}$, for $t \in \mathbf{T}_{-\infty,\infty}$, $L_t = L$ and $H_t = H$ and the group, $\mathbf{G}$, of symmetries of $\mathbf{E}$ is generated by the function (f, g), where $f(l, t) = f(l, t+1)$ and $g(h, t) = g(h, t+1)$, for $(l, t) \in \mathbf{L}$ and $(h, t) \in \mathbf{H}$.

In a stationary economy, individuals are identical up to the calendar time of their birth. It is often possible to allow for intragenerational heterogeneity as long as the aggregate behavior of generations coincides up to the calendar time of their birth.

An economy in which time extends infinitely into the future but not into the past is stationary if and only if it can be extended to a stationary economy.

In a stationary, elementary, exchange economy of overlapping generations, we write

$$z = (z_1, z_2)$$

for the aggregate excess demand function of each generation. Competitive equilibrium relative prices, q^*, are such that

$$z_1(q_{t-1}^*) + z_2(q_t^*) = 0 \,, \quad \text{for } t \in \mathbf{T}_{-\infty,\infty} \,.$$

Definition 16. A stationary elementary economy of overlapping generations is well behaved if and only if the aggregate excess demand correspondence is single valued, and hence a function, and continuous, and also $\limsup_{q_t \to \infty} z_1(q_t) = \limsup_{q_t \to \infty} z_2(q_t) = \infty$.

Recall that competitive equilibrium relative prices, q^*, for a stationary, elementary, exchange economy of overlapping generations, are a cycle of order n if and only if the associated allocation is invariant to the subgroup, $\mathbf{G}^n \subseteq \mathbf{G}$, generated by the maps (f_n, g_n), where $f_n(t) = t + n$ and $g_n(h, t) = (h, t + n)$, but not under $\mathbf{G}^m$, for $m < n$. Steady-state equilibrium prices are a cycle of order $n = 1$.

Evidently, the competitive equilibrium allocation association with competitive equilibrium relative prices, q^*, in a stationary, elementary, exchange economy of overlapping generations is invariant to the subgroup $\mathbf{G}_n \subseteq \mathbf{G}$ if

$$q_t^* = q_{t+n}^*, \quad \text{for } t \in \mathbf{T}_{-\infty,\infty}.$$

Any elementary exchange economy of overlapping generations that is stationary and well behaved indeed has steady-state equilibrium relative prices

$$q^{**} = (\ldots, 1, \ldots),$$

which we refer to as the Samuelson steady-state [Samuelson (1958)]. Thus, a time-invariant real rate of interest, $r_t^{**} = 0$, is associated with a competitive equilibrium independently of the time preference of individuals. This follows from the budget constraints in the individual optimization problems and the observation that, at a time invariant relative price of one, when aggregated across the individuals who belong to each generation they coincide with the aggregate feasibility constraint. When the definition of stationarity is modified to allow for a time invariant rate of population growth, n, the real rate of interest at the Samuelson steady-state is n.

In addition to the Samuelson steady-state, an elementary exchange economy of overlapping generation that is stationary and well behaved has steady-state relative prices $\bar{q}^* = (\ldots, \bar{\rho}, \ldots)$ obtained as solution to the equation

$$z_1(\rho) = 0.$$

A solution to this equation exists in a well-behaved economy. The function $z_1 : I = \{\rho : \rho > 0\} \to \mathbf{R}$ is continuous, and $\limsup_{\rho \to \infty} z_1 \rho = \infty$, while $\liminf_{\rho \to 0} z_1(\rho) < 0$ since $z_1(\rho) + \rho z_2(\rho) = 0$ and $\limsup_{\rho \to 0} z_2(\rho) = \infty$. We refer to these steady states as autarky since they eliminate trade across generations. Evidently, autarky steady-states may be multiple; also, the Samuelson steady-state may be autarky even though, in a sense that can be made easily precise, typically this is not the case.

In stationary economies with multiple commodities in each period, the definitions and arguments for the existence of the Samuelson and autarky steady-states extend easily [Kehoe and Levine (1985)].

For a stationary, elementary economy of overlapping generations there are robust examples of cycles of order n, for every n [Benhabib and Day (1982), Grandmont (1985)]; as Grandmont has argued, they can be interpreted as endogenous business cycles.

This result is very suggestive. Note, however that under the assumptions of Theorem 5a, which are standard, all cyclical equilibria, with the exception of the autarky steady-states, are Pareto optimal, while the theory of macroeconomic business cycles is traditionally concerned with the welfare losses from cyclical fluctuations; that cyclical behavior is not incompatible with optimality is perhaps an important observation for macroeconomics. Furthermore, if the definition of stationarity is extended to allow for economies with more than one commodity per period, it is difficult to construct robust examples of cycles of order 2. And of course, non-stationary economies of overlapping generations, even with one commodity per period typically have no cyclical equilibria of any order. By contrast, the multiplicity of non-periodic equilibria that we discuss in the next section, and the suboptimality that we discussed in Section 3 are robust properties of exchange economies of overlapping generations with multiple commodities as well as intertemporal heterogeneity. The main contribution of the literature on cyclical equilibria is that it establishes the important, suggestive principle that simple dynamic models can have very complex dynamic behavior at equilibrium.

6. Indeterminacy

An economy displays indeterminacy if and only if it has an uncountable infinity of distinct competitive equilibria. Competitive equilibria are distinct if and only if the associated allocations are distinct.

Indeterminacy arises in exchange economies of overlapping generations. And it may be robust to perturbations in the structure of the economy, the utility functions and initial endowments of individuals.

Example 12. Consider an elementary stationary exchange economy of overlapping generations. One individual is born each period, $(1, t) = t$. The utility function of an individual is $u^t = x_t + (1/\alpha)\delta^{\alpha-1}x_{t+1}^{\alpha}$, $\alpha < 1$, and his initial endowment is $w^t = (\ldots, 0, w_t^t = 1, w_{t+1}^t = \varepsilon, 0, \ldots)$, $\varepsilon > 0$. The excess demand of individual t as a function of the relative price of the consumption good in the two periods of life of the individual is $z^t = (z_1^t, z_2^t) = (q_t\varepsilon - \delta q_t^{\alpha/(\alpha-1)}, \delta q^{1/(\alpha-1)} - \varepsilon)$. Competitive equilibrium relative prices are thus obtained as solutions to the nonlinear difference equation

$$\delta q_{t-1}^{1/(\alpha-1)} = \varepsilon - q_t \varepsilon + \delta q_t^{\alpha/(\alpha-1)}, \quad \text{for } t \in \mathbf{T}_{-\infty,\infty}.$$

For $\varepsilon = 0$, this reduces to

$$q_{t-1} = q_t^{\alpha}, \quad \text{for } t \in \mathbf{T}_{-\infty,\infty}.$$

Evidently, for any $\bar{q}_1 > 0$, there exists an equilibrium $q^*(\bar{q}_1)$ with $q_0^*(\bar{q}_1) = \bar{q}_1$; solving explicitly, we obtain q_t^*

$$q_t^*(\bar{q}_1) = \bar{q}_1^{(\alpha^{1-t})}, \quad \text{for } t \in \mathbf{T}_{-\infty,\infty}.$$

Indeterminacy does not arise only in economies that extend infinitely into the future as well as into the past. This is evident since, for Lemma 1, an economy in which time extends infinitely into the future as well as into the past is equivalent to an economy in which time extends infinitely into the future but not into the past.

Example 13. Consider an economy in which time extends infinitely into the future but not into the past. Two commodities are available each period, (l, t), for $l = 1, 2$, and two individuals are born, (h, t), for $h = 1, 2$. The utility function of individual $(1, t)$ is $u^{(1,t)} = x_{(1,t)} + (1/\alpha)\delta^{\alpha-1}x_{(1,t+1)}^{\alpha}$, $\alpha < 1$, and his initial endowment is $w^{(1,t)} = (\ldots, 0, w_t^{(1,t)} = (1, 0), w_{t+1}^{(1,t)} = (\varepsilon, 0), 0, \ldots)$, $\varepsilon > 0$. The utility function of individual $(2, t)$ is $u^{(2,t)} = (1/\alpha)\delta^{\alpha-1}x_{(2,t)}^{\alpha} + x_{(2,t+1)}$ and his initial endowment is $w^{(2,t)} = (\ldots, 0, w_t^{(2,t)} = (0, \varepsilon), w_{(t+1)}^{(2,t)} = (0, 1), 0, \ldots)$. In addition, an individual, $(3, 1) = 0$, is born in the first period, whose utility function is $u^0 = (1/\alpha)\delta^{\alpha-1}x_{(1,1)}^{\alpha} + x_{(2,1)}$ and whose endowment is $w^0 = (w_1^0 = (\varepsilon, 1), 0, \ldots)$. That this economy has a continuum of equilibria and thus displays indeterminacy follows by observing that it is equivalent to the elementary stationary economy in Example 12. It suffices to identify $(2, t)$ with commodity $(1, 1-t)$ and individual $(2, t)$ with individual $(1, 1-t)$, for $t \in \mathbf{T}_{1,\infty}$. For any $k > 0$, the prices defined by $p_{(1,t)}^* = k^{(\alpha^{1-t})}$ and $p_{(2,t)}^* = k^{(\alpha^t)}$ are indeed competitive equilibrium prices.

Theorem 9 [Geanakoplos and Polemarchakis (1982)]. *In a well behaved, stationary, elementary economy of overlapping generations such that $z(1) \neq 0$, there exists a non-degenerate closed interval $\mathbf{I}^* \subset (0, \infty)$ such that for $\bar{q}_0 \in \mathbf{I}^*$* there exists competitive equilibrium relative prices $q^*(\bar{q}_0)$ with $q_0^* = \bar{q}_0$.

Proof. Without loss of generality we may suppose that $z_1(1) < 0$ and hence $z_2(1) > 0$. Since $\limsup_{q \to \infty} z_1 = \infty$, there exists $\bar{q} > 1$ such that $z'(\bar{q}) = 0$. Since the excess demand function is bounded below, there exist $1 > \bar{\bar{q}} > \bar{q}$ such that $z_1(\bar{q}) = -\alpha < 0$ while $z_1(q) \geq -\alpha$ for $q \in \mathbf{Q}$. Since $\limsup_{q \to 0} z_2(q) = \infty$, there exists $\bar{\bar{q}} > 1$ such that $z_2(\bar{\bar{q}}) = \alpha$. Consider the set $\mathbf{K} = \{(z_1, z_2): -\alpha \leq z_1 \leq 0, 0 \leq z_2 \leq \alpha\}$. Observe that for fixed $\bar{z}_1 \in [-\alpha, 0]$ there exists $\bar{z}_2 \in [0, \alpha]$ such that $(\bar{z}_1, \bar{z}_2) \in \mathbf{K}$ and, similarly, for fixed $\bar{z}_2 \in [0, \alpha]$ there exists $\bar{z}_1 \in [-\alpha, 0]$ such that $(\bar{z}_1, \bar{z}_2) \in \mathbf{K}$. Let $\mathbf{I} = \{q \in (0,1): z(q) \in \mathbf{K}\} \subset (0,1)$. Since $\alpha < 0$, the set $\mathbf{I}$ has non-empty interior.

Choose $\bar{q}_0 \in \mathbf{I}$. Since $z(\bar{q}_0) \in \mathbf{K}$, $-\alpha \leq z_1(\bar{q}_0) \leq 0$ while $0 \leq z_2(\bar{q}_0) \leq \alpha$. Let $\bar{z}_2 = -z_1(\bar{q}_0)$ and $\bar{z}_1 = -z_2(\bar{q}_0)$. It follows that $0 \leq \bar{z}_2 \leq \alpha$ and $-\alpha \leq \bar{z}_1 \leq 0$. Hence there exist $q_1^* \in \mathbf{I}$ and $q_{-1}^* \in \mathbf{I}$ such that $z_2(q_{-1}^*) = \bar{z}_2 = -\bar{z}_1$ and $z_1(q_1^*) = \bar{z}_1 = -\bar{z}_2(\bar{q}_0)$. It follows that for $q_0^* = \bar{q}_0$, $z_2(q_{-1}^*) + z_1(q_0^*) = 0$ and $z_2(q_0^*) + z_1(q_1^*) = 0$ or, equivalently, the markets at $t = 0$ and $t = 1$ are at equilibrium. Most importantly, since $q_1^* \in \mathbf{I}$ and also $q_{-1}^* \in \mathbf{I}$ there exist $q_2^* \in \mathbf{I}$ and $q_{-2}^* \in \mathbf{I}$ such that the markets at $t = -1$ and $t = 2$ are in equilibrium. Proceeding in this manner, we construct equilibrium relative prices q^* with $q_0^* = \bar{q}_0$. □

Figure 35.1 illustrates the construction.

In stationary economies with L commodities per period, the degree of indeterminacy, the dimension of an open set of distinct equilibrium allocations generically does not exceed $2L - 1$ [Brown and Geanakoplos (1982), Bona and Santos (1989)]; or $L - 1$ if time extends infinitely into the future but not into the past. Indeed, there is a method for constructing robust examples of indeterminacy of any dimension $0 \leq d \leq L - 1$, for stationary economies into time $\mathbf{T}_{1,\infty}$ [Kehoe and Levine (1985)].

The proof of Theorem 9 conveys the idea that indeterminacy in economies of

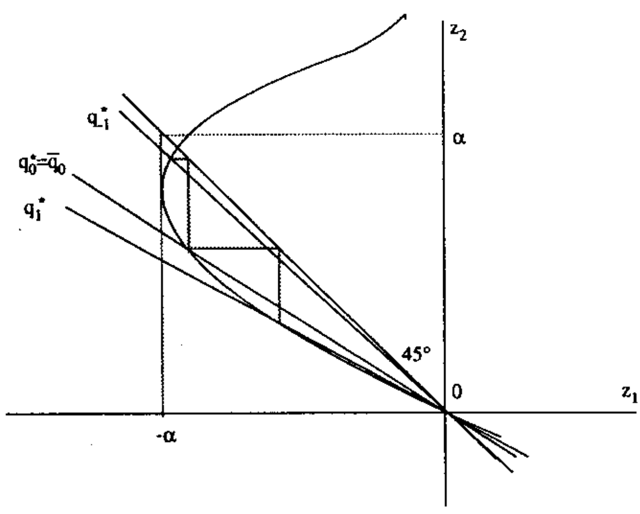

Figure 35.1. The indeterminacy of competitive equilibria in stationary elementary exchange economies of overlapping generations.

overlapping generations can be understood as lack of market clearing at infinity, "$t = \pm\infty$".

7. Implications for macroeconomics

Keynesian macroeconomics is based in part on the fundamental idea that changes in expectations or "animal spirits" can affect economic activity at equilibrium, including the level of output and employment. It asserts, moreover, that publicly announced government policy also has predictable and significant consequences for economic activity and that, therefore, the government should intervene actively in the market place if investor "optimism" is not sufficient to maintain employment at an optimal level.

The Keynesian view of the indeterminacy of equilibrium and the efficacy of public policy has met a long and steady resistance, culminating in the sharpest attack of all, from the so-called new classicals, who have argued that the methodological premises of individual optimization and market clearing, considered together with rational expectations, are logically inconsistent with animal spirits and the non-neutrality of public monetary and bond-financed fiscal policy.

The foundation of the new classical paradigm is the model of Arrow and Debreu, in which it is typically possible to prove that competitive equilibrium allocations are Pareto optimal; also determinate, the equilibrium set is finite. The hypothesis of market clearing fixes the expectations of rational investors. In that model, however, economic activity has a definite beginning and end. Alternatively, economic activity may be better described as a process without a definite and possibly without a definite beginning, as it is in the model of overlapping generations of Allais and Samuelson. In a world without a definite end, the possibility arises that what happens "today" is underdetermined because it depends on what individuals "tomorrow" expect to happen the day after tomorrow, etc.

Consider an economy of overlapping generations with a simple demographic structure and one commodity per period. Net aggregate revenue in the economy can be interpreted as fiat money whose initial stock, $\tau = M$, is held by individuals in the first period of their life at $t = 1$. Commodity prices are $p = (p_1, \ldots, p_t, \ldots)$ and the prices of money is 1 every period. Real balances at $t = 1$ are thus M/p_1. It is helpful to interpret the model as a simple production economy. The endowment in the first period of an individual's life can be thought of as labor, l_t, that can be transformed into output, y_t, according to $y_t = l_t$. We can then think of any purchases of good by individual in the second period of their life, the "old", as demand for real output to be produced by those in the first period of their life, the "young". Individuals derive utility from leisure in their youth and consumption in their old age. Notice that the quantity equation, $p_t y_t = M$ holds for this economy at equilibrium with velocity equal to one.

The indeterminacy of equilibrium has the direct implication that optimistic expectations, which are fulfilled, by themselves can cause the economy's output to expand or contract. The Keynesian story of animal spirits causing economic growth or decline can be told without involving irrationality or the failure of market clearing.

In fact, the indeterminacy of equilibrium is especially striking when seen as a response to public, yet unanticipated, policy changes. Suppose the economy is in equilibrium p^* when at time $\bar{t}$ the government undertake some expenditures, financed either by the lump-sum taxation of the young or by printing money. How should rational individuals respond? The environment has been changed and there is no reason for them to anticipate $(p^*_{\bar{t}+1}, p^*_{\bar{t}+2}, \ldots)$. Indeed, in models with more than one commodity, there may be no equilibrium in the new environment with $p_{\bar{t}+1} = p^*_{\bar{t}+1}$, $p_{\bar{t}+2} = p^*_{\bar{t}+2}$, etc. There is an ambiguity in what can be rationally anticipated.

We argue that it is possible to explain the differences between Keynesian and monetarist policy predictions by the assumptions each makes about expectational responses to policy and not by the supposed adherence of one to optimization, market clearing and rational expectations and the supposed denial by the other of all three.

Consider the government policy of printing a small amount of money, ΔM, to be spend on its own consumption of real output or, equivalently, to be given to the old generations at $\bar{t} = 1$ to spend on its consumption. Imagine that individuals are convinced that this policy is not inflationary, that p_1^* will remain the equilibrium price level at the initial period of the new equilibrium. This will give the old generation at $t = 1$ consumption level $(M + \Delta M)/p_1^*$. As long as ΔM is sufficiently small, and the equilibrium p^* was one of the suboptimal equilibria different from the Samuelson steady-state, there is indeed a new equilibrium $\bar{p}^*$ with $\bar{p}_1^* = p_1^*$. Output has risen by $\Delta M/p_1^*$ and in fact the policy may be Pareto improving. On the other hand, imagine individuals are convinced that the real interest rate, $r_1^* = (p_1^*/p_2^*) - 1$, will remain unchanged. In this case, price expectations are a function of p_1. Recalling the initial period market-clearing equation, it is clear that prices rise proportionately to the growth of the money stock. The result is "forced savings"; output is unchanged and the generation old at $t = 1$ pays for the government's consumption.

This model is only a crude approximation of the differences between Keynesian and monetarist assumptions about expectations and policy. Nevertheless it conveys the idea that when equilibrium prices are not locally unique there is no natural assumption to make about how expectations are affected by policy.

References*[1]

Aiyagari, S.R. (1981) 'Putty-clay durable capital in stochastic overlapping generations models', Ph.D. Dissertation, University of Minnesota.

Aiyagari, S.R. (1985) 'Observational equivalence of the overlapping generations and the discounted dynamics programming frameworks for one-sector growth', *Journal of Economic Theory*, 35: 201–221.

Aiyagari, S.R. (1987) 'Optimality and monetary equilibria in stationary overlapping generations models with long-lived agents: growth versus discounting', *Journal of Economic Theory*, 43: 292–313.

Aiyagari, S.R. (1988) 'Nonmonetary steady states in stationary overlapping generations models with long-lived agents and discounting: multiplicity, optimality and consumption smoothing', *Journal of Economic Theory*, 45: 101–127.

Aiyagari, S.R. (1989a) "Can there be short-period deterministic cycles when people are long-lived?', *Quarterly Journal of Economics*, 104: 163–185.

Aiyagari, S.R. (1989b) 'Equilibrium existence in an overlapping generations model with altruistic preferences', *Journal of Economic Theory*, 47: 130–152.

Aiyargari, S.R. (1989c) "Walras' law and nonoptimal equilibria in overlapping generations models', Working Paper No. 393, Federal Reserve Bank of Minneapolis.

Aiyargari, S.R. and D. Peled (1989) 'Dominant root characterization of Pareto optimality and the existence of optimal equilibria in stochastic overlapping generations models', Working Paper, Federal Reserve Bank of Minneapolis.

Aliprantis, C., D.J. Brown and D. Burkinshaw (1989) 'Equilibria in exchange economies with a countable number of agents', *Journal of Mathematical Analysis and Applications*, 142: 250–299.

Aliprantis , C., D.J. Brown and D. Burkinshaw (1990) 'Valuation and optimality in the overlapping generations model', *International Economic Review*, 31: 275–288.

Aliprantis, C. and O. Burkinshaw (1990) 'An overlapping generations model core equivalence theorem', *Journal of Economic Theory*, 50: 362–380.

Aliprantis, C.D. and C.R. Plott (1990) 'Competitive equilibria in overlapping generations experiments', Discussion paper no. 746, Department of Economics, California Institute of Technology.

Allais, M.A. (1947) *Economie et intérêt*. Paris: Imprimerie Nationale.

*Arrow, K.J. (1951) 'An extension of the basic theorems of classical welfare economics', in: J. Neyman, ed., *Proceedings of the second Berkeley symposium on mathematical statistics and probability*, University of California Press, pp. 507–532.

*Arrow, K.J. (1953) 'Le rôle des valeurs boursières pour la répartition la meilleure des risques', *Econométrie*, 11: 41–48.

*Arrow, K.J. and G. Debreu (1954) 'Existence of an equilibrium for a competitive economy', *Econometrica*, 22: 265–290.

*Arrow, K.J. and F. Hahn (1971) *General competitive analysis*. Holden Day.

Asimakopulos, A. (1967) 'The biological interest rate and the social utility function', *American Economic Review*, 57: 185–189.

Azariadis, C. (1981) 'Self-fulfilling prophesies', *Journal of Economic Theory*, 25: 380–396.

Azariadis, C. and R. Guesnerie (1982) 'Prophéties créatrices et persistence des théories', *Revue Economique*, 33: 787–805.

Azariadis, C. and R. Guesnerie (1986) 'Sunspots and cycles', *Review of Economic Studies*, 53: 725–737.

Azariadis, C. and A. Drazen (1990) 'Investment externalities in economic development', *Quarterly Journal of Economics*, 105: 501–526.

Balasko, Y., D. Cass and K. Shell (1980) 'Existence of competitive equilibrium in a general overlapping generations model', *Journal of Economic Theory*, 23: 307–322.

Balasko, Y., D. Cass and K. Shell (1988) 'Market participation and sunspot equilibria', Working Paper No. 88-08, CARESS, University of Pennsylvania.

*References marked with an asterisk contain background information.

[1]The model of overlapping generations has been used extensively in macroeconomics, monetary economics, public finance, development economics and other applications. In the spirit of our survey, we have limited references to theoretical contributions.

Balasko, Y. and K. Shell (1980) 'The overlapping generations model, I. The case of pure exchange without money', *Journal of Economic Theory*, 23: 281–306.
Balasko, Y. and K. Shell (1981a) 'The overlapping generations model, II. The case of pure exchange with money', *Journal of Economic theory*, 24: 112–142.
Balasko, Y. and K. Shell (1981b) 'The overlapping generations model, III. The case of log-linear utility functions', *Journal of Economic Theory*, 24: 143–152.
Balasko, Y. and K. Shell (1986) 'Lump-sum taxes and transfers: public debt in the overlapping-generations model', in W. Heller, R. Starr and D. Starret, eds., *Essays in honor of Kenneth J. Arrow, Volume II: Equilibrium analysis*. Cambridge University Press, pp. 121–153.
Barta, S.M. (1982) "Capital accumulation in a stochastic decentralized economy', *Journal of Economic Theory*, 26: 124–142.
Baxter, M. (1989) 'Money and market incompleteness in overlapping generations models', *Journal of Monetary Economics*, 24: 69–91.
Becker, G. and N. Tomes (1979) 'An equilibrium theory of the distribution of income and intergenerational mobility', *Journal of Political Economy*, 87: 1153–1189.
Benassy, J.-P. (1989a) 'Microeconomic foundations and properties of a macroeconomic model with imperfect competition', Discussion Paper no. 8927, CEPREMAP.
Benassy, J.-P. (1989b) 'Optimal government policy in a microeconomic model with imperfect competition and rational expectations', Discussion paper. no. 8928, CEPREMAP.
Benassy, J.-P. (1991) 'Incomplete markets and the suboptimality of rational expectations', Discussion paper no. 9101, CEPREMAP.
Benassy, J.-P. and M.C. Blad (1989) 'On learning and rational expectations in an overlapping generations model', *Journal of Economic Dynamics and Control*, 13: 379–400.
Benhabib, J. (1980) 'Adaptive monetary policy and rational expectations', *Journal of Economic Theory*: 261–266.
Benhabib, J. (1981) 'Uncertainty and chaos in overlapping generations models', *IEEE Conference on Decision and Control.*
Benhabib, J. and R.H. Day (1980) 'Erratic accumulation', *Economic Letters*, 6: 113–117.
Benhabib, J. and R. Day (1981) 'Rational choice and erratic behavior', *Review of Economic Studies*, 48: 459–472.
Benhabib, J. and R.H. Day (1982) 'A characterization of erratic dynamics in the overlapping generations model', *Journal of Economic Dynamics and Control*, 4: 37–55.
Benhabib, J. and G. Laroque (1988) 'On competitive cycles in productive economics', *Journal of Economic Theory*, 45: 145–170.
Benhabib, J. and K. Nishimura (1979) 'The Hopf bifurcation and the existence and stability of closed orbits in multisector models of optimum economic growth', *Journal of Economic Theory*, 21: 421–444.
Benhabib, J. and K. Nishimura (1985) 'Competitive equilibrium cycles', *Journal of Economic Theory*, 35: 284–306.
Benhabib, J. and K. Nishimura (1987) 'Stochastic equilibrium oscillations', Research Report No. 87-13, C.V. Starr Center for Applied Economics, New York University.
Benveniste, L. (1983) 'Dynamic models of capital accumulation and generational distribution: a united structure', Working Paper No. 83-16, CARESS, University of Pennsylvania.
Benveniste, L. (1986) 'Pricing optimal distributions to overlapping generations: a corollary to efficiency pricing', *The Review of Economic Studies*, 53: 301–306.
Benveniste, L. and D. Cass (1986) 'On the existence of optimal stationary equilibria with a fixed supply of fiat money: I. The case of a simple consumer', *Journal of Political Economy*, 94: 402–417.
*Benveniste, L. and D. Gale (1975) 'An extension of Cass' characterization of infinite efficient production programs', *Journal of Economic theory*, 10: 229–238.
Bernhardt, D. (1989) 'Money and loans', *The Review of Economic Studies*, 56: 89–100.
Bernheim, D. and D. Ray (1984a) 'Altruistic growth economies I: Existence of bequest equilibria', Manuscript, Department of Economics, Stanford University.
Bernheim, D. and D. Ray (1984b) 'Altruistic growth economies II: Properties of bequest equilibria', Manuscript, Department of Economics, Stanford University.
Bertocchi, G. (1990) 'Safe debt and risky capital', Manuscript, Department of Economics, Brown University.

*Bewley, T. (1972) 'Existence of equilibria in economies with infinitely many commodities', *Journal of Economic Theory*, 4: 514–540.

Bewley, T. (1981) 'The indeterminacy of interest rates', Discussion Paper No. 491, The Center for Mathematical Studies in Economics and Management Science, Northwestern University.

Bewley, T. (1983) 'Dynamic implications of the form of the budget constraint', Manuscript, Department of Economics, Yale University.

Black, F. (1974) 'Uniqueness of the price level in monetary growth models with rational expectations', *Journal of Economic Theory*, 7: 53–65.

Blackorby, C. and R.R. Russell (1990) 'On the observational equivalence of models with infinitely lived agents and models with overlapping generations', Manuscript, Department of Economics, University of British Columbia.

Böhm, V. and H. Keiding (1985) 'Inflation and welfare in international trade', Discussion Paper No. 8601, CORE, Université Catholique de Louvain.

Böhm, V. and M. Puhakka (1988) 'Rationing and optimality in overlapping generations models', *Scandinavian Journal of Economics*, 90: 225–232.

Boldrin, M. and M. Woodford (1990) 'Equilibrium models displaying endogenous fluctuations and chaos', *Journal of Monetary Economics*, 25: 189–222.

Bona, J. and M.S. Santos (1989) 'On the structure of the equilibrium price set of overlapping generations economies', *Journal of Mathematical Economies*, 18: 209–230.

Borglin, A. and H. Keiding (1985) 'Efficiency criteria in infinite-horizon models: an axiomatic approach', Working Paper, Department of Economics, University of Gothenburg.

Bose, A. (1974a), 'Pareto optimal consumptions loans with and without production', Ph.D. Dissertation, University of Rochester.

Bose, A. (1974b) 'Pareto optimality and efficient capital accumulation', Discussion Paper No. 74-4, Department of Economics, University of Rochester.

Brock, W. (1988) 'Overlapping generations models with money and transactions costs', Working Paper, Social Systems Research Institute, University of Wisconsin.

Brock, W. and J.A. Scheinkman (1980) 'Some remarks on monetary policy in an overlapping generations model', in: J.H. Kareken and N. Wallace, eds., *Models of Monetary Economies*. Federal Reserve Bank of Minneapolis, pp. 211–232.

Brown, D.J. and J.D. Geanakoplos (1982) 'Understanding overlapping generations economies as lack of market clearing at infinity', Discussion Paper, Cowles Foundation, Yale University.

Brown, D.J. and J.D. Geanakoplos (1985) 'Comparative statics and local indeterminacy in overlapping generations economies: an application of the multiplicative ergodic theorem', Discussion Paper No. 773, Cowles Foundation, Yale University.

Bryant, J. (1979) 'The political economy of overlapping generations', Research Department Staff Report No. 36, Federal Reserve Bank of Minneapolis.

Bryant, J. (1980) 'Nontransferable interest-bearing national debt', *Journal of Finance*, 35: 1027–1031.

Bryant, J. (1983) 'The role of overlapping generations models in monetary economics: a comment', *Carnegie-Rochester Conference Series on Public Policy*, 18: 45–50.

Bryant, J. and N. Wallace (1979) 'The inefficiency of interest-bearing national debt, *Journal of Political Economy*, 87: 365–381.

Buiter, W. (1979) 'Government finance in an overlapping generations model with gifts and bequests', in: George M. Von Furstenberg, ed., *Social security versus private saving*. Ballinger.

Buiter, W. (1981) 'Time preference and international lending and borrowing in an overlapping generations model', *Journal of Political Economy*, 89: 769–797.

Bulow, J. and H.M. Polemarchakis (1983) 'Retroactive money', *Economica*, 50: 301–310.

Burbridge, J. (1983) 'Government debt in an overlapping-generations model with bequests and gifts', *American Economic Review*, 73: 222–227.

Burke, J.L. (1987a) 'Efficient monetary equilibria in overlapping generations economies', Manuscript, Department of Economics, A&M University.

Burke, J.L. (1987b) 'Inactive transfer policies and efficiency in general overlapping generations economies', *Journal of Mathematical Economics*, 16: 201–222.

Burke, J.L. (1987c) 'Bounded foresight, efficiency and the core of overlapping generations economies', Manuscript, Department of Economics, A&M University.

Burke, J.L. (1988) 'On the existence of price equilibria in dynamic economies', *Journal of Economic Theory*, 44: 281–300.

Burke, J.L. (1989) 'Arc-correctedness and non-closedness of efficient allocations in overlapping-generations economies', *Journal of Economic Theory*, 49: 179–188.

Burke, J.L. (1990a) 'Intertemporal general equilibrium theory: the importance of a sequence of markets', Manuscript, Department of Economics, University of Texas, Austin.

Burke, J.L. (1990b) 'The generic existence of equilibrium with patient consumers', Manuscript, Department of Economics, University of Texas, Austin.

Burke J.L. (1990c) 'Finding an optimal quantity of money, national debt and social security in overlapping-generations economies', Manuscript, Department of Economics, University of Texas, Austin.

Burke, J.L. (1990d) 'A benchmark for comparative dynamics and determinacy in overlapping generations economies', *Journal of Economic Theory*, 52: 268–303.

Burke, J.L. (1991) 'Equilibrium growth with many commodities', Manuscript, Department of Economics, University of Texas, Austin.

Burnell, S. (1989) 'Sunspots', in: F.H. Hahn, ed., *The economics of missing markets, information and games*. Oxford University Press.

Calvo, G. (1978) 'On the indeterminacy of interest rates and wages with perfect foresight', *Journal of Economic Theory*, 19: 321–337.

Calvo, G. and D.A. Peel (1983) 'Growth and inflationary finance: variations on a Mudellian theme', *Journal of Political Economy*, 91: 880–887.

*Cass, D. (1972) 'On capital overaccumulation in the aggregative, neoclassical model of economic growth: a complete characterization', *Journal of Economic Theory*, 4: 200–223.

Cass, D. (1979) 'Money in consumption-loan type models: an addendum', in: J. Kareken and N. Wallace, eds., *Models of monetary economies*. Federal Reserve Bank of Minneapolis.

*Cass, D. (1985a) 'On the "Number" of equilibrium allocations with incomplete financial markets', Manuscript, Department of Economics, University of Pennsylvania.

Cass, D. (1985b) 'Optimality with unbounded numbers of households: I. Overlapping (or overlapping generations) structure and the first basic theorem of welfare', in: G. Ritschard and D. Royer, eds., *Optimalité et structures: Mélanges en hommage à Edouard Rossier*. Economica.

Cass, D., M. Okuno and I. Zilcha (1979) 'The role of money in supporting the Pareto optimality of competitive equilibrium in consumption-loan type models', *Journal of Economic Theory*, 20: 41–80.

Cass, D. and K. Shell (1980) 'In defense of a basic approach', in: J. Kareken and N. Wallace, eds., *Models of monetary economics*. Federal Reserve Bank of Minneapolis, pp. 251–260.

Cass, D. and K. Shell (1983) 'Do sunspots matter?', *Journal of Political Economy*, 91: 193–227.

Cass, D. and K. Shell (1988) 'Sunspot equilibrium in an overlapping generations economy with an idealized contingent-commodities market', Working Paper No. 88-33, CARESS, University of Pennsylvania.

Cass, D. and M. Yaari (1966a) 'A re-examination of the pure consumption loans model', *Journal of Political Economy*, 74: 353–367.

Cass, D. and M. Yaari (1966b) 'A note on the role of money in providing sufficiently intermediation', Discussion Paper, Cowles Foundation, Yale University.

Cass, D. and M. Yaari (1967) 'Individual savings, aggregate capital accumulation and efficient growth', in: K. Shell, ed., *Essays in the theory of optimal growth*. MIT Press, pp. 233–268.

Chae, S. (1987) 'Short run core equivalence in an overlapping generations model', *Journal of Economic Theory*, 43: 170–183.

Chae, S. and J.-M. Esteban (forthcoming) 'The complete characterization of sets related with core in a simple overlapping generations model', *International Economic Review*.

Chamley, C. and H. Polemarchakis (1984) 'Assets, general equilibrium and the neutrality of money', *The Review of Economic Studies*, 51: 129–138.

Chattopadhyay, S.K. (1990) 'Optimal improvement for stationary sunspot equilibria in a OLG economy: The central role of the golden rule allocation', Discussion Paper no 90-21, Department of Economics, S.U.N.Y., Stony Brook.

Chiappori, P.A., P.-Y. Geoffard and R. Guesnerie 'Sunspot fluctuations around a steady state: The case of multidimensional, one-step forward looking economic models', *Econometrica*.

Chiappori, P.A. and R. Guesnerie (1986) 'Expectations, indeterminacy and the non-neutrality of money', Manuscript, Centre d'Economie Quantitative et Comparative, EHESS.
Chiappori, P.A. and R. Guesnerie (1988a) 'Stationary sunspot equilibria of order K', in: W.A. Barnett, J. Geweke and K. Shell, eds., *Economic Complexity*, Cambridge University Press, pp. 21–44.
Chiappori, P.A. and R. Guesnerie (1988b) 'Lucas equation, indeterminacy and non-neutrality: An example', Discussion Paper no. 88-06, DELTA.
Chiappori, P.A. and R. Guesnerie (1989) 'Self-fulfilling theories: The sunspot connection', Manuscript, DELTA.
Chiappori, P.A. and R. Guesnerie (1990) 'Anticipation, indetermination et non-neutralité de la monnaie', *Annales d'Economie et de la Statistique*, 19: 1–26.
Clark, S.E. (1981) 'A combinatorial analysis of the overlapping generations model', *The Review of Economic Studies*, 47: 139–145.
Dasgupta, P.A. (1969) 'On the concept of optimum population', *Review of Economic Studies*, 36: 295–318.
d'Aspremont, Cl., R. Dos Santos Ferreira and L.-A. Gérard-Varet (1989) 'Imperfect competition, rational expectations, and unemployment', Discussion Paper no. 9044, CORE, Université Catholique de Louvain.
Davies, J.B. (1981) 'Uncertain lifetime, consumption and dissaving in retirement', *Journal of Political Economy*, 89: 561–577.
Day, R.H. and Y.K. Fan (1976) 'Myopic optimizing, economic growth and the golden rule', *Hong Kong Economic Papers*, 10: 10–20.
Deardoff, A. (1976) 'The optimum growth rate for population: comment', *International Economic Review*, 17: 510–515.
*Debreu, G. (1951) 'The coefficient of resource utilization', *Econometrica*, 19: 271–292.
*Debreu, G. (1954a) 'Valuation equilibrium and Pareto optimum', *Proceedings of the National Academy of Science*, 40: 588–592.
*Debreu, G. (1954b) 'Representation of a preference ordering by a numerical function', in: R.M. Thrall, C.H. Coombs and R.L. Davis, eds., *Decision processes*. Wiley.
*Debreu, G. (1959) *Theory of value*. Yale University Press.
*Debreu, G. (1960) 'Une économie de l'incertain', *Econommie Appliquée*, 13: 111–116.
*Debreu, G. (1962) New concepts and techniques in equilibrium analysis', *International Economic Review*, 3: 257–273.
*Debreu, G. (1970) 'Economies with a finite set of equilibria', *Econometrica*, 38: 387–392.
*Debreu, G. (1972) 'Smooth preferences', *Econometrica*, 40: 603–615.
*Debreu, G. and H. Scarf (1963) 'A limit theorem on the core of an economy', *International Economic Review*, 4: 235–246.
Dechert, W.D. and K. Yamamoto (1985) 'Asset trading in an overlapping generations model with production shocks', Manuscript, Department of Economics, State University of New York.
Deneckere, R. and S. Pelikan (1986) 'Competitive chaos', *Journal of Economic Theory*, 40: 13–25.
Detemple, J., P. Gottardi and H.M. Polemarchakis (1990) 'The relevance of financial policy', Discussion Paper No. 9011, CORE, Université Catholique de Louvain.
Diamond, P.A. (1965) 'National debt in a neoclassical growth model', *The American Economic Review*, 55: 1126–1150.
Drabicki, J.Z. and A. Takayama (1984) 'Money, national debt, and economic growth', *Journal of Economic Theory*, 33: 356–367.
Duffie, D., J.D. Geanakoplos, A. Mas-Colell and A. McLennan (1989) 'Stationary Markov equilibria', Manuscript, Graduate School of Business, Stanford University.
Dutta, J. and H.M. Polemarchakis (1990a) 'Asset markets and equilibrium processes', *Review of Economic Studies*, 57: 229–254.
Dutta, J. and H.M. Polemarchakis (1990b) 'On the aggregation of individual risks', Discussion Paper No. 9020, CORE, Université Catholique de Louvain, Louvain-la-Neuve.
Dutta, J. and H.M. Polemarchakis (1990c) 'Investment and the aggregation of risks', Discussion Paper No. 9020, CORE, Université Catholique de Louvain.
Esteban, J.-M. (1986a) 'A characterization of the core in overlapping generations economies', *Journal of Economic Theory*, 39: 439–456.

Esteban, J.-M. (1986b) 'Bounded core and efficiency in overlapping generations economies', Working Paper No. 69, Department of Economics, Universidad Autonoma de Barcelona.
Esteban, J.-M. and T. Millan (1990) 'Competitive equilibria and the core of overlapping generations economies', *Journal of Economic Theory*, 50: 155–174.
Farmer, R.E.A. (1986a) 'Deficits and cycles', *Journal of Economic Theory*, 40: 77–88.
Farmer, R.E.A. (1986b) 'Is nominal price stickiness irrational?', Working Paper No. 86-02, CARESS, University of Pennsylvania.
Farmer, R.E.A. (1990a) 'Sticky prices', Manuscript, Department of Economics, UCLA.
Farmer, R.E.A. (1990b) 'Dynasty: a simple stochastic growth model', Working Paper No. 598, Department of Economics, UCLA.
Farmer, R.E.A. and M. Woodford (1987) 'Self-fulfilling prophecies and the business cycle', *Cuadernos Economicos de ICE*, 35: 129–160.
Gale, D. (1971) 'General equilibrium with imbalance of trade', *Journal of International Economics*, 1: 141–158.
Gale, D. (1973) 'Pure exchange equilibrium of dynamic economic models', *Journal of Economic Theory*, 5: 12–36.
Galor, O. (1990) 'A two-sector overlapping-generations model: a characterization of the dynamical system', Working Paper No. 90-23, Department of Economics, Brown University.
Galor, O. and H.E. Ryder, 'Dynamic efficiency of steady-state equilibria in an overlapping generations model with productive capital', *Economic Letters*.
Galor, O. and H.E. Ryder (1989) 'Existence, uniqueness and stability of equilibrium in an overlapping-generation model with productive capital', *Journal of Economic Theory*, 49: 360–375.
Galor, O. and H.M. Polemarchakis (1987) 'Intertemporal equilibrium and the transfer paradox', *Review of Economic Studies*, 54: 147–184.
Geanakoplos, J.D. (1980) 'Sraffa's production of commodities by means of commodities: indeterminacy and sub-optimality in neoclassical economics. Chapter 3 in Four Essays on the Model of Arrow–Debreu', Ph.D. Dissertation, Harvard University.
Geanakoplos, J.D. (1987) 'Overlapping generations models of general equilibrium', in: J. Eatwell, M. Milgate and P. Newman, eds., *The new Palgrave: a dictionary of economics*. Macmillan, pp. 767–779.
Geanakoplos, J.D. and H.M. Polemarchakis (1982) 'Walrasian equilibrium: indeterminacy and aggregate policy: I: the case of certainty', Discussion Paper No. 8247, CORE, Université Catholique de Louvain.
Geanakoplos, J.D. and H.M. Polemarchakis (1984) 'Intertemporally separable, overlapping-generations economies', *Journal of Economic Theory*, 34: 207–215.
Geanakoplos, J.D. and H.M. Polemarchakis (1986) 'Walrasian indeterminacy and Keynesian macroeconomics', *Review of Economic Studies*, 53: 755–779.
Geanakoplos, J.D., H.M. Polemarchakis and G. Talmain (1987) 'Multiple currencies, incomplete markets and indeterminacy', Manuscript, Graduate School of Business, Columbia University.
Gigliotti, G.A. (1983) 'Total utility, overlapping generations and optimal population', *The Review of Economic Studies*, 50: 71–86.
Gottardi, P. (1987) 'Asset structures and the irrelevance of government financial policies', Discussion Paper No. 122, Department of Applied Economics, University of Cambridge.
Gottardi, P. (1990a) 'On stationary monetary equilibria in overlapping generations economics with incomplete markets', Discussion Paper no. 155, Department of Applied Economics, University of Cambridge.
Gottardi, P. (1990b) 'On the nonneutrality of money with incomplete markets', Discussion Paper no. 158, Department of Applied Economics, University of Cambridge.
*Grandmont, J.-M. (1977) 'Temporary general equilibrium theory', *Econometrica*, 45: 535–572.
Grandmont, J.-M. (1984) 'Periodic and aperiodic behavior in discrete one-dimensional dynamical systems', Technical Report No. 446, Institute for Mathematical Studies in the Social Sciences, Stanford University.
Grandmont, J.-M. (1985a) 'Cycles concurrentiels endogènes', *Cahiers du Séminaire d'Econométrie*, 27: 49–81.
Grandmont, J.-M. (1985b) 'On endogenous, competitive business cycles', *Econometrica*, 53: 995–1046.

Grandmont, J.-M. (1986) 'Stabilizing competitive business cycles', *Journal of Economic Theory*, 40: 57–76.

Grandmont, J.-M. (1987) 'Local bifurcations and stationary sunspots', Manuscript, CEPREMAP.

Grandmont, J.-M. and G. Laroque (1973) 'Money in the pure consumption loan model', *Journal of Economic Theory*, 6: 382–395.

Grandmont, J.-M. and G. Laroque (1975) 'On money and banking', *Review of Economic Studies*, 42: 207–236.

Grandmont, J.-M. and G. Laroque (1986) 'Stability of cycles and expectations', *Journal of Economic Theory*, 40: 138–151.

Grandmont, J.-M. and G. Laroque (1988) 'Stability, expectations and predetermined variables', Technical Report No. 522, Institute for Mathematical Studies in the Social Sciences, Stanford University.

Green, J.R. (1972) 'The question of collective rationality in Professor Gales's model of trade imbalance', *Journal of International Economics*, 2: 39–55.

Green, R.C. and S.E. Spear (1987) 'Equilibria with incomplete markets and overlapping generations', Manuscript, Carnegie-Mellon University.

Guesnerie, R. (1986) 'Stationary sunspot equilibria in an N-commodity world', *Journal of Economic Theory*, 40: 103–127.

Guesnerie, R. and M. Woodford (1989) 'Stability and cycles with adaptive learning rules', Manuscript, DELTA.

Hagerup, P.A. (1981) 'Asset price variability and risk aversion in an overlapping generations economy', Manuscript, Department of Economics, University of Chicago.

Hahm, S. (1989) 'Asset prices in a simple overlapping generations model with implications for the equity premium and stock price volatility', Manuscript, Department of Economics, Virginia Polytechnic Institute.

*Hahn, F.H. (1965) 'On some problems in proving the existence of equilibrium in a monetary economy', in: F.H. Hahn and F.P.R. Brechling, eds., *The theory of interest rates*. Macmillan, pp. 126–135.

Hammour, M.L. 'Are business cycles exogenous?', Manuscript, Department of Economics, MIT.

Hammour, M.L. 'Social increasing returns in macro models with external effects', Manuscript, Columbia University.

Hammour, M.L. 'Technology bubbles', Manuscript, Department of Economics, Columbia University.

*Hart, O.D. (1975) 'On the optimality of equilibrium when the market structure is incomplete', *Journal of Economic Theory*, 11: 418–443.

Hendricks, K., K. Judd and D. Kovenock (1980) 'A note on the core of the overlapping generations model', *Economic Letters*, 6: 95–97.

*Hildenbrand, W. (1991) 'Comment on "Time preference and an extension of the Fisher–Hicksian equation" by Hirofuni Uzawa', in: L.W. McKenzie and St. Zamagri, eds., *Value and capital fifty years later*. Macmillan and International Economics Association, London.

Homma, M. (1977) 'A characteristic feature of the consumption-loan model', *Journal of Economic Theory*, 16(2): 490–495.

Honkapohja, S. and T. Ito (1979) 'Nontrivial equilibrium in an economy with stochastic rationing', Working Paper No. 322, NBER.

Huberman, G. (1984) 'Capital asset pricing in an overlapping generations model', *Journal of Economic Theory*, 33: 232–248.

Huffman, G.W. (1986a) 'The representative agent, overlapping generations, and asset pricing', *Canadian Journal of Economics*, 19: 511–521.

Huffman, G.W. (1986b) 'Asset pricing with capital accumulation', *International Economic Review*, 76: 565–582.

Huffman, G.W. (1987) 'A dynamic equilibrium model of asset prices and transaction volume', *Journal of Political Economy*, 95: 138–159.

Huo, T.M. (1987) 'Observational equivalence of the overlapping generations and the cash-in-advance economies', *Economics Letters*, 25: 9–13.

Inaba, F.S. (1977) 'Consumption loans under uncertainty', *Journal of Economic Theory*, 15: 26–53.

Jones, L. (1985) 'Existence of equilibrium with infinitely many consumers and infinitely many commodities', *Journal of Mathematical Economics*, 12: 119–138.

Jones, L.E. and R.E. Manuelli (1990) 'Finite lifetimes and growth', Manuscript, M.E.D.S., J.L. Kellogg Graduate School of Management, Northwestern University.

Jordan, J.S. (1988) 'A stability problem with nonstationary overlapping generations equilibria', *Journal of Economic Theory*, 45: 425–433.

Julien, B. (1988) 'Competitive business cycles in an overlapping generations economy with productive investment', *Journal of Economic Theory*, 46: 45–65.

Julien, B. and P. Picard (1989) 'Efficiency wage and macroeconomic policy', Discussion Paper No. 8918, CEPREMAP.

Kareken, J. and N. Wallace (1981) 'On the indeterminacy of equilibrium exchange rates', *Quarterly Journal of Economics*, 96: 207–222.

Karni, E. and I. Zilcha (1984) 'A welfare analysis of steady states in an overlapping generations model with uncertain lifetime', Working paper No. 137, Department of Economics, Johns Hopkins University.

Karni, E. and I. Zilcha (1986) 'Welfare and comparative statistics implication of fair social security: a steady-state analysis', *Journal of Public Economics*, 30: 341–357.

Kehoe, T.J. (1984) 'A numerical investigation of multiplicity of equilibria', Economic Theory Discussion Paper No. 78, University of Cambridge.

Kehoe, T.J. (1985) 'Multiplicity of equilibria and comparative statics', *Quarterly Journal of Economics*, 100: 119–148.

Kehoe, T.J. (1986) 'General equilibrium models with infinite numbers of consumers and goods', Economic Theory Discussion Paper No. 97, University of Cambridge.

Kehoe, T.J. and D.K. Levine (1982) 'Indeterminacy of relative prices in overlapping generations models', Working Paper No. 313, Department of Economics, M.I.T.

Kehoe, T.J. and D.K. Levine (1984a) 'Intertemporal separability in overlapping generations models', *Journal of Economic Theory*, 34: 216–226.

Kehoe, T.J. and D.K. Levine (1984b) 'Regularity in overlapping generations exchange economies', *Journal of Mathematical Economics*, 13: 69–93.

Kehoe, T.J. and D.K. Levine (1985) 'Comparative static and perfect foresight in infinite horizon models', *Econometrica*, 53: 443–454.

Kehoe, T.J. and D.K. Levine (1988) 'The economics of indeterminacy in overlapping generations models', Manuscript, Department of Economics, University of Minnesota.

Kehoe, T.J., D.K. Levine, A. Mas-Colell and M. Woodford (1988) 'Gross substitutability in large-square economies', Manuscript, Department of Economics, University of Minnesota.

Kehoe, T.J., D.K. Levine, A. Mas-Colell and W.R. Zame (1989) 'Determinacy of equilibria in large-square economies', *Journal of Mathematical Economies*, 18: 231–262.

Kim, K.S. and H.M. Wu (1988a) 'Inflation, employment and welfare in the steady state', Manuscript, Department of Economics, Tulane University.

Kim, K.S. and H.M. Wu (1988b) 'Gains from asset trade wtih convertible currencies and incomplete markets', Manuscript, Department of Economics, Tulane University.

Koda, K. (1984) 'A note on the existence of monetary equilibria in overlapping generations models with storage', *Journal of Economic Theory*, 34: 388–395.

Labadie, P.A. (1986) 'Comparative dynamics and risk premia in an overlapping generations model', *Review of Economic Studies*, 53: 139–152.

Labadie, P.A. (1984) 'A test of risk premia behavior in an overlapping generations model', Manuscript, School of Business, Columbia University.

Laitner, J. (1979a) 'Bequests, golden-age capital accumulation, and government debt', *Economica*, 46: 403–414.

Laitner, J. (1979b) 'Household bequest, perfect expectations, and the national distribution of wealth', *Econometrica*, 47: 1175–1194.

Laitner, J. (1979c) 'Household bequest behavior and the national distribution of wealth', *The Review of Economic Studies*, 46, 467–484.

Laitner, J. (1980) 'Intergenerational preference differences and optimal national saving', *Journal of Economic Theory*, 22: 55–66.

Laitner, J. (1981a) 'The steady states of a stochastic growth model', *Journal of Economic Theory*, 24: 377–392.

Laitner, J. (1981b) 'The stability of steady states in perfect foresight models', *Econometrica*, 49: 319–334.

Laitner, J. (1982) 'Monopoly and long-run capital accumulation', *Bell Journal of Economics*, 143–157.

Laitner, J. (1984) 'Transition time paths for overlapping generations models', *Journal of Economic Dynamics and Control*, 7: 111–129.

Laitner, J. (1985) 'Stationary equilibrium transition rules for an overlapping generations model with uncertainty', *Journal of Economic Theory*, 35: 83–108.

Laitner, J. (1986a) 'Sunspot equilibra and stability', Manuscript, Department of Economics, University of Michigan.

Laitner, J. (1986b) 'The existence of a solution for an overlapping generations model with both gifts and bequests', Manuscript, Department of Economics, University of Michigan.

Laitner, J. (1987) 'The dynamic analysis of continuous time life cycle saving growth models', *Journal of Economic Dynamics and Control*, 11: 331–357.

Laitner, J. (1988) 'Bequests, gifts, and social security', *Review of Economic Studies*, 55: 275–299.

Laitner, J. (1989) 'Dynamic determinacy and the existence of sunspot equilibria', *Journal of Economic Theory*, 47: 39–50.

Laitner, J. (1990) 'Tax changes and phase diagrams for an overlapping generations model', *Journal of Political Economy*, 98: 193–220.

Lenninger, W. (1983) 'The existence of perfect equilibria in a model of growth with altruism between generation', *Review of Economic Studies*, 53: 349–367.

Lerner, A.P. (1959a) 'Consumption-loan interest and money', *Journal of Political Economy*, 77: 512–517.

Lerner, A.P. (1959b) 'Rejoinder', *Journal of Political Economy*, 77: 523–525.

Leroy, S.F. and N. Raymon (1987) 'A monetarist model of inflation', *Journal of Economic Theory*, 42: 275–310.

Loury, G.C. (1981) 'Intergenerational transfers and the distribution of earnings', *Econometrica*, 49: 843–867.

Lucas, R.E. (1972) 'Expectations and the neutrality of money', *Journal of Economic Theory*, 4: 103–124.

*Malinvaud, E. (1953) 'Capital accumulation and efficient allocation of resources', *Econometrica*, 21: 233–268.

*Malinvaud, E. (1962) 'Efficient capital accumulation: a corrigendum', *Econometrica*, 30: 570–573.

Malinvaud, E. (1986) 'Maurice Allais, précurseur méconnu des modèles à générations renouvelées', in: M. Boiteux, Th. de Montbrial and B. Municer, eds., *Marchés, capital et incertitudes: Essais en l'honneur de Maurice Allais*.

Malinvaud, E. (1987) 'The overlapping generations model in 1947', *Journal of Economic Literature*, 25: 103–105.

Manueli, R. (1984) 'A note on the relationship between existence and optimality of monetary equilibrium in stochastic overlapping generations model', Manuscript, Department of Economics, Stanford University.

Manueli, R. (1986) 'Existence of currency equilibrium in stochastic overlapping generations models: the pure endowment case', Manuscript, J.L. Kellogg Graduate School of Management, Northwestern University.

Martins, A.C. (1975) 'The private and the social optimum quantity of public debt in a pure consumption loans model', Ph.D. Dissertation, University of Chicago.

McCallum, B.T. (1983) 'The role of overlapping-generations models in monetary economics', *Carnegie-Rochester Series in Public Policy*, 19: 9–44.

McCallum, B. (1987) 'The optimal inflation rate in an overlapping generations economy with land', in: W. Barnett and K. Singleton, eds., *New approaches to monetary economics*. Cambridge University Press.

*McKenzie, L.W. (1954) 'On equilibrium in Graham's model of world trade and other competitive systems', *Econometrica*, 22: 147–161.

*McKenzie, L.W. (1959) 'On the existence of general equilibrium for a competitive market', *Econometrica*, 27: 54–71.

*McKenzie, L.W. (1961) 'On the existence of general equilibrium: some corrections', *Econometrica*, 29: 247–248.
Meckling, W.H. (1960) 'An exact consumption-loan model of interest: a comment', *Journal of Political Economy*, 68: 72–75.
Millan, T. (1981) 'On the existence of optimal competitive equilibria in the overlapping generations model', Ph.D. Thesis, University of Minnesota.
Mitra, T. (1988) 'Public debt in the overlapping generations model', Manuscript, Department of Economics, Cornell University.
Muench, T.J. (1977) 'Optimality, the interaction of spot and futures markets, and the non-neutrality of money in the Lucas model', *Journal of Economic Theory*, 15: 325–344.
Muller, W.J. and M. Woodford, 'Stationary overlapping generations economies with production and infinite lived consumers: I. Existence of equilibrium', *Journal of Economic Theory*.
Muller, W.J. and M. Woodford, 'Stationary overlapping generations economies with production and infinite-lived consumers: II. Determinacy of equilibrium', *Journal of Economic Theory*.
Muller, W.J. and M. Woodford (1988) 'Determinacy of equilibrium in stationary economies with both finite and infinite-lived consumers', *Journal of Economic Theory*, 46: 255–290.
Ni, X. (1989) 'Asset pricing in general overlapping generations economies', Manuscript.
Nielsen, M.T. (1988a) 'The Pareto domination of irrational expectations over rational expectations', *Journal of Economic Theory*, 46: 322–334.
Nielsen, M.T. (1988b) 'Expectations and learning in an overlapping generations economy', Ph.D. Dissertation, University of Copenhagen.
*Nikaido, H. (1956) 'On the classical multilateral exchange problem', *Metroeconomica*, 8: 135–145.
Okuno, M. and I. Zilcha (1978) 'Existence of competitive equilibrium in a non-stationary consumption-loan model', Manuscript, Department of Economics, Tel Aviv University.
Okuno, M. and I. Zilcha (1980) 'On the efficiency of a competitive equilibrium in infinite horizon monetary economics', *Review of Economic Studies*, 42: 797–807.
Okuno, M. and I. Zilcha (1981) 'A proof of existence of competitive equilibrium in an overlapping-generation exchange economy with money', *International Economic Review*, 239–252.
Okuno, M. and I. Zilcha (1982) 'Approximately monetary equilibrium in overlapping generation economy', *European Economic Review*, 19: 213–228.
Okuno, M. and I. Zilcha (1983a) 'Optimal stationary overlapping generations model', *Economics Letters*, 11: 331–335.
Okuno, M. and I. Zilcha (1983b) 'Optimal steady-state in stationary consumption-loan type models', *Journal of Economic Theory*, 31: 355–363.
Orosel, G.O. (1989) 'Endogenous stock price variability in an overlapping generations model of short run speculation', Working Paper No. 8808, Department of Economics, University of Vienna.
Ostroy, J. (1984) 'The existence of Walrasian equilibrium in large-square economies', *Journal of Mathematical Economics*, 13: 143–163.
Pasinetti, L. (1981) 'The natural rate of interest', in: *Structural change and economic growth*. Cambridge University Press, Chap. VIII, pp. 156–175.
Patinkin, D. (1983) 'Paul Samuelson's contribution to monetary economics', Research Paper No. 150, The Maurice Falk Institute for Economic Research.
Peck, J. (1988) 'On the existence of sunspot equilibria in an overlapping generations model', *Journal of Economic Theory*, 44: 19–42.
Peck, J. and K. Shell (1985) 'Market uncertainty: sunspot equilibrium in imperfectly competitive economies', Working Paper No. 85-21, CARESS, University of Pennsylvania.
Peck, J. and K. Shell (1989) 'Market uncertainty: correlated equilibrium and sunspot equilibrium in imperfectly competitive economies', Working Paper No. 88-22, CAE, Cornell University.
Peled, D. (1982) 'Informational diversity over time and the optimality of monetary equilibria', *Journal of Economic Theory*, 28: 255–274.
Peled, D. (1984) 'Stationary Pareto optimality of stochastic asset equilibria with overlapping generations', *Journal of Economic Theory*, 34: 396–403.
*Peleg, B. and M.E. Yaari (1970) 'Markets with countably many commodities', *International Economic Review*, 11: 369–377.

*Phelps, E.S. (1965) 'Second essay on the golden rule', *American Economic Review*, 55: 793–814.
Polemarchakis, H.M. (1979) 'Incomplete markets, price regulation and welfare', *American Economic Review*, 69: 662–669.
Polemarchakis, H.M. and L. Weiss (1977) 'On the desirability of "Totally Random" Monetary Policy', *Journal of Economic Theory*, 15: 345–350.
Polemarchakis, H.M. and L. Weiss (1978) 'Fixed wages, layoffs, unemployment compensation and welfare', *American Economic Review*, 68: 909–917.
Prescott, E. (1987) 'Dynamic coalitions: growth and the firm', in: E. Prescott and N. Wallace, eds., *Contractual Arrangements for Intertemporal Trade*, University of Minnesota Press, pp. 146–160.
Prescott, E. and J. Boyd (1987) 'Dynamic coalitions: engines of growth', *American Economic Review, Papers and Proceedings*, pp. 63–67.
Pringle, M. and L. Tesfatsion (1987) 'Pareto efficiency of stationary competitive equilibria in overlapping generation economies', MRG Working Paper No. 8707, Department of Economics, University of Southern California.
Pringle, M. and L. Tesfatsion (1991) 'Intermediation, bubbles and Pareto efficiency in economics with production', Manuscript, Department of Economics, Iowa State University, Ames.
Reichlin, P. (1986) 'Equilibrium cycles in an overlapping-generations economy with production', *Journal of Economic Theory*, 40: 89–102.
Reichlin, P. (1989) 'Endogenous cycles with long lived agents', Manuscript.
Romer, P.M. (1988) 'Capital accumulation in the theory of long-run growth', in: R. Barro, ed., *Modern macroeconomics*. Harvard University Press.
Sakai, Y. (1988) 'Conditional Pareto optimality of stationary equilibrium in a stochastic overlapping generations model', *Journal of Economic Theory*, 44: 209–213.
Salyer, K.D. (1988) 'Comparative dynamics and risk premia in an overlapping generations model: a note', *The Review of Economic Studies*, 55: 667–668.
Samuelson, P.A. (1958) 'An exact consumption-loan model of interest with or without the social contrivance of money', *Journal of Political Economy*, 66: 467–482.
Samuelson, P.A. (1959) 'A "Reply" to A.P. Learner, consumption-loan interest and money', *Journal of Political Economy*, 77: 519–522.
Samuelson, P.A. (1967) 'A turnpike refutation of the golden rule in a welfare maximizing many-year plan', in: K. Shell, ed., *Essays in the theory of optimal economic growth*. MIT Press, pp. 269–280.
Samuelson, P.A. (1975) 'The optimum growth rate for population', *International Economic Review*, 16: 531–538.
Samuelson, P.A. (1976) 'The optimum growth rate for population: Agreement and evaluations', *International Economic Review*, 17: 516–525.
Samuelson, P.A. (1979) 'Land and the rate of interest', in: H.I. Greenfield, A.M. Levenson, W. Hamovitch and E. Rotwein, eds., *Theory for economic efficiency: essays in honor of Abba P. Lerner*. MIT Press.
Santos, M.S. (1989) 'Existence of equilibria for monetary economies', Manuscript, Universidad Autonoma de Barcelona.
Santos, M.S and J.L. Bona, 'On the structure of the equilibrium price set of overlapping-generations economies', *Journal of Economic Theory*.
Sargent, T.J. (1984) 'Consumption, loans and currency', Lecture Notes, University of Minnesota.
Scheinkman, J.A. (1979) 'Discussion of overlapping generations model of fiat money', in: J.H. Karekan and N. Wallace, eds., *Models of Monetary Economics*. Federal Reserve Bank of Minneapolis, pp. 91–96.
Scheinkman, J.A. (1980) 'Notes on asset trading in an overlapping generations model', Manuscript, Department of Economics, University of Chicago.
Scheinkman, J.A. (1984) 'General equilibrium models of economic fluctuations: a survey of theory', Manuscript, Department of Economics, University of Chicago.
Schmachtenberg, R. (1987) 'Fiat money in general equilibrium: open-endedness and overlapping-generations structure', Working Paper No. 87-17, CARESS, University of Pennsylvania. Revised 'The structure of monetary economics', Discussion Paper No. 390-89, Department of Economics, University of Mannheim.
Schmachtenberg, R. (1988a) 'Stochastic overlapping generations model with incomplete financial

markets I: Existence of equilibria', Discussion Paper No. 363-88, Department of Economics, University of Mannheim.
Schmachtenberg, R. (1988b) 'Stochastic overlapping generations model with incomplete financial markets II: The impact of outside money,' Manuscript, Department of Economics, University of Mannheim.
Schmachtenberg, R. (1988c) 'Non-existence of stationary equilibria in stochastic overlapping generations models – a note', Discussion Paper No. 365-88, Department of Economics, University of Mannheim.
Shell, K. (1971) 'Notes on the economics of infinity', *Journal of Political Economy*, 79: 1002–1011.
Shell, K. (1977) 'Monnaie et allocation intertemporelle', Manuscript, Séminaire d'Econométrie Roy-Malinvaud.
Shell, K. (1987) 'Sunspot equilibrium', in: J. Eatwell, M. Milgate and P. Newman, eds., *The new Palgrave dictionary of economics*, Macmillan, Vol. 4, pp. 549–551.
Shitovitz, B. (1988) 'On stationary 3-period overlapping generations models', *Journal of Economic Theory*, 46: 402–408.
Shubik, M. (1981) 'Society, land, love or money (a strategic model of how to glue the generations together)', *Journal of Economic Behavior and Organization*, 2: 359–385.
Smith, B.D. (1989) 'Legal restrictions, "sunspots", and cycles', *Journal of Economic Theory*, 47: 369–392.
Spear, S.E. (1984) 'Sufficient conditions for the existence of sunspot equilibria', *Journal of Economic Theory*, 34: 360–370.
Spear, S.E. (1985) 'Rational expectations in the overlapping generations model', *Journal of Economic Theory*, 35: 251–275.
Spear, S.E. (1986a) "Existence of cycle sunspot equilibria in a general overlapping generations model', Manuscript, GSIA, Carnegie-Mellon University.
Spear, S.E. (1986b) 'Rationality chaos and growth', Manuscript, GSIA, Carnegie-Mellon University.
Spear, S.E. (1988a) 'Existence and local uniqueness of functional rational expectations equilibria in dynamic economic models', *Journal of Economic Theory*, 44: 124–155.
Spear, S.E. (1988b) 'Growth, externalities, and sunspots', Manuscript, GSIA, Carnegie-Mellon University.
Spear, S.E. (1989) 'Are sunspots necessary?', *Journal of Political Economy*, 97: 965–973.
Spear, S.E. and R. Green (1987) 'Equilibria with incomplete markets and overlapping generations', Manuscript, GSIA, Carnegie-Mellon University.
Spear, S.E. and S. Srivastava (1983) 'Equilibrium dynamics in a stochastic overlapping generations model', Manuscript, GSIA, Carnegie-Mellon University.
Spear, S.E. and S. Srivastava (1986) 'Markov rational expectations equilibria in an overlapping generations model', *Journal of Economic Theory*, 38: 35–62.
Spear, S.E., S. Srivastava and M. Woodford (1986) 'On the structure of equilibrium in stochastic overlapping generations model', Manuscript, GSIA, Carnegie-Mellon University.
Spear, S.E., S. Srivastava and M. Woodford (1990) 'Indeterminacy of equilibria in stochastic overlapping generations models', *Journal of Economic Theory*, 50: 265–284.
*Sraffa, P. (1960) *Production of commodities by means of commodities*. Cambridge University Press.
Srivastava, S. and S.F. Richard (1988) 'Equilibrium in economics with infinitely many consumers and infinitely many commodities', *Journal of Mathematical Economics*, 17: 19–21.
Starret, D. (1972) 'On the golden rules, the "biological theory of interest", and competitive inefficiency', *Journal of Political Economy*, 80: 276–291.
Stutzer, M. (1980) 'Chaotic dynamics and bifurcations in a macro model', *Journal of Economic Dynamics and Control*, 2: 353–376.
Tilman, G. (1983) 'Stability in a simple pure consumption loan model', *Journal of Economic Theory*, 30: 315–329.
Tillman, G. (1985) 'Existence and stability of rational expectation – equilibria in a simple overlapping generation model', *Journal of Economic Theory*, 36: 333–351.
Tirole, J. (1985) 'Asset bubbles and overlapping generations', *Econometrica*, 53: 1499–1528.
Tirole, J. (1990) 'Intertemporal efficiency, intergenerational transfers, and asset pricing: an introduction', in: *Essays in honor of Edmond Malinvaud*. MIT Press.

Waldo, D.G. (1985) 'Open market operations in an overlapping generations model', *Journal of Political Economy*, 93: 1242–1257.

Wallace, N. (1980) 'The overlapping generations model of fiat money', in: J.H. Kareken and N. Wallace, eds., *Models of monetary economics*. Federal Reserve Bank of Minneapolis, pp. 49–82.

Weil, P. (1987) 'Confidence and the real value of money in an overlapping generations economy', *Quarterly Journal of Economics*, 102: 1–22.

Weil, P. (1989) 'Overlapping families of infinitely-lived agents', *Journal of Public Economics*, 38: 183–198.

Weiss, L. (1984) 'The effects of the money supply on economic welfare in the steady state', *Econometrica*, 48: 565–576.

Wilson, C. (1981) 'Equilibrium in dynamic models with an infinity of agents', *Journal of Economic Theory*, 24: 95–111.

Woodford, M. (1983) 'Infinite-lived consumers in a stationary overlapping generations exchange economy', Manuscript, Department of Economics, MIT.

Woodford, M. (1984) 'Indeterminacy of equilibrium in the overlapping generations model: a survey', Manuscript, Department of Economics, Columbia University.

Woodford, M. (1985) 'Self-fulfilling expectations, finance constraints, and aggregate fluctuations', Manuscript, Department of Economics, Columbia University.

Woodford, M. (1986a) 'Stationary sunspot equilibria', Manuscript, Department of Economics, University of Chicago.

Woodford, M. (1986b) 'Stationary sunspot equilibria: the case of small fluctuations around a deterministic steady state', Manuscript, Department of Economics, Columbia University.

Woodford, M. (1986c) 'Stationary sunspot equilibria in a finance constrained economy', *Journal of Economic Theory*, 40, 128–137.

Woodford, M. (1986d) 'Asset bubbles and fiat money', Manuscript, Department of Economics, Columbia University.

Woodford, M. (1986e) 'Comments on Grandmont's paper', in: H. Sonnenschein, ed., *Models of economic dynamics*. Springer-Verlag.

Woodford, M. (1987) 'Three questions about sunspot equilibria as an explanation of economic fluctuations', *American Economic Review*, 77: 93–98.

Woodford, M. (1988a) 'Imperfect financial intermediation and complex dynamics', in W.A. Barnett, J. Geweke and K. Shell, eds., *Economic complexity*, Cambridge University Press.

Woodford, M. (1988b) 'Expectations, finance and aggregate instability', in M. Kohn and S.C. Tsiang, eds., *Finance constraints, expectations and macroeconomics*, Oxford University Press.

Woodford, M. (1990) 'Learning to believe in sunspots', *Econometrica*, 58: 277–308.

Woodford, M. 'Expectations, finance constraints, and the instability of aggregate expenditure', *Journal of Economic Theory*.

Wright, R. (1987) 'Market structure and competitive equilbrium in dynamic economic models', *Journal of Economic Theory*, 41: 189–201.

Yaari, M.E. (1985) 'Uncertain lifetime life insurance and the theory of the consumer', *Review of Economic Studies*, 32: 137–150.

Yamamoto, K. (1984) 'Asset trading in overlapping generations models: efficiency of competitive equilibrium', Discussion Paper No. 546, Department of Economics, SUNY, Buffalo.

Ye, M.-H., S.C. Smith and M.A. Conte (1989) 'Optimal internal investment in a labor-managed firm with heterogeneous membership', Manuscript, Department of Economics, The George Washington University.

Younès, Y. (1987) 'Equivalence between the core and the set of competitive equilibria in overlapping generations and incomplete market models with many agents', Manuscript, Department of Economics, University of Pennsylvania.

Zame, W.R. (1986) 'Markets with a continuum of traders and infinitely many commodities', Manuscript, Institute for Mathematics and its Applications, SUNY, Buffalo.

Zilcha, I. (1981) 'Consumption-loan type models: the Samuelson case', *Economic Letters*, 7: 201–205.

Zilcha, I. (1987) 'Dynamic efficiency in overlapping generations models with stochastic production', Working Paper No. 23-87, The Forder Institute for Economic Research, Tel Aviv University.