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## AN INTRODUCTION TO GENERAL EQUILIBRIUM WITH INCOMPLETE ASSET MARKETS\*

John GEANAKOPLoS

*Cowles Foundation, Yale University, New Haven, CT 06520-2125, USA*

I survey the major results in the theory of general equilibrium with incomplete asset markets. I also introduce the papers in this volume, and offer a few suggestions for further work.

### 1. Introduction

The theory of general equilibrium with incomplete asset markets (GEI) studies the pricing of securities and commodities, and the interactions of perfectly competitive asset markets and commodity markets in determining consumption and investment. Since financial economics is fundamentally concerned with the pricing of securities, and since macroeconomics is fundamentally concerned with the real effects of monetary assets, the GEI model provides a framework for a microeconomic analysis that touches a variety of fields.

The theory of general equilibrium with incomplete asset markets is much more general than the Arrow–Debreu model, but it maintains the same methodological approach: agents optimize, their expectations are rational (that is, they have perfect conditional foresight, though they do not know the state of nature in advance), markets clear, and all market transactions are anonymously arranged under conditions of perfect competition. Yet, as we shall see, many of the lessons of the Arrow–Debreu model are apparently reversed.

Many new phenomena that cannot be described in the Arrow–Debreu model come to light in the incomplete asset markets model. These include the distinction between real and financial assets, the limitations (for understanding local uniqueness and efficiency) of representative agent and single commodity models, the non-neutrality (and occasional neutrality) of money, the importance of diversification, the dependence of asset prices on covar-

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iances and not variances, the ‘random walk’ or martingale property of asset prices, the relevance (or possible irrelevance) of corporate financial policy, the rationale for lenient bankruptcy and default penalties, the speculative, precautionary, and transactions demands for money, the ambiguity of firm objectives, and a positive role for government redirection of private investment and risk taking decisions. If for no other reason, the GEI model is important for freeing one’s thinking from the automatic association of competitive equilibrium with Pareto optimality that has become standard in models derived from Arrow–Debreu. This is not without significance, for example, for the occasional claims of some new classical macroeconomists and finance theorists that the methodological approach just mentioned in itself implies market efficiency.

There is little doubt that permitting the incompleteness of asset markets is a step in the direction of realism. For a quarter of a century, scores of economists have complained about the absurdity of allowing all agents (including the unborn) to meet together at one moment in time, and to trade assets that allow for every conceivable contingency, for all future time. There are at least three reasons why important assets might not be marketed. First, on account of asymmetric information, there might be events for which two types of agents could have agreed to insure each other except that the event is unobservable to one type. Second, many of the most interested traders might not have access to the markets, either because they are not yet born, or not sufficiently sophisticated, or do not have the money to trade at the appropriate moment in time. (The lack of credit markets, due to the usual moral hazard reasons, indirectly causes the failure of many other asset markets.) Young workers especially might wish they had the opportunity to insure themselves against events affecting their marginal productivity, or the future price of homes, etc. Third, the transactions costs in establishing and specifying any particular asset market might not be covered by the profit an entrepreneur could earn by opening it.<sup>1</sup> The flexibility of allowing any asset structure permits the GEI model to incorporate as special cases many of the macroeconomic and financial models that seemed orthogonal to the microeconomic tradition as expressed in the Arrow–Debreu model. For example, the prototypical financial model, the capital asset pricing model, can easily be analyzed via the GEI approach. The overlapping generations model with incomplete markets has been used as a source of myriad macroeconomic examples which are gradually coming to be understood as part of a systematic theory. Questions concerning the role of financial assets and

<sup>1</sup>A fourth reason that asset markets are incomplete has to do with the bounded rationality of the agents. It is impossible to think of all the future contingencies it would be necessary to specify to have complete asset markets. In keeping with the Arrow–Debreu methodological premise of full rationality, however, the GEI model implicitly assumes that all contingencies are foreseen, even if they are not insured.

exchange rates between different currencies can be advantageously treated from the GEI point of view. So can the problem of price supports for farmers and investment subsidies for producers.

The simplest example of a GEI model may be said to explain the difference between single agent economies, as modeled by the Kuhn–Tucker theorem, and the multiagent economies of Arrow–Debreu. In the Kuhn–Tucker model, different ‘agents’ (or activities) are permitted to trade ‘utiles’, while in the Arrow–Debreu model, they are kept separate by a prohibition against such trades, that is, by an extra budget constraint for each agent. The immediate effect of this prohibition is to destroy the property of competitive equilibria in the Kuhn–Tucker model that they maximize total utility, or total output, or total surplus. Moreover, equilibria are no longer typically unique. But of course competitive equilibria in the Arrow–Debreu model turn out to satisfy a weaker efficiency criterion, namely Pareto optimality, which has proved to be more useful than total surplus and has served the defenders of laissez-faire capitalism just as well. And although uniqueness fails, local uniqueness typically holds. The GEI model studies the character of economic activity when there may be more than one missing market, and more than one budget constraint.

GEI analysis has made five fundamental contributions to economic theory that I will discuss in this introduction. First, it has provided a framework for understanding the pricing of assets, and the significance of the asset span when short sales are permitted. Some results which can be derived in this framework are (1) the martingale property of asset prices, (2) the Modigliani–Miller principle (MMP) that corporate financed policy is irrelevant when there is no bankruptcy, (3) the failure of MMP when there is bankruptcy and limited liability, and (4) the dependence of asset prices on covariances as opposed to variances. Second, it has thrown up mathematical problems so delicate that they have forced economists for the first time to abandon that old war horse – Brouwer’s fixed point theorem – and to invent (or borrow from mathematics) a new methodology for constructing existence proofs for general equilibrium that do not rely on convexity and which promise to have wide applications outside GEI. Third, it has demonstrated a significant difference between real and financial assets. Fourth, it has shown that default and bankruptcy can be understood as equilibrium, as opposed to disequilibrium phenomenon. Fifth, it has, to my mind, greatly increased the presumption against Pareto efficiency of the market process.

One important insight provided by the GEI model is the crucial distinction between financial assets and real assets. In the Arrow–Debreu model (or the equivalent Arrow complete securities model), there is not much difference; rational investors pay attention to the purchasing power both confer, and trade them accordingly. With incomplete markets, rationality of expectations does not fix the real purchasing power of money. GEI models with monetary

assets typically have many dimensions of indeterminacy, whereas with real assets local uniqueness is the rule. This indeterminacy is more pronounced when the assets are denominated in different currencies. It becomes impossible, for example, to speak of market determined exchange rates.

The battle over the viability of *laissez-faire* is almost as old as modern economics, and the GEI model adds what may turn out to be an influential reinterpretation of the meaning of efficiency. Equilibria in the GEI model not only are Pareto suboptimal, but even the assets that exist are typically used inefficiently, according to a new definition of constrained optimality. Any argument against active intervention must rely, it seems now, on the central planner not understanding the model, i.e. being ignorant not only of the state of nature, but also of the way in which agents' characteristics depend on the state. Negative optimality results have also been reached in the burgeoning literature on contracts and incentive compatibility. Indeed the two literatures are intimately related since asymmetric information and missing markets go hand in hand. A major difference, however, is that in the GEI models the market interactions are perfectly competitive, whereas in the contracts literature (including for example the famous principal-agent problem) there is usually not a continuum of identical agents on each side of every market transaction. The presence of competition has always been the hallmark of efficient markets, and therefore the failure of efficiency in the GEI model is all the more surprising.

Incidentally, though many markets cannot appropriately be taken to be competitive (and thus the GEI model is not applicable), when it is appropriate, the assumption of pure competition affords an ease of generality and elegance which is difficult to find in non-competitive environments.

Default is a quintessentially general equilibrium phenomenon. In a world in which promises can exceed physical endowments, each default can begin a chain reaction. A creditor in one market where payment does not occur is deprived of the means of delivery in another market where he is the debtor, thereby causing a further default in some other market, etc. The indirect effects of default might be as important as the direct effects, but they are missed in partial equilibrium models. In GEI both the direct and indirect effects of default can be captured, and the welfare implications of default studied.

Despite its broad generality, the GEI model has been shown to rest on the same secure axiomatic foundation as the Arrow-Debreu model. To prove this has required the use of new mathematical tools in economics, or at least their application in novel ways. New proofs and simplifications of recently invented proofs are being discovered at a rapid rate. Reflecting the central role in GEI of the asset span, the most important mathematical object in GEI analysis is the Grassmanian manifold, which is the collection of all  $J$ -dimensional subspaces of  $\mathbb{R}^S$ . It is well known that the Grassmanian is not

convex, and that has necessitated the invention of more powerful fixed point theorems, which in turn has allowed for consideration of richer economic models. As usual in the progress of a new branch of economic theory, the development of the conceptual framework and the mathematical analysis go hand in hand, each illuminating and directing the other.

Perhaps the most unexplored part of the GEI model is a theory explaining which markets are open and which are closed. This may be viewed either as a challenge for a research programme (which is now well under way, as we shall see in section 7), or as a critique of the current theory. As a criticism it has force because most of the propositions derived for GEI models hold 'generically', i.e. for randomly chosen endowments or asset structures. The idea is that perhaps the most natural asset structures in any realistic context lie in the set of measure zero not covered by the general propositions. One could examine this claim in particular cases by studying which asset markets in fact exist. On a more general level, one could try to extend the generic results to all economies, paying close attention to the way the conclusions change, without confronting the question of which are the available assets in any particular context. To a substantial extent this extension has proved possible, and the conclusions often are surprisingly robust. For example, the real indeterminacy of equilibria with financial assets generically has dimension  $S-1$ , one less than the number of states, independent of the number of assets  $B$ , so long as  $B$  is less than  $S$ . But it can be shown that for *any* incomplete asset structure, the dimension of real indeterminacy is typically at least  $S-B$ .

Pioneering articles about asset economies were written in pure theory by Arrow (1953) and Diamond (1967), and in finance by Modigliani and Miller (1958), Sharpe (1964), Lintner (1965), Mossin (1966), and Black and Scholes (1973). The archetype GEI model, with multiple commodities and incomplete asset markets, was first formulated by Radner (1972). He drew attention to two of the fundamental problems, namely the questions of existence of equilibrium when short sales are permitted, and the appropriate goals of the firm when 'spanning' fails and profit maximization is not unambiguously defined, that have been at the center of the theory ever since. In response to the second question, Drèze (1974) proposed a definition of firm optimization that he argued is consistent with the interests of final shareholders. Later Grossman and Hart (1979) proposed a different criterion for firm decision making that they said would be in the interests of the original shareholders, given certain expectations. Radner's existence puzzle seemed to be a more intractable problem, especially after Hart's counterexample. Hart (1975) focused attention on the discontinuity in the span of the assets when the prices vary. He used this phenomenon to construct examples of economies where there is no competitive equilibrium, and other economies in which one equilibrium Pareto dominates another because at the second equilibrium,

the asset span is smaller. He suggested that in addition to the problem of non-existence of equilibrium, a radical change in the definition of efficiency was necessary to apply to the GEI models.

After this flurry of interest in the 1970s, the pure theory of GEI stood still for half a decade. The existence puzzle and the lack of a solid basis for evaluating the efficiency of GEI equilibrium stymied further abstract theorizing. But in spite of these theoretical difficulties, the use of special GEI models accelerated in finance and macroeconomics. In finance there was an explosion of option pricing studies and work on arbitrage pricing. [See, for example, Ross (1976a, b) Cox, Ross and Rubinstein (1979), Cox, Ingersoll and Ross (1985), and Huberman (1982).] And in macroeconomics the incomplete markets approach was developed by Lucas (1978, 1980, 1982) by Prescott and Mehra (1980), by Svensson (1985), and by Lucas and Stokey (1987), among others. In finance the existence problem was largely ignored, while in macroeconomics attention was confined to representative agent models for which standard techniques guarantee the existence (and Pareto optimality) of equilibrium.

Suddenly in the middle 1980s the pure theory of GEI fell into place. In two provocative and influential papers, Cass (1984, 1985) showed that existence of equilibrium could be guaranteed if all the assets promise delivery in fiat money, and he gave an example showing that with such financial assets there could be a multiplicity of equilibrium. Almost simultaneously Werner (1985) also gave a proof of existence of equilibrium with financial assets, and Geanakoplos and Polemarchakis (1986) showed the same for economies with real assets that promise delivery in the same consumption good. Duffie (1987) then extended the existence results for purely financial assets to multiperiod GEI economies. Following a hint in Diamond (1967) about constrained efficiency, and successive refinements of the definition and examples by Diamond (1980), Loong and Zeckhauser (1982), Newbery and Stiglitz (1982), Stiglitz (1982), and Greenwald and Stiglitz (1984), a reasonable definition of GEI (constrained) efficiency became available in 1986. Geanakoplos and Polemarchakis proved that GEI equilibria are generically constrained inefficient.

Hart's counterexample to existence of equilibrium was shown to be degenerate by McManus (1984), Repullo (1986), and Magill and Shafer (this issue). Then in a major breakthrough Duffie and Shafer (1985) showed that for generic endowments and asset structures, GEI equilibrium exists.

With a handle on existence and (in)efficiency for a broad class of GEI economies, work in the pure theory of GEI could turn to completely new questions. Following up on the example of Cass, Geanakoplos and Mas-Colell (1989) and Balasko and Cass (1989) characterized the multiplicity of equilibria that arises with monetary assets. Dubey, Geanakoplos and Shubik (1988) began the investigation of general equilibrium default in the GEI model.

This volume represents a collection of some of the most interesting new results to emerge in GEI analysis over the past two years, that is since the breakthroughs of the middle 1980s. In the first three papers, Lasry, Husseini and Magill (LHM), Geanakoplos and Shafer (GS), and Hirsch, Magill and Mas-Colell (HMM) present new approaches to the equilibrium existence problem that do not rely on convexity if the domain of endogenous variables. In paper no. 4, Polemarchakis and Ku (PK) raise another puzzle to GEI equilibrium existence which has yet to be resolved (see section 7).

In paper no. 5, by Geanakoplos, Magill, Quinzii and Drèze (GMQD), the generic inefficiency of GEI equilibrium is extended to production. In paper no. 6, Geanakoplos and Polemarchakis (GP) characterize the kind of knowledge a central planner would need to have in order to effect Pareto improving changes. Paper no. 7, by Magill and Shafer (MS), gives both an existence and an efficiency result. If there are more assets than states, and if the assets are non-degenerate, then generically in the endowments equilibria exist and they are all fully Pareto optimal. The Magill–Shafer paper was the first one written for this volume.

In paper no. 8, Balasko, Cass and Siconolfi (BCS) show that with monetary assets, indeterminacy can arise even if some agents do have access to a complete set of asset markets, provided there are enough other agents who do not. In paper no. 9 Werner continues his work on GEI with financial assets by considering a multiperiod economy in which assets may be retraded. He explains the difficulty in giving a general formula for the dimension of indeterminacy, but succeeds in finding one in two polar cases.

In section 2 I describe the GEI model, and in section 3 I describe the GEI theory of asset pricing. In section 4 I discuss the equilibrium existence problem, which is the context for papers nos. 1–4; in section 5 I discuss efficiency and papers nos. 5–7, and in section 6 I describe the GEI theory of financial assets and papers nos. 8 and 9. In section 7 I offer some suggestions for extending GEI analysis to capture more fully some aspects of money, production, and default.

## **2. The model**

In the Arrow–Debreu model of general equilibrium, all commodities are traded at once, no matter when they are consumed, or under what state of nature. All consumers face only one budget constraint. The distinguishing feature of the GEI model is that consumers face a multiplicity of budget constraints, at different times and under different states of nature. To transfer wealth between budget constraints, consumers must hold assets.

Consider a world in which there are two time periods, and  $S + 1$  states of nature,  $S$  of which are resolved between period zero and period one. Utility

for any agent  $h$  depends on consumption of  $L$  commodities at the date-events,  $u^h: \mathbb{R}_+^{L(S+1)} \rightarrow \mathbb{R}$ . Agents have endowments  $e^h \in \mathbb{R}_+^{L(S+1)}$ . The data for the economy is exactly the same as for an Arrow–Debreu economy.<sup>2</sup> The difference lies in how trade is conducted.

An asset is represented by a vector  $a \in \mathbb{R}^{LS}$  denoting the promised payoff of each commodity in each state in period 1. The collection of all  $J$  assets is given by the  $SL \times J$  matrix  $A$ . Agents transfer wealth between periods, and between states of nature, only by holding assets. The GEI theory has a different character depending on the form of the assets. If the assets are endogenous, for example, if they represent the production choices of firms, then it is convenient to call them stocks. Agents will then have endowments  $\bar{\theta}_j^h$  of the stocks and  $\sum_h \bar{\theta}_j^h = 1$ , for each  $j$ . If the assets are specified exogenously, representing a limitation on the kinds of contracts that can be written, then they are called securities. The initial endowments of securities is taken to be zero. For most of this introduction we take  $\bar{\theta}^h = 0$  and deal only with securities. If a security delivers only money (a good which directly affects no agent's utility) then it is called a financial security. If it never pays out any money, it is called a real security. If in every state  $s$  there is a commodity bundle such that every asset delivers a scalar multiple of that bundle, then the assets are called numeraire assets. If there is only one consumption good in each state, then the assets are always numeraire assets.

Given commodity prices  $p = (p_0, p_1, \dots, p_S) \in \mathbb{R}_+^{L(S+1)}$  and asset prices  $q = (q_1, \dots, q_J)$ , the GEI budget set of agent  $h$  is written

$$B^h(p, q) = \left\{ (x, \theta) \in \mathbb{R}_+^{L(S+1)} \times \mathbb{R}^J \mid p_0 \cdot (x_0 - e_0^h) + q \cdot \theta = 0 \right.$$

$$\left. \text{and } \forall s = 1, \dots, S, p_s(x_s - e_s^h) = p_s \cdot \sum_j A_s^j \theta_j \right\}.$$

We call  $(p, q, (x^h, \theta^h)_{h=1}^H)$  a GEI for the GEI economy  $((u^h, e^h)_{h=1}^H, A)$  if

$$(a) \quad \sum_{h=1}^H (x^h - e^h) = 0,$$

<sup>2</sup>All utilities are assumed to be strictly monotonic in each good, quasi-concave, and continuous (the usual continuity conditions), or else smooth, strictly monotonic, with marginal utility approaching  $\infty$  as consumption of any good approaches 0, and quasi-concave, with negative Gaussian curvature (the usual smoothness conditions).



$$(b) \quad \sum_{h=1}^H \theta^h = 0,$$

$$(c) \quad (x^h, \theta^h) \in B^h(p, q), \quad h=1, \dots, H,$$

$$(d) \quad (x, \theta) \in B^h(p, q) \Rightarrow u^h(x) \leq u^h(x^h), \quad h=1, \dots, H.$$

In equilibrium, agents face the same prices, choose optimally in their budget sets, and demand equals supply. Let us pause to make a few observations about the model. Notice that there is a separate budget constraint for each date-event, and that expenditure can exceed receipts from endowment income in one date-event only if the difference can be made up by the revenue from asset holdings. As a result there will be not 1, but  $S+1$  Walras Laws. Moreover there are  $S+1$  independent homogeneity properties for demand in prices: doubling the vector  $p_s$  for any  $s$  does not affect market clearing. Notice also that for simplicity we have assumed that each agent has access to the same asset markets.

Take the case where there is only one asset, a financial security that delivers one dollar in both states 1 and 2. Agents can save by purchasing the asset, giving up wealth at date zero, in exchange for money at date one. Similarly, they can borrow by selling the asset at date zero, collecting money  $-q\theta$ , and later paying back  $-\theta$  dollars in each state. When  $\theta^h + \bar{\theta}^h < 0$ , we say that agent  $h$  has gone short in the asset, selling something he does not originally own. Trade in securities which are not in positive supply cannot take place unless some agents are allowed to go short. The definition of GEI explicitly allows for all possible short sales by letting agents choose  $\theta \in \mathbb{R}^J$ , instead of restricting their choices to  $\mathbb{R}_+^J$ . Short sales prohibitions for some assets could easily be incorporated into the model. The main effect would be that many of the asset pricing formulas that we discuss in the next section would become complicated.

If in our two-state example there is not another asset, then consumers can save or borrow, but they cannot insure themselves against uncertainty between the two states. In the absence of the usual insurance markets, we shall see that many of the standard questions of microeconomic theory have different answers. Does equilibrium exist? If so, how many equilibria are there? Are equilibrium allocations Pareto optimal? If not, then is it at least true that any competitive equilibrium provides an efficient level of savings, given that there are no insurance markets? If there is a single firm, what objectives are optimal from the social point of view, and from its owner's point of view?

We have seen that there are a number of reasons why the asset market at

date-event 0 might not be complete. Chief among these is asymmetric information. A security that promises 1 dollar in state 1 and nothing in state 2 will not be traded unless both buyers and sellers (or one side and an enforcer) can distinguish between the states.<sup>3</sup> Similarly an agent who alone can observe his endowment  $e^h$  may not be able to sell the asset  $\tilde{e}^h = (e_1^h, \dots, e_S^h)$  if some  $\tilde{e}_s^h$  are much bigger than others. In the GEI model presented above the asset structure is taken to be exogenous, perhaps reflecting the underlying causes of incompleteness such as asymmetric information and transactions costs. (See section 7 for a brief discussion of how the asset structure might be made endogenous.) From the point of view of asymmetric information, the assets which pay exactly the same thing in every state take on a particular significance. They are often called riskless assets, although when there are multiple consumption goods in each state, relative prices may differ across the states, and the purchasing power commanded by ownership of the asset may depend on the state.

In GEI equilibrium all agents optimize with respect to budget sets defined by the same prices. This has several noteworthy implications. First, it implies that all agents have perfect (conditional) expectations, in keeping with the rational expectations hypothesis. [They may or may not also be assumed to attach the same probability to each state. A utility of the form  $u^h(x) = \sum_{s=0}^S \pi_s^h v_s^h(x_s)$  is consistent with the GEI model. The  $\pi^h$  may or may not be identical, although in rational expectations models the presumption is often that they are identical.] The hypothesis that agents are correct in their conditional forecasts of  $p_s$  is in keeping with a long tradition in economic theory. The same kind of hypothesis is made in Nash equilibrium for games with uncertainty, and private information. If  $S$  is a total description of the possible states of the world, then the price uncertainty should already be reflected in  $S$ , through the different probabilities that agents assign to  $S$ . Of course there is the alternative view, held in temporary equilibrium theory (TGE), which puts almost no restrictions on the compatibility of agents' forecasts (except for a mild restriction on the non-disjointness of the supports of the expectations; observe that in GEI, the supports are identical). There is a modeling trade-off here. The TGE approach seems to strain less the credibility of the implicit powers of calculation of the agents. This power is especially significant in the GEI model in that at time 0 there is not even an imaginary auctioneer who is supposed to be calling out the prices of commodities for states of nature at time 1. On the other hand, in the TGE model the price expectations are fixed exogenously (perhaps as an arbitrary continuous function of time 0 market conditions). To the extent that they are allowed to differ from the GEI expectations, they are wrong. In such

<sup>3</sup>Actually a more detailed analysis would be valuable on this point. It may suffice, for example, if an outside agency can distinguish the two states with some probability

circumstances it is difficult to make sense of efficiency questions, because every possible answer is confounded with the irrationality of agents' plans.

Second, all agents keep all their promises to deliver on assets – there is no bankruptcy or default (but see section 7). Third, as in the Arrow–Debreu model, there is implicitly a perfect credit market from the beginning of a date-event to its end. Agents can keep their promises to deliver goods even if their endowments do not include those particular goods, provided they can afford to trade for the promised goods on the spot markets. Fourth, the agents know the spot prices and the state of nature when they carry out their spot market trade. Since one of the reasons for the incompleteness of markets is the ignorance of some traders about the state of nature at the moment assets deliver, this last point calls for some comment. The main problem is that competitive equilibrium does not provide an explanation of the process of price formation. The single period condenses a sequence of exchanges over which information is revealed. It is probably worthwhile to consider an explicit model of price formation such as the Shapley–Shubik mechanism, in which agents act before they know the prices. Agents might then not be aware of all the spot prices before they decided on their bids and offers. The conclusions from such a model are likely to be similar to these from the GEI model.<sup>4</sup>

When there is only one good in each state, and all the asset pay offs are exogenously fixed, or else allowed to vary in constant proportions, the GEI model is particularly simple. There is no trade on the spot markets, so it is always possible to work backwards from the utility of consumption to an induced utility for the assets. One might then forget the original structure, and simply analyze the trade for assets as if they were goods in an Arrow–Debreu economy, as Diamond (1967) did.

A particularly famous example of such a simple one-good economy is the so-called capital asset pricing model (CAPM). One assumes

- (i) that each agent has quadratic utility for period 1 consumption:

$$u^h(x) = v^h(x_0) + \sum_s \pi_s^h (x_s - \frac{1}{2} \beta^h x_s^2) \quad \text{and}$$

- (ii) each  $\pi^h = \pi$  is independent of  $h$  and  
 (iii)  $A^j$ ,  $j = 1, \dots, J$ , is exogenously given and  $\tilde{e}^h \in \text{span}[A^1, \dots, A^J]$  for all  $h$ ,  
 and  
 (iv) there is a riskless asset,  $\tilde{1} \in (1, \dots, 1) \text{span}[A^1, \dots, A^J]$ .

<sup>4</sup>Incidentally, in such a model agents would also have correct forecasts of the prices conditional on knowing the state of nature. They just would not know either the price or the state at the moment they committed themselves to make purchases.

Under assumptions (i)–(iii), the induced utility for a portfolio of assets depends only on its mean and variance, calculated with respect to the common probabilities,  $\pi$ . The mean–variance analysis of portfolio choice is a long standing tradition. But even here there is a virtue in retaining the full fledged GEI structure. As we have just said, by thinking of the assets themselves as commodities, one reduces the analysis to an Arrow–Debreu economy except that there is no lower bound on consumption. It is well known that one can construct examples of such economies where equilibrium does not exist [see Nielson (1985)]. Note, however, that by retaining the original structure of endowments and consumption goods, one can deduce immediately from the lower bound on consumption sets that consumers cannot go arbitrarily short in any asset when there is only one consumption good in each state (assuming w.l.o.g. that the asset payoffs are linearly independent), thereby guaranteeing the existence of equilibrium. The reduced form of the CAPM model also ignores all the information about commodity allocations. As we shall see later, one can derive a remarkable efficiency property for CAPM equilibrium allocations in the original consumption economy. However, as section 5 makes clear, the CAPM model is non-generic in GEI.

When there are multiple goods in every state, it is impossible to establish an a priori lower bound for how short a consumer can go in any asset, even if all the assets deliver in the same numeraire commodity. In the definition of competitive equilibrium a consumer is allowed to cover his short sales from the sales revenues of his endowment, even if he does not physically have the goods on hand. Thus consumers can go arbitrarily short provided that their endowments are not co-linear with the asset payoffs, and the relative spot prices are sufficiently skewed. Yet in a series of papers that came out simultaneously, by Cass (1984), Werner (1985), and Duffie (1987), for financial assets, and by Geanakoplos and Polemarchakis (1986) for real numeraire assets, it was shown that if preferences satisfied a weak monotonicity requirement, then numeraire asset GEI economies always have competitive equilibria, even with no a priori restriction on short sales. This was the first general result suggesting that the GEI model could be put on the same rigorous footing as the Arrow–Debreu model, and it marked the starting point for the recent flood of research into general equilibrium with incomplete markets models.

### 3. Some basic properties of GEI and asset pricing

In this section we see how useful the idea of asset span is for understanding the pricing of assets, the connection between GEI and Arrow–Debreu, and the Modigliani–Miller principle.

We begin by introducing some useful notation. Let  $x =$

$(x_0, x_1, \dots, x_S) \in \mathbb{R}^{L(S+1)}$ , where each  $x_s \in \mathbb{R}^L$ . Then by  $\tilde{x}$  we mean  $(x_1, \dots, x_S) \in \mathbb{R}^{LS}$ . If  $\tilde{p}$  and  $\tilde{x}$  are elements of  $\mathbb{R}^{LS}$ , then by  $\tilde{p} \square \tilde{x}$  we mean the vector  $(\tilde{p}_1 \cdot \tilde{x}_1, \dots, \tilde{p}_S \cdot \tilde{x}_S) \in \mathbb{R}^S$ . If  $A$  is an  $LS \times J$  matrix with  $j$ th column denoted  $A^j \in \mathbb{R}^{LS}$ , then by  $\tilde{p} \square A$  we mean the  $S \times J$  matrix whose  $j$ th column is  $\tilde{p} \square A^j \in \mathbb{R}^S$ . Finally, if  $V$  is a  $K \times J$  matrix, then by  $\text{sp}[V]$  we mean the span of all the columns of  $V$ , i.e., the smallest linear subspace in  $\mathbb{R}^K$  containing every column of  $V$ .

With this notation in mind, we proceed to give an equivalent, more abstract definition of GEI, based on the span of the assets.

*Definition.* Let  $((u^h, e^h)_{h=1}^H, A)$  be a GEI economy. Let  $p \in \mathbb{R}_+^{L(S+1)}$ ,  $x^h \in \mathbb{R}_+^{L(S+1)}$ ,  $h=1, \dots, H$ . We say that  $(p, (x^h)_{h=1}^H)$  is a GEI\* iff

- (i)  $\sum_{h=1}^H (x^h - e^h) = 0$ ,
- (ii)  $x^h \in \hat{B}^h(p) \equiv \{x \in \mathbb{R}_+^{L(S+1)} \mid p \cdot (x - e^h) = 0 \text{ and } \tilde{p} \square (\tilde{x} - \tilde{e}^h) \in \text{sp}[\tilde{p} \square A]\}$ ,
- (iii)  $x \in \hat{B}^h(p) \Rightarrow u^h(x) \leq u^h(x^h)$ .

The following formalization of the equivalence of GEI and GEI\* is an easy consequence of Farkas' Theorem.

*Lemma 1.* Consider the GEI economy  $((u^h, e^h)_{h=1}^H, A)$  satisfying the usual continuity conditions. Let  $(p, q, (x^h, \theta^h)_{h=1}^H)$  be a GEI. Then there is a vector  $v \in \mathbb{R}_+^S$  such that  $q' = v[\tilde{p} \square A]$ . Moreover, there is  $\hat{p} \in \mathbb{R}_+^{L(S+1)}$  such that  $(\hat{p}, (x^h)_{h=1}^H)$  is a GEI\*. Conversely, if  $(p, (x^h)_{h=1}^H)$  is a GEI\*, then there is  $(\theta^h)_{h=1}^H$  and  $q' = \tilde{p}'A$  such that  $(p, q, (x^h, \theta^h)_{h=1}^H)$  is a GEI.

Lemma 1 has a number of important consequences for asset pricing. One corollary is that asset pricing can be thought of as a linear functional. If  $\gamma_1 A^1 + \gamma_2 A^2 = A^3$ , then by Lemma 1 we must have that  $\gamma_1 q_1 + \gamma_2 q_2 = q_3$ . This has been called in the financial literature 'the no-arbitrage condition'. Note that the possibility of unlimited short sales is essential for the result.

*Theorem 1.* Let  $((u^h, e^h)_{h=1}^H, A)$  satisfy the usual continuity conditions. Suppose that the dimension of  $\text{sp}[\tilde{p} \square A]$  is  $S$  for all  $\tilde{p} \in \mathbb{R}_+^{LS}$ . Then the set of GEI equilibrium commodity allocations  $(x^h)_{h=1}^H$  is the same as the set of Arrow–Debreu equilibrium allocations for the economy  $(u^h, e^h)_{h=1}^H$ .

The proof follows immediately from noting that the definition of GEI\* reduces to Arrow–Debreu equilibrium when  $\text{sp}[\tilde{p} \square A]$  has full dimension. Notice that rescaling any  $p_s$  does not change the dimension of  $\text{sp}[p \square A]$ . One consequence of Theorem 1 and the generic finiteness of Arrow–Debreu

equilibria [Debreu (1970)] is that if  $(u^h, e^h)_{h=1}^H$  satisfy the usual smoothness conditions, and if  $A$  is such that  $\text{sp}[\tilde{p} \square A]$  is  $\mathbb{R}^S$  for all  $\tilde{p} \in \mathbb{R}_{++}^{LS}$ , then for generic  $e^1$  there are only a finite number of GEI allocations  $(x^h)_{h=1}^H$  for  $((u^h, e^h)_{h=1}^H, A)$ .

Arrow (1953) proved Theorem 1 in the special case where  $A$  consists of  $S$  ‘elementary Arrow securities’, each of which promises delivery of some good in a different state. He pointed out that in the state contingent commodity model, all  $L(S+1)$  commodities must be traded at date 0 to obtain optimality, while in the GEI model it is only necessary to trade  $S+L$  goods at date 0, and then just  $L$  more when the state is realized. The case where  $\text{sp}[\tilde{p} \square A] = \mathbb{R}^S$  is called the situation of complete (asset) markets. When  $J < S$ , we necessarily have incomplete (asset) markets.

Another consequence of Theorem 1 is that when asset markets are already complete, introducing a new asset does not disturb the equilibrium. In fact the price of the new asset can be computed unambiguously from  $\tilde{p}$  and  $v$ .

If time lasts for many periods  $T$  instead of just for two, the analogue to Theorem 1 would require spanning between each date-event and its immediate successors. Let  $b$  be the maximal number of branches from any node in the date-event tree. The number of nodes might be as high as  $b^T - 1$ . The ingenious contribution of Black and Scholes (1973) was to observe that spanning could be achieved just by trading (and re trading) the same  $b$  long-lived assets. The re trading of assets in a multiperiod model greatly enlarges the dimension of their span. [See also Harrison and Kreps (1979).] In this introduction I confine myself mostly to the two-period GEI.

The significance of Lemma 1 can be more sharply expressed by observing that the assets always matter only through their collective span, and not their individual form.

*Theorem 2. Let  $((u^h, e^h)_{h=1}^H, A)$  and  $((u^h, e^h)_{h=1}^H, B)$  be two GEI economies satisfying the usual continuity conditions. Let  $(p, q, (x^h, \theta^h)_{h=1}^H)$  be a GEI for the first economy, and suppose that  $\text{sp}[\tilde{p} \square A] = \text{sp}[\tilde{p} \square B]$ . Then there is  $\hat{q}, (\hat{\theta}^h)_{h=1}^H$ , such that  $(p, \hat{q}, (x^h, \hat{\theta}^h)_{h=1}^H)$  is a GEI for the second economy.*

Again the proof follows immediately from Lemma 1. One implication of this theorem is the celebrated Modigliani–Miller proposition (MMP) in finance [see Modigliani and Miller (1958)]. Suppose that  $\tilde{z} \in \mathbb{R}_+^{LS}$  is a vector (to be thought of as the state contingent output of a firm). Suppose that  $A^1 + A^2 = \tilde{z} = B^1 + B^2$  are two ways of dividing the output of the firm, say between debt and equity. If

$$\text{sp} \{ \tilde{p} \square A^1, \tilde{p} \square A^2 \} = \text{sp} \{ \tilde{p} \square B^1, \tilde{p} \square B^2 \},$$

then the redistribution of firm profits between debt and equity will not disturb the original equilibrium.

MMP can be interpreted in two quite distinct ways. In the first, it says that if there are two different firms in the same economy, with the same random output  $\tilde{z}$ , that finance their operations differently between debt and equity, then the sum of the values of debt and equity must be the same in both cases. This follows immediately from the linearity of the prices, as in Lemma 1.

In the second interpretation, the question is whether, if some industry changed the mix of its financing between debt and equity, there would be a change in real outcomes. To take the central special case, let  $\tilde{p} \square A^1 = (k, \dots, k)$ , and let  $\tilde{p} \square B^1 = \gamma(k, \dots, k)$ ,  $\gamma > 1$ . We interpret  $\tilde{p} \square A^1$  as the payoff to the bondholders, when the firm issues  $k$  bonds and delivers all that it promises, and  $\tilde{p} \square B^1$  as the payoff when the firm delivers all that it promises on  $\gamma k$  bonds. Then the above spanning equality necessarily holds, and we get the irrelevance of firm financing.

There is a converse to Theorem 2 which I do not state formally. But if  $\text{sp}[\tilde{p} \square A] \neq \text{sp}[\tilde{p} \square B]$ , then ‘except for accidents’, the equilibrium allocation  $(x^h)_{h=1}^H$  for the economy with asset structure  $A$  will not be part of an equilibrium with asset structure  $B$ .

This has a very important consequence for the MMP when we consider limited liability. Suppose that

$$(\tilde{p} \square A^1)_s = \min[k, \tilde{p}_s \cdot \tilde{z}_s] \quad \text{for } s=1, \dots, S,$$

$$(\tilde{p} \square B^1)_s = \min[\gamma k, \tilde{p}_s \cdot \tilde{z}_s] \quad \text{for } s=1, \dots, S.$$

In this situation the bondholders are paid only out of the earnings  $\tilde{p}_s \cdot \tilde{z}_s$  of the firm. The payments to the shareholders,  $\tilde{p}_s \cdot \tilde{z}_s - \tilde{p}_s A_s^1$  and  $\tilde{p}_s \cdot \tilde{z}_s - \tilde{p}_s B_s^1$  are not allowed to be negative. If  $\# S > 2$ , and for some  $s$   $\gamma k > (\tilde{p} \square \tilde{z})_s$ , then almost surely  $\text{sp}\{\tilde{p} \square A^1, \tilde{p} \square A^2\} \neq \text{sp}\{\tilde{p} \square B^1, \tilde{p} \square B^2\}$ , and the firms’ financial policy will have a real effect. As a concrete example, consider

$$\tilde{p} \square \tilde{z} = \begin{bmatrix} 12 \\ 11 \\ 1 \end{bmatrix}, \quad \tilde{p} \square A = \begin{bmatrix} 8 & 4 \\ 8 & 3 \\ 1 & 0 \end{bmatrix}, \quad \tilde{p} \square B = \begin{bmatrix} 9 & 3 \\ 9 & 2 \\ 1 & 0 \end{bmatrix}.$$

Thus when there is limited liability and the possibility of bankruptcy, MMP is generally invalid, even ignoring the distortions caused by institutional factors like the court costs and settlement costs of bankruptcy. [See Stiglitz (1974), Hellwig (1981), Dubey and Geanakoplos (1989c).]

‘Efficient financial’ markets are often said to be characterized by price processes that follow random walks. Let us make the important observation

that the martingale property of asset prices (that  $q_j$  is the discounted expected payoff of the asset) holds for any GEI equilibrium, no matter how close or far the associated commodity allocation is from Pareto optimality.

*Theorem 3.* Let  $(p, q, (x^h, \theta^h)_{h=1}^H)$  be a GEI. Then there is a probability measure  $\mu$  on  $S$  and a discount rate of  $\delta > 0$  such that after discounting, each asset follows a martingale w.r.t.  $\mu$ , i.e., for each  $j = 1, \dots, J$ ,

$$q_j = \delta E_\mu [p_s \cdot A_s^j] \equiv \delta \sum_{s=1}^S \mu_s p_s \cdot A_s^j.$$

Once again the proof follows immediately from Lemma 1, taking  $\mu_{s'} = v_{s'}/\sum_{s=1}^S v_s$ , for  $s' = 1, \dots, S$ , and  $\delta = \sum_{s=1}^S v_s$ . [The idea behind Theorem 3 is due to Ross (1978) and Harrison and Kreps (1979).] In section 5 we shall see that typically GEI are Pareto suboptimal in a strong sense. Thus the phrase ‘efficient markets’ must not be misunderstood to mean efficient or Pareto optimal allocations. Note also that  $\mu$  typically does not correspond to any agent’s measure of beliefs, even when they all agree.

A central relationship in the capital asset pricing model, the fundamental model of finance, is the security market line, which suggests that the return to an asset is linearly and negatively related to the covariance of the asset payoff and the ‘market’ payoff. [See Sharpe (1964) and Lintner (1965).] The fact that it is the covariance, and not the variance, which is important to the pricing of assets (despite the ‘variance aversion’ of the agents) is one of the enduring lessons of CAPM.

The connection between asset prices and covariances, as opposed to variances, is a lesson which holds generally for all GEI equilibria, provided that we are allowed to substitute an arbitrary asset for the market asset.

Let  $\pi$  be an arbitrary strictly positive probability measure on  $\{1, \dots, S\}$ . For any vectors  $\beta, \gamma \in \mathbb{R}^S$ , define  $E_\pi \beta \equiv \sum_{s=1}^S \pi_s \beta_s$ , and  $\text{Cov}_\pi(\beta, \gamma) = \sum_{s=1}^S \pi_s (\beta_s - E_\pi \beta)(\gamma_s - E_\pi \gamma)$ . Call  $\beta$  riskless if  $\beta_s = k$  for  $s = 1, \dots, S$ , for some  $k > 0$ .

*Theorem 4.* Let  $((u^h, e^h)_{h=1}^H, A)$  be a GEI economy, and let  $(\bar{p}, q, (x^h, \theta^h)_{h=1}^H)$  be a GEI. Let  $\pi$  be an arbitrary, strictly positive probability on  $\{1, \dots, S\}$ . Then there is a renormalization of prices  $\bar{p}$  to  $p$  so that  $(p, q, (x^h, \theta^h)_{h=1}^H)$  is a GEI, and vectors  $\mu, z \in \text{sp}[A]$ , such that  $\bar{p} \square z$  is riskless, and such that for any  $r \in \text{sp}[A]$  with  $q(r) = 1$ ,

$$[E_\pi(\bar{p} \square r) - E_\pi(\bar{p} \square z)] = \frac{\text{Cov}_\pi(\bar{p} \square \mu, \bar{p} \square r)}{\text{Var}_\pi(\bar{p} \square \mu)} [E_\pi(\bar{p} \square \mu) - E_\pi(\bar{p} \square z)].$$

Note that  $E_\pi(\bar{p} \square z)$ ,  $E_\pi(\bar{p} \square \mu)$ , and  $\text{Var}_\pi(\bar{p} \square \mu)$  are constants, independent of



$r$ . So the above equality expresses a linear relation between  $E_\pi(\tilde{p} \square r)$  and  $\text{Cov}_\pi(\tilde{p} \square \mu, \tilde{p} \square r)$ .

In the CAPM, which was described in section 2 as a GEI economy in which additional properties (i)–(iv) hold, Theorem 4 can be sharpened. First,  $z$  can be taken to be the ‘riskless’ asset  $\tilde{1}$ . Second,  $\mu$  can be taken to be the market  $\sum_{h=1}^H \tilde{e}^h$ . Third, with these choices of  $z$  and  $\mu$ ,  $\pi$  must be taken to be the objective probabilities specified by CAPM for the states.

#### 4. The existence problem: Generic existence

##### 4.1. The existence problem

The fundamental significance of the assets  $A$  lies in their span,  $\text{sp}[\tilde{p} \square A]$ , at least when short sales are permitted (as we saw in section 3). This important economic insight can be expressed mathematically via the so-called Grassmanian manifold: the collection of all  $J$ -dimensional subspaces of  $\mathbb{R}^S$ . In this section we shall begin to see how the Grassmanian manifold has been used to settle the puzzle of (generic) existence of GEI. Moreover, since the Grassmanian manifold is not convex, we shall see that a new kind of fixed point analysis, more powerful than Brouwer’s fixed point theorem, had to be formulated in order to cope with the GEI existence problem.

Since the foundational work in mathematical economics of the 1950s by Nash, Arrow, Debreu and McKenzie, new models have not achieved the status of full fledged theories, as opposed to examples, until they have been proved to be consistent across a variety of settings and parameter values. This helps to explain the explosive growth of interest in the GEI model since the first equilibrium existence proofs for GEI models with numeraire of financial securities were given several years ago. Demonstrating existence in the general case with arbitrary assets was more difficult, especially in view of the Hart counterexample (presented below). The major breakthroughs for generic existence came in a paper by Duffie and Shafer (1985), which used the Grassmanian manifold, and in the first three papers in this issue, HLM, GS and HMM, which used the Grassmanian manifold and introduced new paradigms for solving economic systems of equations.

In the following pages, I describe the existence problem, starting from the simplest possible example. This is the most technical part of the introduction, and the reader mainly interested in applications of GEI may move on to section 5.

*Example.* Consider a simplified version of an example due to Hart (1975). Let there be two states of nature, one commodity in states 0 and 2, and two commodities in state 1. Utilities of type  $A$  and  $B$  are  $U^A(x_0, x_1, y_1, x_2) = \log x_1 + 2 \log y_1 + \log x_2$ ,  $U^B(x_0, x_1, y_1, x_2) = \log x_1 + \log y_1 + 2 \log x_2$ , where subscripts refer to the state. Notice that neither consumer cares for  $x_0$ .

Endowments are  $(\bar{x}_0^A, \bar{x}_1^A, \bar{y}_1^A, \bar{x}_2^A) = (0, 1, 1, 2)$ ,  $(\bar{x}_0^B, \bar{x}_1^B, \bar{y}_1^B, \bar{x}_2^B) = (0, 1, 2, 1)$ , where  $(x_1, y_1)$  represent the two goods in state 1. There are two assets, one which pays  $A^1 = (1, 0, 1)$ , i.e., one unit of the  $x$ -good in both states and the other pays  $A^2 = (0, 1, 1)$ , i.e., one unit of good  $y_1$  in the first state, and one unit of  $x_2$  in the second state. Neither consumer begins with any endowment of either asset. The assets are to be thought of as potential contracts, not as physical outputs. Equilibrium is defined as in section 2.

There can be no equilibrium in the example, as can be seen from the definition of GEI\*. If the prices  $p$  are such that the assets yield independent payoffs, then the markets are effectively complete. But there is a unique Arrow–Debreu price vector  $p = (0, 1, 1, 1)$ , which means that the assets have the same yield in both states, a contradiction. Similarly, if the assets have co-linear yields, then the only equilibrium possible is the pure spot market equilibrium in which the assets are not traded, and the spot prices are then  $p = (0, 1, \frac{7}{8}, 1)$ . In this case the assets do have independent payoffs, another contradiction.

Why does equilibrium fail to exist in this example? Radner (1972) had already recognized that the problem stems from the possible unboundedness of short sales. Recall that in the existence proof for Arrow–Debreu equilibrium, it is required that consumption sets be bounded from below; in GEI the purchase of assets can be arbitrarily negative. Radner showed that any a priori constraint on short sales would indeed guarantee existence. (An interesting mathematical problem is to characterize the set of allocations that can be obtained as limits of ‘Radner’ equilibria, as the short sale constraint is relaxed to  $\infty$ . Evidently in this example none of the limiting allocations are genuine GEI.)

Hart (1975) attributed the non-existence to lack of continuity of the GEI\* demand function. For any  $p \in \mathbb{R}_+^{L(S+1)}$ , let  $\hat{x}^h(p) = \arg \max \{u^h(x) \mid x \in \hat{B}^h(p)\}$ . Then it is easy to show (as we shall remark later) that at any  $p$  for which the dimension of  $\text{sp}[\hat{p} \square A]$  is  $J$  (that is, ‘full’),  $\hat{x}^h$  is continuous. But for  $p$  at which  $\text{sp}[\hat{p} \square A]$  drops rank (that is, has dimension less than  $J$ ), demand  $\hat{x}^h$  may not be continuous. This possible discontinuity is related, of course, to the possible unboundedness of short sales. Let  $p^{(n)} \rightarrow p$ , but  $\hat{x}^h(p^{(n)}) \not\rightarrow \hat{x}^h(p)$ . Constructing the associated  $q^{(n)}$  and  $q$  from Lemma 1,  $q^{(n)} \rightarrow q$ . But if  $(\hat{x}^h(p^{(n)}), \theta^h(p^{(n)}, q^{(n)}))$  is the optimal consumption and portfolio choice in  $B^h(p^{(n)}, q^{(n)})$ , then we must have that for some asset  $j$ ,  $\liminf_n \theta_j^h(p^{(n)}, q^{(n)}) = -\infty$ .

Let us see graphically (see fig. 1) why there is no GEI\* equilibrium in our example. Fix  $P_{x_0} = 0$ ,  $P_{x_1} = P_{x_2} = 1$ , and consider the GEI\* demand for  $y_1$ . As long as  $P_{y_1} \neq 1$ , the assets are independent, and since there are as many assets as states, that means markets are effectively complete and demand for  $y_1$  is  $(3 + P_{y_1})/2P_{y_1} + (2 + 2P_{y_1})/4P_{y_1} = 2/P_{y_1} + 1$ . At  $P_{y_1} = 1$ , agents cannot transfer wealth into or out of state 1, so their demand for  $y_1$  is  $4/3 + 3/2 =$

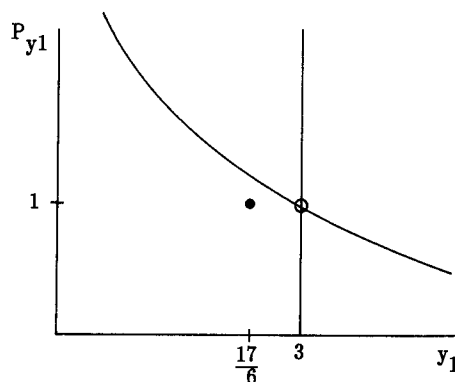


Fig. 1

17/6. Supply for  $y_1$  is fixed at 3. Hence GEI\* demand is discontinuous. This explains the non-existence of equilibrium in the Hart example. It also gives a hint why no such example can be robust.

Notice that as  $p_{y_1} \rightarrow 0$ ,  $y_1(p_{y_1}) \rightarrow \infty$ . More importantly, notice that the discontinuity in the GEI\* demand is removable. Almost everywhere it is equal to the Arrow–Debreu demand  $\sum_{h=1}^H x_w^h(p)$ . Since the Arrow–Debreu demand is not the genuine demand for the economy, let us call it a ‘pseudo-demand’. Let us call a price vector where pseudo-demand equals supply a pseudo-equilibrium. Since Arrow–Debreu (pseudo) demand is continuous, and satisfies Walras’ Law and the boundary condition  $p_i \rightarrow 0 \Rightarrow |z(p)| \rightarrow \infty$  there must always be a pseudo-equilibrium. Since the GEI\* demand is the same as the pseudo-demand except for a small set of points, one strategy for proving the existence of GEI equilibria ‘except for accidents’ is to show that for a generic choice of economies, all pseudo-equilibrium price vectors occur at points where pseudo-demand and GEI\* demand are the same.

In the seventh paper in this issue Magill and Shafer indeed prove that if there are at least as many assets as states, and if the asset structure  $A$  is non-degenerate, then for generic endowments of agent 1 the set of Arrow–Debreu allocations  $(x^h)_{h=1}^H$  is identical to the set of GEI allocations. This result is similar to one obtained by Repullo (1986), and McManus (1984).

But when there are fewer assets than states,  $J < S$ , two new problems arise which seem at first sight to be insurmountable. First, GEI\* demand never coincides with Arrow–Debreu demand, so it is not clear what continuous function can play the role of the pseudo-demand. Second, the discontinuity arising where the dimension of  $\text{sp}[\tilde{p} \square A]$  drops rank is usually not removable. Consider the following picture (fig. 2) of a non-removable singularity, which could arise for example in an Arrow–Debreu economy with non-convex preferences.

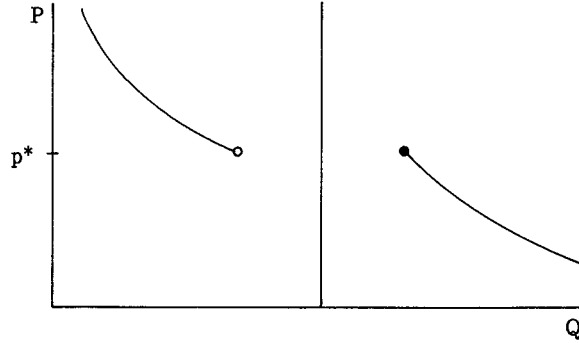


Fig. 2

There is only one point  $p^*$  at which demand is discontinuous, yet no perturbation of the economy will restore equilibrium.

In the three papers that begin this issue, HLM, GS and HMM, it is shown that despite the non-removable discontinuities, GEI nearly always exists; given any economy where there is no equilibrium, the smallest perturbation to  $e^1$  and  $A$  will restore existence. How this can happen we shall shortly begin to see.

4.2. *Existence for numeraire asset economies and the Cass trick*

Let us begin the analysis of existence with an elementary special case which does not confront us with the puzzles of discontinuities. Suppose that for each  $s$ , all the assets in  $A$  promise delivery in units of the same commodity bundle. Then it is easy to show that the dimension of  $\text{sp}[\tilde{p} \square A]$  is the same for all  $\tilde{p} \in \mathbb{R}_+^{L^s}$ . It follows that the GEI\* demands  $\hat{x}^h(p)$  are continuous. As we have said, a number of authors [Cass (1984), Werner (1985), Geanakoplos and Polemarchakis (1986), Duffie (1987)] showed that such numeraire asset economies (or the related financial asset economies) must have GEI. But though straightforward, the proofs are not quite completely obvious. For instance, let  $\hat{z}(p) \equiv \sum_{h=1}^H \hat{x}^h(p) - \sum_{h=1}^H e^h$ . Then  $\hat{z}$  satisfies Walras' Law and continuity (for numeraire asset GEI economies) but it does not necessarily satisfy the boundary condition. (In the example, let  $p_{x_0} = 1$ , and  $p_{x_1} = p_{y_1} = p_{x_2} = \varepsilon$ . As  $\varepsilon \rightarrow 0$ , GEI\* demand is unaffected.) Hence the usual Walrasian approach to showing existence, which depends crucially on the boundary condition, cannot be applied.

One way round this difficulty is through yet another equivalent definition of GEI due to Cass.

For  $p \in \mathbb{R}_+^{L(s+1)}$ , let  $B_w^h(p) = \{x \in \mathbb{R}_+^{L(s+1)} \mid p \cdot x \leq p \cdot e^h\}$  be the Walrasian budget set of agent  $h$ , and let  $x_w^h(p) = \arg \max_x \{u^h(x) \mid x \in B_w^h(p)\}$  be the Walrasian demand. Consider the system of equations

$$x_w^1(p) + \sum_{h=2}^H \hat{x}^h(p) = \sum_{h=1}^H e^h.$$

Any solution  $(p, x_w^1(p), (\hat{x}^h(p))_{h=2}^H)$  of this system of equations is called a  $\text{GEI}_w^*$ . The crucial insight of Cass is that if  $(p, (x^h)_{h=1}^H)$  is a  $\text{GEI}_w^*$ , then it must also be a  $\text{GEI}^*$ . The point is that if  $\tilde{p} \square (x^h - e^h) \in \text{sp}[\tilde{p} \square A]$  for all  $h \geq 2$ , and  $\sum_{h=1}^H (x^h - e^h) = 0$ , then  $\tilde{p} \square (x^1 - e^1) \in \text{sp}[\tilde{p} \square A]$ . Conversely, if  $(p, (x^h)_{h=1}^H)$  is a  $\text{GEI}^*$ , then by a simple separating hyperplane argument one can show that it is possible to rescale  $p$  so that  $(\hat{p}, (x^h)_{h=1}^H)$  is a  $\text{GEI}_w^*$ . Thus we have yet a third equivalent definition of  $\text{GEI}$  allocations.

The advantage of Cass' definition of  $\text{GEI}$  is that it leaves the first agent unconstrained. As  $p_{s^\ell} \rightarrow 0$  for some commodity  $s^\ell$ , the demand of agent 1 will tend to infinity because of monotonicity. Since the demand by the other agents for commodities (though not for assets) is bounded below by zero, aggregate demand must tend to infinity. This demonstrates the existence of  $\text{GEI}$  for numeraire asset economies.

#### 4.3. The Grassmanian approach to pseudo-equilibrium

In order to analyze the existence problem for  $J < S$  and non-numeraire assets it seems necessary to introduce a new mathematical object, the Grassmanian manifold  $\mathcal{G} = \mathcal{G}(S, J)$ . The set  $\mathcal{G}$  is the collection of all  $J$ -dimensional subspaces of  $\mathbb{R}^S$ . It is well-known that  $\mathcal{G}$  can be given a system of local neighbourhoods with respect to which it becomes a smooth, compact manifold of dimension  $J(S - J)$ . However,  $\mathcal{G}$  is not convex. Yet it is the object on which all four papers [Duffie and Shafer (1985) and the three in this volume] build their existence proofs.

To get some idea why  $\mathcal{G}$  is a manifold [of dimension  $J(S - J)$ ], consider  $L = \text{sp}[M]$  where  $M$  is an  $S \times J$  matrix of the special form  $M = \begin{bmatrix} I \\ M_2 \end{bmatrix}$ , with  $I$  the  $J \times J$  identity matrix, and  $M_2$  an arbitrary  $(S - J) \times J$  matrix. As  $M_2$  varies in  $\mathbb{R}^{(S - J)J}$ , it parameterizes a  $(S - J)J$  dimensional open set in  $\mathcal{G}$ . Observe also that any  $L \in \mathcal{G}$  is equal to the span of a matrix with  $J$  rows forming the  $J \times J$  identity matrix. To get some idea why  $\mathcal{G}$  is compact, note that it can be described as the collection of all sets of  $J$  orthonormal vectors in  $\mathbb{R}^S$ .

With these preliminary remarks about  $\mathcal{G}$  out of the way, let us proceed to describe the general notion of pseudo-demand [see Duffie and Shafer (1985), and HLM]. For any  $h \in H$ ,  $p \in \mathbb{R}_+^{L(S+1)}$ ,  $L \in \mathcal{G}$ , let

$$B^h(p, L) = \{x \in \mathbb{R}_+^{L(S+1)} \mid p \cdot x \leq p \cdot e^h \text{ and } \tilde{p} \square (x - e^h) \in L\}.$$

Let us call the pseudo-demand for agent  $h$  the vector  $x^h(p, L)$  that maximizes  $u^h$  in  $B^h(p, L)$ . Once it is understood that  $\mathcal{G}$  is a manifold, it is easy to show that  $x^h$  is a continuous function. The discontinuity in  $\text{GEI}^*$  demand is

avoided by restricting attention to subspaces  $L \in \mathcal{G}$  of full dimension  $J$ . A ‘full’ GEI equilibrium is described by  $(p, L) \in \mathbb{R}_+^{L(S+1)} \times \mathcal{G}$  satisfying

$$z(p, L) = (x_w^1(p) - e^1) + \sum_{h=2}^H (x^h(p, L) - e^h) = 0, \quad (1)$$

$$L = \text{sp} [\tilde{p} \square A]. \quad (2)$$

Of course, as we have seen, a full equilibrium need not exist. Eq. (2) is a discontinuous relation, since  $\text{sp} [\tilde{p} \square A]$  changes discontinuously when its dimension jumps. But let us replace (2) by

$$L \supset \text{sp} [\tilde{p} \square A]. \quad (2')$$

Any  $(p, L)$  satisfying (1) and (2') is called a *pseudo-equilibrium*. Notice that if  $\text{sp} [\tilde{p}_n \square A]$  approaches  $L$  as  $p_n \rightarrow p$ , then even if  $\text{sp} [\tilde{p} \square A]$  drops in dimension, still it must be contained in  $L$ . So the collection of  $(p, L)$  satisfying (2') is closed in  $\mathbb{R}_+^{L(S+1)} \times \mathcal{G}$ .

The following theorem is proved in each of the four papers Duffie and Shafer (1986), HLM, GS and HMM.

*Theorem 5 (loosely).* For any economy  $((u^h, e^h), A)$  satisfying the usual continuity conditions, a pseudo-equilibrium exists.

Moreover, the following theorem is also shown [see Duffie and Shafer (1985) and, for a briefer proof, GS].

*Theorem 6.* If  $(u^h, e^h)_{h=1}^H$  satisfies the usual smoothness conditions, then for generic asset structures  $A$  and endowments  $e^1$ , all pseudo-equilibria of the economy  $((u^h, e^h)_{h=1}^H, A)$  are full GEI. Moreover, for such generic  $(e^1, A)$ , the set of pseudo-equilibria is finite, and there are no equilibria which are not full.

Combining the two theorems yields the generic existence of GEI, and the generic local uniqueness of GEI.

The proof of Theorem 5 is not obvious, and requires a new approach to equilibrium existence. The standard Kakutani fixed point method is apparently not applicable. Note that since  $\mathcal{G}$  is not convex, the domain  $\mathbb{R}_+^{L(S+1)} \times \mathcal{G}$  is not convex. Note also that condition (2') cannot be described in a convenient way as the zero of a smooth function.

#### 4.4. The vector bundle approach to equilibrium

HLM in paper no. 1 of this issue and HMM in paper no. 3 describe pseudo-

equilibrium as the intersection of a properly chosen section of a vector bundle and the zero bundle. The reformulation of equilibrium in terms of vector bundles and zero sections promises to be an enduring idea with applications to other economic models. Of course this is not the place to give a precise definition of a vector bundle. In essence it is the graph of a correspondence whose domain is a manifold, and whose value at each point of the domain is a vector space. A ‘section’ of the vector bundle is simply a continuous selection of one element in the set corresponding to each element of the domain. A fundamental example for equilibrium analysis is the following: let  $\mathcal{S}_+^{L-1}$  be the positive part of the  $(L-1)$ -sphere. Then  $V = \{(p, z) \in \mathcal{S}_+^{L-1} \times \mathbb{R}^L \mid p \cdot z = 0\}$  is a vector bundle over  $\mathcal{S}_+^{L-1}$ . The graph of the Walrasian excess demand  $\{(p, z(p)) \mid p \in \mathcal{S}_+^{L-1}\}$  is a section of  $V$ , and a Walrasian equilibrium is a point  $(p, z(p))$  with  $z(p) = 0$ .

Now let

$$V = \{(p, L, z, y_1, \dots, y_J) \in \mathbb{R}_+^{L(S+1)} \times \mathcal{G} \times \mathbb{R}^{L(S+1)} \times \mathbb{R}^{SJ} \mid p \cdot z = 0, y_j \perp L, \\ j = 1, \dots, J\}.$$

Then  $V$  is a vector bundle over  $\mathbb{R}_+^{L(S+1)} \times \mathcal{G}$ . Let  $f_j(p, L) \equiv \text{Proj}_{L^\perp}(\tilde{p} \square A^j)$ . Then  $\{(p, L, z(p, L), (f_1(p, L), \dots, f_J(p, L))) \mid (p, L) \in \mathbb{R}_+^{L(S+1)} \times \mathcal{G}\}$  is a section of  $V$ , and pseudo-equilibrium is a  $(p, L)$  with  $z(p, L) = 0$  and  $f_j(p, L) = 0, j = 1, \dots, J$ . For if at all  $j, f_j(p, L) = 0$ , then  $\tilde{p} \square A^j \in L$ , and  $L \supset \text{sp}[\tilde{p} \square A]$ .

By using the theory of characteristic classes (HLM) or intersection theory (HMM), and the special structure of  $\mathcal{G}$ , it is shown that every section of  $V$  must intersect the zero section. Hence a pseudo-equilibrium exists

#### 4.5. Solving systems of simultaneous equations on non-convex domains

The approach taken by GS in paper no. 2 in this issue begins with the set  $N = \{(p, L) \in \mathbb{R}_+^{L(S+1)} \times \mathcal{G} \mid L \supset \text{sp}[\tilde{p} \square A]\}$ . They show the existence of  $(p, L) \in N$  with  $z(p, L) = 0$ . Condition (2') is thus used to reformulate the domain of the function  $z$  from the usual price simplex to  $N$ . Otherwise the problem is similar to the Walrasian existence problem. The difficulty here is that  $N$  need not be convex. The contribution of GS is to show via degree theory that  $z$  has a zero in  $N$  even though  $N$  is not convex provided that  $N$  satisfies three other easy-to-check properties.

We can express these three properties most easily as follows. For each price vector  $p$ , let  $\Phi(p) = \{L \in \mathcal{G} \mid (p, L) \in N\}$ . Then  $N = \text{graph } \Phi$ . Moreover, let  $\Psi(p) = \{z(p, L) \mid L \in \Phi(p)\}$ . Observe that  $\Psi$  is a correspondence mapping most price vectors  $p$  into the set of all excess-demands, and possibly some price vectors into a set of pseudo excess demands.  $\Psi$  may not be convex-valued. But if (i)  $\Phi$  is upper hemi-continuous, (ii) for some (small) open set of prices

$\Phi$  is a function, and (iii) graph  $\Phi = N$  is a manifold of the same dimension as the price space, then there must be a  $p$  with  $0 \in \Psi(p)$ . These properties are indeed easy to check. We already saw that condition (2) is preserved under limits, so  $\Phi$  is surely upper hemi-continuous. Whenever  $p$  is such that  $\text{sp}[\tilde{p} \square A]$  is full, then  $\Phi(p)$  is a single point. For generic asset structures  $A$ , (ii) is thus easily confirmed. Finally, it is also easy to show that for generic  $A$ , (iii) must hold.

The GS approach to existence also promises to have applications outside of GEI. Indeed GS use the approach to prove the existence of marginal cost pricing equilibria. Note also that if  $\Psi$  is upper hemi-continuous and convex, compact valued, then it can be approximated by  $\hat{\Psi}$  satisfying (i), (ii), (iii), so GS, like HLM and HMM, generalizes the usual Walrasian framework.

It is perhaps instructive to compare the GEI existence problem (or marginal cost existence problem) to the insoluble problem of existence of Walrasian equilibrium with non-convex preferences, as exhibited in fig. 2. There the demand ‘holes’ could be filled in by connecting the two endpoints to form an upper semi-continuous, convex-valued demand. [Evidently the resulting  $\Psi$  satisfies (i), (ii), and (iii).] By Kakutani’s fixed point theorem, there must be a pseudo-equilibrium. Unfortunately, since there were many pseudo-demands that had to be added (albeit at one price), even after perturbing the problem chances are high that the pseudo-equilibrium will not be a genuine equilibrium. By contrast, in the GEI problem the extra pseudo-demand points that are added [points of the form  $(p, L) \in N$  with  $L \not\equiv \text{sp}[\tilde{p} \square A]$ ] have dimension less graph  $\Psi$ , hence the probability is 1 that after perturbing the system each pseudo-equilibrium is a genuine equilibrium.<sup>5</sup> That is the content of Theorem 6.

#### 4.6. Robust non-existence of GEI

The proof that generically pseudo-equilibria are GEI relies on the fact that

<sup>5</sup>The reader may wonder why in the GEI model it is not possible to fill in the holes in demand with a lower dimensional set of points to obtain a pseudo-demand correspondence that is convex-valued. Consider the following example of a correspondence  $\Psi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ .  $\Psi$  is single-valued and continuous at every point  $(x_1, x_2) \in \mathbb{R}^2$  not lying on the line with  $x_2 = 0$ .  $\Psi$  is empty for each  $(x_1, x_2)$  with  $x_2 = 0$ . It will be seen that the only way to make  $\Psi$  upper hemi-continuous is by adding extra points to make  $\Psi$  a genuine correspondence. These extra points form a 1-dimensional set. But the resulting  $\hat{\Psi}$  is not convex-valued.

For  $x_2 \neq 0$ , let

$$\Psi(x_1, x_2) = \begin{cases} (0, 0) & \text{if } x_1 \leq 0, \\ (x_1/x_2, x_1^2/x_2^2) & \text{if } 0 < x_1 \leq x_2, \\ (1, 1) & \text{if } x_1 > \max\{0, x_2\}. \end{cases}$$

Now define

$$\hat{\Psi}(x_1, 0) = \begin{cases} (0, 0) & \text{if } x_1 < 0, \\ \{(t, t^2) \mid 0 \leq t \leq 1\} & \text{if } x_1 = 0, \\ (1, 1) & \text{if } x_1 > 0 \end{cases}$$



the set of  $(p, A)$  at which  $\text{sp}[\tilde{p} \square A]$  drops dimension is a low dimensional set. This follows quite straightforwardly from the linearity of  $\tilde{p} \square A$ .

In the fourth paper of this issue, PK note that for assets like options, the payoffs are not linear in the prices. For example, the payoff to a call option in each state  $s$  is a function of the form  $\max\{p - c, 0\}$ . This may be zero over a large interval, and non-zero over another large interval. Hence, when assets take the form of options, there will be open sets in the domain of prices at which the span of the assets drops rank. PK use this observation to construct a robust example of non-existence of GEI with options.

### 5. Constrained inefficiency

It is a very old idea – propounded for example by Kenneth Arrow among others – that when markets are missing competitive equilibrium should not be expected to be Pareto optimal. There may be a useful role for the government or other institutions to act in place of the missing markets. The GEI model strengthens the role for intervention in a non-trivial way. In the Arrow–Debreu world, if a market is missing, a central planner can improve the final allocation, but only in effect by *replacing* the missing market. An illuminating real world example is the so-called ‘operation bubble’ that is in place in some midwestern American states, where polluters must buy the right to pollute the air on a government-run market. In the Arrow–Debreu model, if the planner has no ability to affect the allocation of goods that would have been traded on the missing market, then it should not intervene at all. If, for example, apples cannot be traded on the market (or if some subset of agents is barred from trading in apples) there is no reason for the government to induce firms to change the mix of pears and oranges they are selling. In the GEI model this is not the case. If some kinds of ‘risk’ cannot be traded, then the government indeed ought to intervene and induce consumers and producers to alter their trades of the existing assets.

Traditional conceptions of market efficiency have always evaluated competitive allocations in terms of the physically feasible alternatives. The conventional conclusion, deriving from Adam Smith, Pareto, Hicks, Samuelson, Arrow, and Debreu is that in the absence of externalities competitive equilibria are Pareto efficient. In discussing the efficiency of GEI equilibria, we shall find it helpful to restrict the set of alternative allocations to what is feasible given the constraints under which the market is operating, in addition to the usual physical feasibility conditions.

It is almost universally acknowledged that asset markets are incomplete. Yet many economists suggest that the same factors, such as asymmetric information or transactions costs, which keep the market from introducing new assets must also impede any central planner from intervening advantageously [see, for instance, Ross (1976b)]. The notion of constrained feasible

introduced by Diamond (1967) was meant to capture precisely the allocations that can be physically achieved via market interactions with the existing assets. Diamond found that in a model with a single consumption good (and multiplicative uncertainty in production), all GEI are constrained efficient (i.e., undominated by constrained feasible allocations).

When there is only one consumption good,  $L=1$ , equilibria are necessarily constrained efficient, as Diamond (1967) noted. They are generically not Pareto optimal, however. One interesting special case is the CAPM model satisfying assumptions (i)–(iv) described in section 2. It can be shown that interior equilibria of that model are fully Pareto optimal [see Mossin (1973), Rubinstein (1974) and Geanakoplos and Shubik (1989)]. CAPM is clearly non-generic, however, for almost any perturbation of the endowments will violate (iii), and destroy the full optimality of equilibria.<sup>6</sup>

A host of authors noted that Diamond's proposition about constrained efficiency does not extend to economies with more than one consumption good. Hart (1975) gave a famous example of a GEI economy with multiple consumption commodities that has two GEI, one of which Pareto dominates the other. Diamond (1980), Stiglitz (1982), and Newbery and Stiglitz (1982), all presented examples with two commodities in which local changes in holdings of existing assets and production decisions could effect Pareto improvements. Geanakoplos and Polemarchakis (1986) proved that generically all GEI equilibria are constrained inefficient in the following sense.

*Theorem 7. Suppose that a planner could intervene only at date zero before nature moves, inducing agents to hold non-maximizing portfolios of existing assets and consumption goods. Subsequently, in each state in time 1, markets clear competitively. If  $L \geq 2$ , then for generic utilities and endowments, for every GEI there is an intervention which is Pareto improving, provided that  $1 < H < (S-1)(L-1)$ .*

The reason for the inefficiency is that for generic utilities (for which different agents have different marginal propensities to consume), a redistribution of assets redistributes wealth in any given state, leading to a change in relative prices. This relative price change yields yet another redistribution of wealth which is not necessarily achievable through the market. By judiciously choosing the right portfolio adjustments, the government can use the pecuniary externality to make everyone better off.

<sup>6</sup>It is interesting to note that CAPM can be extended to  $L > 1$ , maintaining all the same conclusions. The only difference is that the appropriate definition of 'riskless' asset depends on the preferences and endowments of the agents. In general its payoff will differ from state to state. [See Geanakoplos and Shubik (1989).] Thus the major policy recommendation of the classical, one commodity CAPM model, that the government should endeavor to create a riskless asset, is seen to be problematic in the many good CAPM model. To create the correct riskless asset the government would have to know the tastes and state-dependent endowments of all the agents in the economy. (Perhaps this explains why some governments have made no such efforts.)

The hypothesis that  $H < (L-1)(S-1)$  is needed in Geanakoplos and Polemarchakis (1986) because they restricted attention to interventions in which the planner had to maintain budget balancing for each agent at time 0 (no transfers were allowed). Mas-Colell (1987) gave an example showing that without transfers the upper bound on  $H$  is necessary for the theorem. But with transfers the same kind of proof as in Geanakoplos and Polemarchakis (1986) would go through for arbitrary  $H$ .

In the fifth paper in this issue, GMQD extend the constrained inefficiency result to firms. Here transfers are permitted. The government is allowed, for example, to subsidize a firm to produce food in states where the relative prices will adjust strongly enough to insure the poor. GMQD show that generically in endowments (for any utilities) there is always a subsidy program and change in production that makes everybody better off.

The GMQD constrained inefficiency theorem holds for all utilities because the change in relative output alters relative prices even when all consumers have identical marginal propensities to consume. GMQD require a more stringent upper bound on  $H$  than in Geanakoplos and Polemarchakis (1986) because of the complications that can arise with the choice of firm optimization. The goals of the firm in GMQD are given by the Drèze criteria, which are the most favorable for efficiency if effects of production on relative prices are negligible.

Taken together, the results in Geanakoplos and Polemarchakis (1986) and GMQD seem to make a strong case for government intervention. Yet proponents of laissez-faire might appeal to another problem. In both papers the government was assumed to know the model, though it did not have any special ability to forecast the correct state  $s \in S$ . But knowing the model means knowing the state contingent utilities of the individuals, and also knowing the probabilities they assign to each state. In the sixth paper of this issue, GP analyze how much the planner needs to know to make the correct intervention. GP show that this information can be recovered from the individual excess demand functions, provided that these specify the changes in the demands for assets that would occur if expected future spot prices changed. If only the demand for assets as a function of contemporary asset prices and contemporary commodity prices is observable, then the government can never be sure it is making a Pareto improving intervention.

In the seventh paper of this issue MS show that when  $J \geq S$ , and assets are non-degenerate, then generically in endowments all GEI are Arrow–Debreu, and hence Pareto efficient.

## **6. Money, financial assets, and indeterminacy**

So far we have assumed that all agents' utilities are strictly monotonic in

each commodity, and that endowments are also strictly positive. The first assumption rules out fiat money as one of the commodities.

If we dropped the first assumption and allowed for some  $(L+1)$ th commodity that did not affect any agent's utility, a familiar backward induction argument would show that in equilibrium its price must be zero, even if it were storable. In the last period no agent would end up holding and consuming a positive quantity of it if it had a positive price. Its price in any  $s=1, \dots, S$  would have to be zero. Similarly for  $s=0$ , no agent would consume it if its costs were non-zero, and no agent would hold it even if it were storable, since there would be no gain from selling it in period 1.

In order to introduce fiat money into our model, as a preliminary step we allow both for commodities that enter no agent's utility, and are zero in aggregate supply. Such goods have some of the character of fiat money.

At first glance it might seem that if each agent had zero endowment of a good, and if it had no effect on utility, then there would be no role for it to play in the GEI model. But that is far from accurate. As we have seen, agents' promised deliveries of commodities can exceed their endowments, provided they can afford to purchase the promised commodities by selling goods they do own. The interesting role for our preliminary 'fiat money' to play in the GEI model is as the good promised for delivery by the assets. In practice, we observe that many real world assets, like insurance, do provide for state contingent dollar payoffs.

Let us describe a pure monetary asset structure by adding an  $(L+1)$ th commodity to the model with  $e_{s,L+1}^h=0$  and  $u^h$  independent of  $x_{s,L+1}^h$ , for all  $h$  and  $s$ . Let us replace  $A$  by an  $S \times J$  matrix  $R$  describing the promised 'money' payoffs (of good  $L+1$ ) in each of the  $S$  states, for each of the  $J$  assets. GEI is defined exactly as before, except that  $L+1$  replaces  $L$ . But we can take advantage of the special structure of this economy to simplify the definition of equilibrium.

First, we shall confine attention to equilibria in which 'money' has a positive price in each state. By the homogeneity property of GEI, we may therefore also confine attention to GEI for which  $p_{s,L+1}=1$  for all  $s=0, 1, \dots, S$  without missing out on any GEI commodity allocations  $(x^h)_{h=1}^H$ . Second, since agents' consumption sets prevent them from consuming negative amounts of money, as long as  $p_{s,L+1}>0$ , all agents will demand  $x_{s,L+1}^h=0$ , which is precisely the supply. In the market clearing conditions defining GEI, the equations corresponding to  $x_{s,L+1}$  are redundant. Hence we can entirely suppress goods  $(s, L+1)$  from the definition of a monetary GEI. Thus a GEI monetary equilibrium for the pure monetary asset GEI economy  $((u^h, e^h)_{h=1}^H, R)$  is a  $(p, q, (x^h, \theta^h)_{h=1}^H)$  exactly as before, with the only difference that we replace  $B^h(p, q)$  by

$$B_M^h(p, q) = \{(x, \theta) \in \mathbb{R}_+^{LS} \times \mathbb{R}^J \mid p_0 \cdot (x_0 - e_0^h) + q \cdot \theta = 0, \text{ and}$$

$$\forall s \geq 1, p_s \cdot (x_s - e_s^h) = R_s \cdot \theta\}.$$

Since payoffs are all in terms of money, which has price one,  $p_s$  does not enter the right-hand side of  $B_M^h(p, q)$ .

Under this definition no commodity, including the money good, can be stored. But it is easy to show that if one of the assets in  $R$  promises the same constant  $k$  in each state, then holding that asset is just like storing money. In this model money is the unit of account, it can be regarded as a store of value, and it must also compete with other commodities in agents' portfolios. It apparently has no transactions role, but we shall reconsider this in section 7.

The following theorem is proved in Geanakoplos and Mas-Collel (1989). [A similar result is in Balasko and Cass (1989). The first example of real indeterminacy with financial assets was given by Cass (1985).]

*Theorem 8. Let  $((u^h, e^h)_{h=1}^H, R)$  be a pure monetary asset GEI economy, satisfying the usual smoothness conditions. Let there be at least one asset,  $J \geq 1$ , and let the asset market be incomplete,  $J < S$ . Let there be at least  $J + 1$  traders,  $H \geq J + 1$ . Let  $R$  be in general position, i.e., every submatrix of  $R$  has maximal rank. Then for a generic assignment of endowments, there are  $S - 1$  dimensions of real indeterminacy, i.e., the set of equilibrium allocations  $(x^h)_{h=1}^H$  contains the image of a  $C^1$  one-to-one function with domain  $\mathbb{R}^{S-1}$ .*

There is something surprising about this result. If there are  $S$  assets, then, as we saw in Theorem 1, markets are complete and the equilibrium allocations correspond to the locally unique (zero dimensional) Arrow–Debreu equilibria. If just one asset is missing, the degree of indeterminacy jumps to  $S - 1$  dimensions, and stays there no matter how many assets there are.

The reason for indeterminacy is that all possible rates of inflation across the  $S$  states of nature are consistent with rational expectations and market clearing. When  $J = S$ , these different rates of inflation do not entail real effects on consumption. But if  $J < S$ , and  $H \geq J + 1$ , then except for balanced inflation, differing rates of inflation almost always do have real effects.

It is worth emphasizing the striking difference between complete markets and incomplete markets, and also the importance of moving away from the ‘representative-agent model’ in which  $H = 1$ . In this theorem the hypothesis  $H \geq J + 1$  plays a crucial role: the theorem only applies generically in the endowments, excluding cases where the endowments themselves form a Pareto optimal allocation, and cases where the  $H$  agents have identical starting points if their utilities are identical. If  $H = 1$ , then of course there is a unique GEI allocation.

I believe that the study of monetary theory is itself incomplete unless it

recognizes the implications of incomplete markets. Theorem 8, however, just begins the analysis.

In paper no. 8 in this issue, BCS extend the significance of the result by making the following kind of argument. There are some critics who argue that the set of potentially available assets is complete, or effectively complete. But almost nobody suggests that every agent has access to all of these assets. BCS show that if there is a group of  $H \geq J+1$  agents as above who have access only to assets represented by the  $S \times J$  matrix  $R$ ,  $J < S$ , then no matter how many other agents  $H'$  there are in the economy, and even if all of them have access to a complete set of assets  $[R, R']$ , there will still typically be  $S-1$  dimensions of indeterminacy.

One of the most difficult problems in GEI analysis is to describe the equilibrium set when the economy extends over multiple time periods, and assets may be long-lived and retraded each period. In paper no. 9 in this issue, Werner takes two polar cases of the multiperiod model with financial assets and calculates the dimension of indeterminacy in each one. BCS point out some problems of existence with restricted participation in the multiperiod model.

Werner also gives a formula for the dimension of indeterminacy in the two-period GEI model when the assets  $R$  are not in general position [a similar formula appears in Geanakoplos and Mas-Colell (1989)]. General position is a generic condition, and for such asset structures the dimension of indeterminacy is constant at  $S-1$ . But as I mentioned earlier, it is possible to extend the analysis to non-generic asset structures as well. So long as there is a non-zero payoff in every state, the formula shows that the dimension of indeterminacy is at least  $S-J$ .

## 7. The future of GEI

In my opinion GEI analysis, though becoming mature, is far from its final stage of development. Just as enlarging the Arrow–Debreu model by adding assets created a host of interesting new questions and solutions, so I believe that further extensions of the GEI model will bring it into much closer contact with monetary theory and financial economics. I allow myself the luxury of speculating below on three promising avenues for further exploration: money production and default.

The current GEI model is insufficient, as we have seen, because in order for our preliminary ‘fiat money’ to have positive value, it must be in zero aggregate supply. The way to repair this defect, I believe, is to model explicitly the way money enters the system through a banking system. Macroeconomic models that involve agent optimization have tended to ignore the banking sector, relying on ‘helicopter money’ or initial endowments to inject money into the system. Adding a bank sector to the GEI

model, with its explicit intertemporal structure, creates an ideal environment to study the factors determining an endogenous rate of interest on loans, and an endogenous value and velocity of money.

A critical test for GEI monetary theory, and an obvious programme for further research, is to describe how the structure of the set of GEI monetary equilibria changes as the model is enriched to capture more of the properties of money. One role for money to play is a 'medium of exchange'. A crude, and well-known device, for representing this function of money is to impose a cash-in-advance constraint for all transactions. A bank could then be added to the model in a simple manner: at each state  $s$  there is  $M_s$  of available bank money that agents can borrow at an endogenous interest rate. (At  $s=0$  there might be two stocks of money, one for short-term loans, and the other for long-term loans.) In equilibrium the interest rates will adjust so that demand for money equals supply.<sup>7</sup> Then the GEI model describes a subtle interplay of factors generating the equilibrium rates of interest: speculative demand (portfolio demand), precautionary demand, transactions demand, etc. It may well be that in equilibrium holding cash balances earns a rate of return strictly less than holding some other assets. [For preliminary studies of cash-in-advance, GEI models, see Dubey and Geanakoplos (1989a) and Magill and Quinzii (1989).]

Let us quickly note that adding a cash-in-advance constraint, which obliges agents to hold money if they want to make purchases, does *not* by itself break the backward induction proof that money cannot have positive value in a finite horizon model if it is initially in positive supply.<sup>8</sup> Another very important step would be to merge systematically GEI analysis with the existing infinite horizon studies of monetary equilibrium (e.g., overlapping generations economies, Lucas style infinitely lived representative agent models, etc.) to see whether the presence of outside money alters the set of equilibria.

I have not written much so far about production in the GEI model, since only one paper in this issue is concerned with it. [See also Duffie and Shafer (1986b).] Evidently the question of what the goal of the firm should be with incomplete markets is widely thought to be one of the bugaboos of GEI analysis. On the contrary, I think that the GEI framework is much more hospitable an environment for a study of the firm than is the complete markets setting. In the first place, in Arrow–Debreu there is unanimity about the firms' goals, but there is also no trade in the stock market.

<sup>7</sup>This simple description of the banking sector as a passive lender of a fixed stock of money has been used by Martin Shubik in game theoretic analyses of transactions

<sup>8</sup>The presence of bank loans and an endogenous interest rate does, however, break the backward induction hypothesis. If individuals privately own  $M_1$  units of money, and the bank loans  $M_0$ , then an interest rate of  $\theta = M_1/M_0$  will allow for a positive price of money in equilibrium. See Dubey and Geanakoplos (1989d).

In the second place, there are many logically consistent utilities one could assign to firms. For example, one could require that a particular agent (type) always owns at least 50% of the firm, and always directs the firm to act in his (type's) interest. Or one could assume that all firms had multiplicative, ray technologies as in Diamond (1967). Then unanimity would hold again (but shares would nevertheless be traded). Or one could just arbitrarily assign a utility, like the Drèze criterion, or the Grossman–Hart criterion, etc., which captures some notion that the firm is acting in the interests of its shareholders. [Of course, as Radner (1972) pointed out, one cannot assign the firms arbitrary utilities and still have existence, even if short sales are prohibited.] The upshot is that it may be necessary to assign utilities to firms (which might well depend on who the owners are) just as general equilibrium has always assigned utilities to individuals. This can be done without jeopardizing the existence of equilibrium, and without limiting the ability of the theory to adjudicate such questions as whether corporate financial policy has real effects, etc.

One of the most important phenomena that must be modeled is the financing of production. Up until now general equilibrium models have cheated by assuming that the shareholders automatically provided the cash to finance all investments. In fact firms raise money for capital expenditures not just through retained earnings, but by issuing new shares, or bonds, or by borrowing from banks. Capital expenditures and investment are much more volatile than consumption expenditures, and it is therefore at the level of firms' demand for money (and the effects of their efforts to raise capital on the portfolio decisions of households) that the implications of monetary policy should be studied.

Once one has the possibility of borrowing money from a bank, one also should permit the possibility of failure to repay (bankruptcy). This holds as well for default on the payments of assets. We have already seen that in the basic GEI model the budget constraint is always satisfied, so default and bankruptcy are ruled out by assumption. Indeed they are often thought to be incompatible with equilibrium analysis. But in fact the GEI model can move closer to reality by accommodating them, yet at the same time preserving perfect competition, rational expectations, and the general existence of equilibrium. [See Dubey, Geanakoplos and Shubik (1989) for details.]

At first glance default might seem to be incompatible with perfect competition, since rational agents would have to calculate the probabilities of their loans defaulting, and conjecture how these probabilities would change as they increased their loans. The right way to maintain perfect competition, and also to move closer to reality, is to suppose that there is financial intermediation, so that all defaults on any asset are spread out proportionately among all the owners of the asset. Each lender, regarding himself as one of a continuum, could quite rationally conjecture that the proportion of



defaults is independent of his decisions about how much to lend. (Let us denote by  $K_{s\ell j}$  the fraction of promised payments that agents rationally expect to receive of good  $\ell$  in state  $s$  for asset  $j$ .)

In order to allow for strategic default, but maintain some incentive for repayment, individuals who default must be punished. The simplest method of modeling these penalties without getting bogged down in a morass of institutional details is to suppose that the law prescribes exogenous penalties  $\lambda_{s\ell j}^h > 0$ , where  $\lambda_{s\ell j}^h$  is the utility loss to agent  $h$  for each unit default of good  $\ell$  in state  $s$  on asset  $j$ .

Using the exogenous  $\lambda$  and the endogenous  $K$ , let me briefly indicate how to describe GEI with default. Agents must rationally calculate that they will only receive  $K_{s\ell j} A_{s\ell}^j$  of good  $s\ell$  for each unit purchased of asset  $j$ . They are permitted to deliver whatever they want on their own promises, but they are penalized according to  $\lambda$  for every unit they fail to deliver. In equilibrium the aggregate proportion of deliveries on  $s\ell j$  out of total promises on  $s\ell j$  must indeed be  $K_{s\ell j}$ .

Dubey, Geanakoplos and Shubik (1989) show that under quite general conditions, for any default penalties  $\lambda > 0$ , a  $GEI_\lambda$  equilibrium exists. For sufficiently harsh  $\lambda$ , there will be no default and  $GEI_\lambda$  reduces to GEI. For sufficiently lenient  $\lambda$ , there is equilibrium default. One interesting point is that on account of superior risk sharing, when markets are incomplete it may be Pareto superior for a society to choose lenient penalties and large defaults.

Another interesting problem arises when we combine the strategic default of individuals with the default of firms. With limited liability, firms cannot be punished for defaulting. On the other hand, if they are expected to default, they will have trouble raising money. The precise circumstances under which equilibrium exists with limited liability could be a revealing guide to the proper formation of bankruptcy law.

The GEI model specifies the asset structure exogenously. Some critics have called this a grave drawback to GEI analysis. As we have said earlier, to the extent that one can derive results independent of the asset structure, it is not a weakness of the model but a strength. On the other side of the coin, the extensions to GEI indicated above go a long way toward making the formation of the asset structure endogenous. The possibility of default is a crucial ingredient. If this is combined with asymmetric information, then I believe a natural model explaining why some assets would not be traded even if they were available could easily be developed. Production is a major source of asset endogeneity. The payoffs from the shares of firms, the securities they issue, and derivative securities like options, are all endogenous. Finally, there have also been several recent studies for the formation of assets with set-up costs and transactions costs [see Allen and Gale (1987), Duffie and Jackson (1988)].

Once a GEI model with endogenous asset formation is developed, it will

become still more useful to refine the measures of suboptimality and indeterminacy that we use. Any reasonable model of asset formation will leave the markets incomplete, and presumably the equilibria will be sub-optimal and (with financial assets) indeterminate. If we had a rough estimate of the *size* of the suboptimality or indeterminacy, we could speculate about conditions in which socially more important assets tended to be marketed before socially less important assets.

Let me conclude by mentioning four technical problems suggested by the papers in this issue. First, as PK show, the existence of GEI with options is a serious problem. [See also Krasa (1987).] I believe that the proper modeling of liquidity constraints (not a priori bounds on asset trades) will eliminate this problem. In general, existence of equilibrium obtains more easily once one allows for liquidity constraints. [See Dubey and Geanakoplos (1989a,b,c).] But that raises a second problem, namely the structure of liquidity constrained equilibria, (and short sale constrained equilibria).

Third, in paper no. 8 BCS developed a GEI model with differential participation. They confined attention to numeraire assets. With more general assets the existence problem would again have emerged, because the existence proofs in papers nos. 1–3 are not directly applicable, since they all use the Cass trick, which only works when all agents have access to the same markets. [See Younes (1988).] Finally, let me repeat that whereas most of GEI analysis carries over unchanged when time is extended more than two periods, there are still open questions concerning the existence and multiplicity of equilibria with long-lived, retraded assets, and differential participation. A multi-period GEI model with production also allows for an interesting interplay between incomplete asset markets, and incomplete markets for intermediate goods.

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