6

Number

> It was Frege who first forced both philosophers and mathematicians to acknowledge the lack of any philosphical account of the nature and epistemological basis of mathematics. He himself constructed a complete system of philosophy of mathematics . . . . [T]he philosophical system, considered as a unitary theory, collapsed when . . . shown to be incapable of fulfillment . . . by Russell's discovery of the set-theoretic paradoxes. . . . [M]uch as we now owe to Frege . . ., it would now be impossible for anyone to consider himself a whole-hearted follower. . . .
>
> Michael Dummett
> *Elements of Intuitionism*

These excerpts express what appears to be the prevalent attitude toward logicism among leading contemporary philosophers of mathematics. Despite this, I am still inclined to hold a logicist position. In what follows I will employ the theory of PRPs to defend it. Along the way I will reply to the standard criticisms of logicism, none of which hits its mark in my opinion. I begin by considering logicism in the context of arithmetic. This after all was what Frege himself was concerned with, and it is here that the doctrine is most defensible.

32. A Neo-Fregean Analysis

Ask a practicing mathematician what the Peano postulates for number theory are. If he does not have a philosophical or historical ax to grind, in the majority of cases he will state the following:

(1) 0 is a natural number.
(2) Natural numbers have unique successors.
(3) 0 is not the successor of anything.
(4) If the successor of x = the successor of y, then $x = y$.

(5) For all properties z, if 0 has z and if each successor of anything having z itself has z, then every natural number has z.[1]

Indeed, this informal statement of the Peano postulates is given in book after book on number theory. Now it is clear that the most direct and natural formalization of (1)–(5) goes as follows:

(1′) $NN0$
(2′) $NNx \supset NNx'$
(3′) $\neg(\exists x)\, 0 = x'$
(4′) $x' = y' \supset x = y$
(5′) $(\forall z)((0 \mathbin{\Delta} z \mathbin{\&} (\forall x)(x \mathbin{\Delta} z \supset x' \mathbin{\Delta} z)) \supset (\forall x)(NNx \supset x \mathbin{\Delta} z))$.

These then are what I will call the Peano postulates. In doing so I believe I am being faithful to actual informal mathematical practice.

Consider the following neo-Fregean analysis of natural number:

$0 =_{\mathrm{df}}$ the property of being a property with no instances.

the successor of $x =_{\mathrm{df}}$ the property of being a property with one more instance than the instances of x.

x is a natural number *iff*$_{\mathrm{df}}$ x has each property z that is had both by 0 and by the successors of things that have z.

Symbolized in L_ω with Δ this neo-Fregean analysis becomes:

$$0 =_{\mathrm{df}} [\neg(\exists u)\, u \mathbin{\Delta} y]_y$$

$$x' =_{\mathrm{df}} [(\exists u)(u \mathbin{\Delta} x \mathbin{\&} (\exists v)(v \not\mathbin{\Delta} u \mathbin{\&} y \doteqdot [w \mathbin{\Delta} u \vee w = v]^{uv}_{w}))]^{x}_{y}$$

$$NNx \; \textit{iff}_{\mathrm{df}} \; (\forall z)((0 \mathbin{\Delta} z \mathbin{\&} (\forall y)(y \mathbin{\Delta} z \supset y' \mathbin{\Delta} z)) \supset x \mathbin{\Delta} z)$$

where $y \doteqdot z$ *iff*$_{\mathrm{df}}$ $(\forall w)(w \mathbin{\Delta} y \equiv w \mathbin{\Delta} z)$. According to this analysis, natural numbers are fixed, purely logical objects, as logicists have thought. (They are not mere theoretical posits, as, e.g., Gödel thought.) This is one of the characteristic claims of logicism. Another characteristic claim of logicism is that Peano's postulates can be derived from principles of pure logic. And the following surprising little theorem results if the neo-Fregean definitions are adopted:

Theorem: Peano's postulates are theorems of the intensional logic T2 (i.e., if $\vdash_{\mathrm{PP}} A$, then $\vdash_{\mathrm{T2}} A$).[2]

Proof. Assume the definitions. Then (1′), (2′), and (5′) are immediate consequences of the axioms for quantifiers and truth-functional connectives. In addition, (3′) follows directly from the following instance of axiom $\mathscr{A}9$:

$$[\neg(\exists u)u \,\Delta\, y]_y \neq$$
$$[(\exists u)(u \,\Delta\, x \mathbin{\&} (\exists v)(v \not\Delta\, u \mathbin{\&} y \mathrel{\underset{\cdot}{=}} [w \,\Delta\, u \vee w = v]_w^{uv}))]_y^x.$$

Finally, the following is a theorem obtained by a few applications of axiom $\mathscr{A}10$ and an application of axiom $\mathscr{A}11$:

$$x = y \equiv [(\exists u)(u \,\Delta\, x \mathbin{\&} (\exists v)(v \not\Delta\, u \mathbin{\&} z \mathrel{\underset{\cdot}{=}} [w \,\Delta\, u \vee w = v]_w^{uv}))]_z^x =$$
$$[(\exists u)(u \,\Delta\, y \mathbin{\&} (\exists v)(v \not\Delta\, u \mathbin{\&} z \mathrel{\underset{\cdot}{=}} [w \,\Delta\, u \vee w = v]_w^{uv}))]_z^y.$$

(4′) follows immediately.

33. Reply to Criticisms

So far so good. Now let us see how the neo-Fregean definitions stand up against two of the standard criticisms of Frege's original definitions. (A third criticism, given by Charles Parsons, will be considered in §35.)

The easiest criticism to avoid is that of Robert Hambourger ('A Difficulty with the Frege-Russell Definition of Number'). The criticism is given in four steps. First, it is claimed that the following two propositions must both be true:

(a) There is at least one possible world in which the number 1 exists but in which some object that exists in the actual world does not exist.
(b) One and the same entity is the number 1 in each possible world; that is, it is not the case that one entity is the number 1 in one possible world while a different entity is the number 1 in another possible world. (p. 410)

Secondly, given the extensionality of sets, 'a set that exists in the actual world exists in a second possible world only if everything that belongs to it in the actual world exists in that second world' (p. 413). Thirdly, 'under the definition offered by Frege and Russell, ... 1 is the set of all unit sets' (p. 409).[3] Fourthly, it follows that, if (a) is true, then (b) is false.

This criticism is easy to avoid since it applies only to those analyses of the natural numbers that identify them with *extensional* entities, e.g., sets. However, the neo-Fregean analysis identifies natural numbers with *intensional* entities, namely, properties. According to the analysis, the natural numbers are fixed, purely logical objects. Although it is possible for their contingent instances to be different from what they actually are (for example, $[u$ is a president of USA in $1980]_u \, \Delta$ the successor of $[\neg(\exists u)u \, \Delta \, y]_y$, and yet it is possible that $[u$ is a president of USA in $1980]_u \not\Delta$ the successor of $[\neg(\exists u)u \, \Delta \, y]_y$), their fundamental logical relations necessarily remain unchanged and, hence, so do their mathematical relations. And that is all it takes to avoid Hambourger's criticism.

The next criticism of the logicist definitions is that of Paul Benacerraf ('What Numbers Could Not Be'). Before considering this criticism I must say more about how the neo-Fregean analysis dovetails with the theory of the logical structure of natural language. Consider the following English sentence:

There are 12 apostles.

The first thing to notice is that '12' occurs as a singular term, as the following intuitively valid argument shows:

There are 12 apostles.
$5 + 7 = 12$.
———
$\therefore$ There are $5 + 7$ apostles.

Next observe that in English 'There are *n* *F*s' is well-formed if and only if 'The *F*s are *n*' is also well-formed. (E.g., the following is perfectly good English. Question: 'How many is your party?' Answer: 'We are twelve.') Moreover, 'There are *n* *F*s' and 'The *F*s are *n*' seem to be synonymous. Finally, the following equinumerosity principle is intuitively valid:

There are exactly as many *F*s as *G*s *iff* for some number *n*, there are *n* *F*s and there are *n* *G*s.

and, more generally:

There are exactly as many things that have property *x* as there are things that have property *y* *iff* for some number *n*, there are *n* things that have property *x* and there are *n* things that have property *y*.

An adequate logical syntax for natural language should account for these elementary facts.

Now sentences such as 'There are twelve apostles' are not straightaway representable in first-order languages with intensional abstraction and predication. By contrast, sentences such as 'The apostles are twelve' are. In fact, the treatment of the copula arrived at in chapter 4 and the treatment of plurals[4] tentatively arrived at in chapter 5 lead directly to the following simple, indeed automatic, representation of 'The apostles are twelve':

$$\{v\colon Av\}\,\Delta\,12$$

where $\{v\colon Av\}$ is an extensional abstract contextually defined in terms of Δ, and 12 is a singular term.[5] Therefore, the easiest way to bring, e.g., 'There are twelve apostles' within the scope of first-order logic is to treat it as a (meaning preserving) transformation from 'The apostles are twelve'. This way no new underlying logical structures need to be posited; first-order quantifier logic with predication and intensional abstraction suffices. But now consider the above equinumerosity principle. Given this principle and given the indicated treatment of 'There are twelve apostles' and 'The apostles are twelve', it follows that the singular term 'twelve' must be semantically correlated with a property (call it x) such that x's instances include *only* properties having twelve instances and, for *all* properties having twelve instances, x's instances include at least one property having those twelve instances. The simplest such property is a property whose instances are all and only properties having twelve instances. But this is exactly what the number twelve is defined to be in the neo-Fregean analysis. Therefore, given this analysis, we arrive at an analysis of natural number that easily accounts for all the data from natural logic cited above. In addition, the analysis achieves this without having to posit any new logical structures. And finally, it achieves this in such a way that the Peano postulates are derivable from the provably sound and complete logic T2, a logic that is justified quite independently of issues in philosophy of mathematics.[6]

With these conclusions in hand I am ready to consider Paul Benacerraf's influential criticism of the various analyses of natural number, including Fregean analyses. Omitting whatever implicit premises Benacerraf might have in mind, one may summarize the criticism by means of the following argument:

> There are many different things that, for all we know, the natural numbers could be.
>
> ---
>
> ∴ The natural numbers could not be any of them.[7]

Now although this argument is invalid,[8] its force is to point up a problem: since there are several non-equivalent candidate analyses of the natural numbers and elementary number-theoretic language, we need a rationale for selecting an analysis as the correct one. This is Benacerraf's challenge to us. I will try to meet it by sketching a rationale for selecting the neo-Fregean analysis.

There is no chasm separating elementary number-theoretic language from the idiom of cardinality that is built into the logical structure of natural language. Elementary number-theoretic language is part of natural language. Therefore, the best analysis of elementary number-theoretic language is the one that is part of (entailed by) the correct analysis of the logical syntax of the idiom of cardinality that is built into natural language.[9] For this reason, the problem raised by Benacerraf is a special case of the general problem of finding a rationale for selecting a theory of logical syntax for natural language as the correct one. Thus, the problem is a special case of Quine's indeterminacy problem in the theory of natural language.

What is this indeterminacy supposed to be? A careful analysis of Quine's skeptical attack shows, I believe, that it is at worst a fancy case of underdetermination (though Quine attempts to deny this in 'On the Reasons for the Indeterminacy of Translation'). And many commentators are beginning to see the matter this way. Now underdetermination is a problem that besets virtually all theories regardless of subject matter. Take virtually any subject matter and virtually any body of data concerning that subject matter. Typically there will be several candidate theories that provide acceptable accounts of the data. The rational way to decide among such competing theories is on grounds of naturalness, simplicity, and elegance. If these grounds are used elsewhere in theory to solve the problem of underdetermination, it would be an unreasonable use of a double standard to depart from this practice only in the case of the theory of logical syntax or the philosophy of mathematics.

Quite independently of issues in philosophy of mathematics I had already arrived at a theory of logical syntax that leads directly and almost automatically to the neo-Fregean analysis of natural number. This analysis serves to explain in a simple and natural way a variety of syntactic, semantic, and logical phenomena in natural language. In addition, it does so without having to posit any new logical structures. Finally, this is achieved in such a way that the Peano postulates are derivable from the previously arrived at logical theory. For these (and other) reasons, the neo-Fregean analysis seems simpler and more natural than its competitors. If indeed it is, then we are justified in identifying it as the correct analysis. And that, in my view, is the basis for a solution to the problem raised by Benacerraf's criticism.[10]

34. The Derivation of Mathematics from Logic

I have shown that the Peano postulates (1)–(5) stated above can be derived from principles of pure logic. How, then, from Peano's postulates does one go on to derive the rest of the elementary arithmetical truths expressed by sentences built up from $=$, NN, 0, and $'$? Consider an example. If $A(x)$ is the formula $NNx \,\&\, (x \neq 0 \supset (\exists y)(NNy \,\&\, x = y'))$, how from Peano's postulates does one infer that $(\forall x)((A(0) \,\&\, (\forall x)(A(x) \supset A(x'))) \supset (\forall x)(NNx \supset A(x)))$? In particular, how does one infer it from postulate (5), which says that, for all properties z, if 0 has z and if each successor of anything having z itself has z, then every natural number has z, i.e., $(\forall z)((0 \,\Delta\, z \,\&\, (\forall x)(x \,\Delta\, z \supset x' \,\Delta\, z)) \supset (\forall x)(NNx \supset x \,\Delta\, z))$? What the practicing mathematician typically would do (overtly or covertly) is to apply the following trivial validity:

> x has the property of being a natural number that is distinct from 0 only if x is the successor of some natural number *iff* x is a natural number that is distinct from 0 only if x is the successor of some natural number.

i.e.,

$$x \,\Delta\, [NNx \,\&\, (x \neq 0 \supset (\exists y)(NNy \,\&\, x = y'))]_x \\ \equiv (NNx \,\&\, (x \neq 0 \supset (\exists y)(NNy \,\&\, x = y'))).$$

Given this trivial validity, the arithmetical truth I wanted to derive follows immediately from postulate (5).

Now let me be very clear here. I am not talking about what the philosophically stern extensionalist logician would say ought to be done. I am talking about what a practicing mathematician in fact would typically do in making the inference from his own working property-theoretic statement of Peano's postulates, i.e., what his actual thought step would typically be as he reasons. I claim that what the practicing mathematician would typically do is to make the inference in question as a *purely logical* step. It would be perverse to insist that in making the inference he appeals to some further non-logical principle that he had omitted by oversight from his list of mathematical axioms (i.e., from his property-theoretic Peano postulates).

But notice that if we are given the single additional validity

$$x \,\Delta\, [NNx \,\&\, (x \neq 0 \supset (\exists y)(NNy \,\&\, x = y'))]_x \\ \equiv (NNx \,\&\, (x \neq 0 \supset (\exists y)(NNy \,\&\, x = y')))$$

then the complete theory for the structure $\langle NN, =, 0, ' \rangle$ is derivable from the pure logic T2.[11] One might argue against this that the above principle is not logically valid because, after all, closely related principles might somewhere down the line produce logical paradoxes. However, this is like arguing, e.g., that 'it is true that snow is white *iff* snow is white' is not logically valid because closely related sentences might give rise to the Epimenides paradox. Or even worse, that this is not oxygen because closely related gases might be noxious to humans. What is important here is that the particular principle in question is logically valid. And it unquestionably is.

Now I call formulas of the form

$$\langle v_1, \ldots, v_j \rangle \,\Delta\, [A]_{v_1 \ldots v_j} \equiv A(v_1, \ldots, v_j)$$

principles of predication. Consider the trivial principles of predication that are needed to derive from Peano's postulates classical number theory (with $+$ and $\cdot$) and real and complex analysis.[12] As in the above case, it is clear that these few principles of predication are used in making purely logical inferences. They are not newly discovered non-logical principles that the mathematician forgot to include among his mathematical axioms. The particular principles of predication used are unquestionably logically valid. And so it is that from the pure logic T2 and a few additional validities classical mathematics can be derived.

35. Reply to Criticisms

Constructivism

The constructivist would object to the above derivation of classical mathematics on the grounds that some of the principles of predication used are non-constructive in character and, hence, that they are not valid. But a trap of sorts has been set for the constructivist. Recall the principle of predication discussed earlier:

$$x \,\Delta\, [NNx \,\&\, (x \neq 0 \supset (\exists y)(NNy \,\&\, x = y'))]_x$$
$$\equiv (NNx \,\&\, (x \neq 0 \supset (\exists y)(NNy \,\&\, x = y'))).$$

Since the range of all variables in the intensional abstract $[NNx \,\&\, (x \neq 0 \supset (\exists y)(NNy \,\&\, x = y'))]_x$ is restricted to natural numbers, this principle is the very paradigm of a constructive principle as conceived by the typical constructivist. Therefore, the constructivist has no choice but to accept it. Hence, if it is the case that when properly analysed NN, 0, and $'$ turn out to be non-constructive notions, then the constructivist is committed to accepting a non-constructive principle of predication. And if he is committed here, it is hard to see what grounds he could have for drawing the line when it comes to the classical theory of the real numbers. So in this way the consistency of the typical constructivist's philosophy of mathematics turns on an issue in philosophical analysis and the theory of logical syntax for natural language, namely, the issue of how various numerical constructions found in natural language are to be properly analysed. However, I have already given evidence for concluding that the neo-Fregean definitions are the right ones. And according to these definitions the natural numbers and the arithmetical operations are distinctly non-constructive. So unless the constructivist can meet the challenge either to discredit the evidence or to produce a better theory of the logical syntax of natural language (frequently the constructivist does not even seem to be aware of this challenge), it would appear that his philosophy of mathematics is not consistent.

Set Theory

Perhaps the most common objection to logicism these days is that the logicist construction of classical mathematics makes use of set theory and yet set theory is not part of logic. I agree that set theory is not part of logic; indeed, that was part of the thesis of the preceding chapter. But this fact is irrelevant to the logicist construction I am advocating, for this construction makes no use of set theory. Rather, the background theory is logic, specifically, first-order logic with intensional abstraction and predication.

Axiom of Infinity

The next criticism of logicism is that it needs to assume an axiom of infinity yet such an axiom lies outside the province of logic *per se*. While this criticism does tell against several logicist constructions (most notable of which is the type-theoretic construction in *Principia Mathematica*), it does not apply to the one proposed here. The various principles of predication that are employed in this construction are as close as I come to assuming an axiom of infinity. Consider, e.g., the principle of predication:

$$x \,\Delta\, [NNx]_x \equiv NNx.$$

To see that this trivial validity is not an axiom of infinity, adjoin it to elementary first-order extensional logic with identity. The existence of a property having infinitely many instances still cannot be inferred. It can be inferred only when appropriate auxiliary laws for NN—specifically, $NN0$ and $(\forall x)(x \neq \overbrace{x'''\cdots'}^{n})$, for all $n \geq 1$—are adjoined. The reason for this is that these auxiliary laws for NN, and not the principle of predication $x \,\Delta\, [NNx]_x \equiv NNx$, are what insure the existence of the infinitely many entities satisfying the predicate NN. Therefore, the auxiliary laws for NN are what yield the infinite ontological commitment. The principle of predication merely insures that the entities satisfying the predicate NN are indeed instances of the property that is expressed by NN. To be sure, given the neo-Fregean definitions, the auxiliary laws for NN can be derived in the intensional logic T2. But this is no embarrassment to logicism, for the infinite abstract ontology of intensional logic was already justified (chapter 1) quite independently of issues in philosophy of mathematics. Hence, unlike most logicist constructions, the neo-Fregean construction needs no special axiom of infinity, nor does it make any ontological commitments motivated by extra-logical considerations.

Incompleteness

The next criticism is based on Gödel's first incompleteness theorem.

The criticism is this: since, given Gödel's theorem, there exists no complete recursive axiomatization of number theory, not all truths of mathematics can be derived from logical validities; hence, logicism is false.

Notice, however, that this criticism goes through only when it is assumed that the validities have a complete recursive axiomatization. But this assumption seems to me to be false. For, given the logicist analysis of number, what Gödel's theorem shows is just that the validities have no complete recursive axiomatization.

This defense of logicism depends on how the concept of a logical validity is defined. I will take up this topic in §47, where the concept is defined formally. Given that definition, however, the defense of logicism is sustained. Although at this stage of the book it is not possible to go over this material in detail, I can say a few words about the underlying philosophical issue.

Informally speaking, the valid propositions are those that must be true in virtue of their logical form. What is the logical form of a proposition? Given the general framework of PRPs, there are two opposing views. On the first view the logical form of a proposition is merely the abstract shape of its decomposition tree as determined by the inverses of the type 2 fundamental logical operations (Conj, Neg, ...), where the particular identity of the various nodes in this decomposition tree is disregarded. The second view is just like the first except that the identity of the purely logical nodes (e.g., nodes occupied by identity, necessary equivalence, predication, etc.) is counted in. According to the first view of logical form, the valid propositions do have a complete recursive axiomatization. According to the second view, they do not. The second notion is, I believe, the right one. For observe that it is only on the second view that elementary truths involving identity and necessary equivalence—e.g., the propositions that $(\forall x)\, x = x$ and that $(\forall x)\, x \approx_N x$—are counted as valid. However, if the purely logical relations identity and necessary equivalence are counted in, then surely there can be no rational grounds for not counting in the purely logical relation predication.[13] But when the predication relation is counted in, two important consequences follow immediately. First, given the neo-Fregean analysis of number, the truths of mathematics turn out to be validities. Second, there exists no complete recursive axiomatization of the validities.

Logicism in no way requires that there should be a complete recursive characterization of the validities; it requires only that the truths of mathematics should be validities.[14] And given the neo-Fregean definitions and the definition of validity, this is so.

Analyticity

While on the topic of validity, I should mention a criticism of logicism derived from Quine's attack on the notion of analyticity. Logicists claim that mathematics is analytic. The Quinean criticism is simply that the concept of analyticity is undefinable; hence, the logicist claim is not meaningful and, as such, is not true. However, the notion of analyticity can be rigorously defined within the theory of PRPs (see §47). Appropriately enough, analytic propositions are exactly those that are valid. Consequently, the truths of mathematics are analytic in a clearly defined sense. And this conclusion is arrived at without significantly distorting the original Kantian usage of the term 'analytic'.

The Failure to Find an Intuitive Complete System of Predication Principles

Consider the particular principles of predication that are used in the derivation of classical mathematics from Peano's postulates. Philosophers who would doubt these principles often do so as a result of the following faulty line of reasoning: the easiest syntactic generalization on these principles gives rise to an inconsistent system of logic; therefore, the principles themselves are called into doubt.

This line of reasoning is faulty, for it is based on the assumption that the easiest syntactic generalizations on sentences that express validities should lead to valid general principles of logic. This is an unjustified assumption. (Indeed, the tendency to make easy syntactic generalizations is partly what makes one so susceptible to the paradoxes in the first place.) To be sure, some parts of logic behave rather like this. When this is so, work goes very smoothly for logicians. However, given its essential incompleteness, the logic for the predication relation does not in general behave in this formally orderly way. In fact, there might be no intuitive complete system of predication principles, not even a very complicated one. Simply because a number of particular principles of predication are obviously valid, why should they not at the same time be full-blooded creatures of this incompleteness in logic in just the sense that they always defy syntactic generalization? Their resistance to syntactic generalization provides no more evidence for their invalidity than does the unprovability of an intuitively true "Gödel sentence" provide evidence for its falsehood.

To my knowledge no one has shown that an intuitive complete system of predication principles does not exist. In historical time the search for such a system is very young. There are several systems of logic that without unreasonable distortion may be viewed as generalizations on intuitively valid principles of predication. To find a more nearly perfect system requires finding appropriate features to generalize on. It would be no disaster, however, if there were none. General systems of logic are nice, but they are not required in order to have knowledge of particular validities. Logicism in no way depends on the existence of a companion general system of logic. The paradoxes indicate difficulties in the science of logic but not in logicism, for logicism is a doctrine in the philosophy of mathematics that concerns only the logical status of mathematics.

Large Numbers

A quite specific line of attack is that logicism is committed to the sort of unconditioned totality that leads to the paradoxes:

> ...[T]he application of numbers must be so wide that, if *all* concepts (or extensions of concepts) numerically equivalent to a concept F are members of $N_x Fx$ [i.e., the number of Fs], then it is by no means certain that $N_x Fx$ is not the sort of 'unconditioned totality' that leads to the paradoxes.
>
> ...[I]n the most natural systems of set theory, such as those based on Zermelo's axioms, the existence of ordinary equivalence classes is easily proved, while if anything at all falls under F, the non-existence of Frege's $N_x Fx$ follows. (p. 185, Charles Parsons, 'Frege's Theory of Number')

I will give four responses to this criticism. Since each is promising, I will not settle on any one of them.

(1) For a moment assume the whole of the Zermelo-style theory of properties, including the axioms that are used to prove Cantor's theorem and, in turn, the theorem that there exists no property that everything has. Even in this case, it appears that there is no difficulty in defining numbers in the neo-Fregean way. Reminiscent of Hambourger's problem, the problem here of a clash with Zermelo-based theories appears to arise only if natural numbers are identified with entities that must be picked out by reference to their

instances or members. The problem does not appear to arise if natural numbers are identified with entities that are picked out, not by reference to their instances or members, but rather by use of a canonical rigid designator such as an intensional abstract. Let me explain.

Consider an arbitrary natural number, e.g., 1. Suppose that 1 is indeed the property of being a property having one more instance than the instances of 0; i.e., suppose that $1 = [(\exists u)(u \,\Delta\, 0 \;\&\; (\exists v)(v \not\Delta\, u \;\&\; y \mathrel{\dot{=}} [w \,\Delta\, u \vee w = v]^{uv}_{w}))]_{y}$. Assume further that every singleton property in fact has this property. Take the union of this property, i.e., the property of being an instance of an instance of 1. In a Zermelo-style theory of properties everything has this new property. However, in a Zermelo-style theory of properties it is also a theorem that there exists no property that everything has. Hence a contradiction. But notice that this argument makes use of an assumption, namely, the assumption that every singleton property has the property with which 1 has been identified. This assumption would hold if the extension of the predication relation were exactly what one would naively take it to be, i.e., if $\mathscr{G}(\bar{\Delta}) = \{xy \in \mathscr{D} : x \in \mathscr{G}(y)\}$. But the Zermelo-style resolution of the paradoxes in property theory modifies things just here. I see no way to derive the above assumption in a Zermelo-style theory of properties. Thus, I conjecture that the neo-Fregean analysis need not clash with Zermelo-style theories.[15]

(2) Even if I should be wrong about the foregoing, there is a way to preserve the neo-Fregean analysis within a more or less Zermelo-based theory of properties, namely, within the von Neumann-style theory.[16] The only thing that needs to be changed is the definition of NN. The new definition of NN is then derived from Dedekind's definition, not Frege's. Finite and transfinite arithmetic are still derivable. In addition, every singleton property that has properties at all has the property with which 1 is identified. (Likewise for doubleton properties and the number 2; and so on.) This is possible because natural numbers now are proper objects (i.e., unsafe properties). Hence, natural numbers are equinumerous with, e.g., $[x = x]_x$, a property that *all* safe objects have. So the threatened clash between the neo-Fregean analysis of number and this more or less Zermelo-based theory is avoided.

(3) A third way to respond to Parsons' criticism within the setting of a Zermelo-style theory goes as follows. Recall the reso-

lution of the paradoxes in property theory modifies things just here. sidered in §26. (Parsons himself favors such a resolution in 'The Liar Paradox'.) According to this resolution, in all contexts of speech and thought there is an implicit limitation on the universe of discourse. If this line is adopted, then it should be applied to numerical expressions and concepts as well. Hence, in the neo-Fregean definitions all variables within intensional abstracts ought to be grounded, i.e., they ought to be restricted in their range to a given antecedently fixed universe of discourse u. So, for example, relative to a context in which u is the implicit universe of discourse, the neo-Fregean definition of zero might go as follows:

$$0_u =_{\mathrm{df}} [x \,\Delta\, u \;\&\; \neg(\exists y)(y \,\Delta\, u \;\&\; y \,\Delta\, x)]^u_x.$$

Since in the context of arithmetic (finite or transfinite) the identity of u plays no special role (as long as u is sufficiently large for the purposes at hand), explicit occurrences of u may be suppressed.

(4) A fourth response to Parsons' attack is just to reject the Zermelo-based logics for the predication relation. After all, the Zermelo-based logics are not all that natural. And there are rather natural alternatives. For example, the logic for the predication relation that is based on Quine's NF has a variety of attractive features. E.g., according to it everything has the property of being something; i.e., the concept of a thing applies to all things. Even if one has reservations about the existence of a universal set, these reservations do not obviously carry over to properties or to concepts. There seems nothing absurd in holding that everything has the property of being something, that the concept of a thing applies to all things, including itself. This picture of properties and concepts is in fact very much like the view that Gödel himself arrived at.[17] Now in the logic for the predication relation that is based on NF, not only does everything have the property of being a thing, but also every property that is a property of exactly one thing has the property of being a property having one more instance than the properties having no instances; e.g., $(\forall x)[y = x]^x_y \,\Delta\, 1$. In view of this, Zermelo-based theories can hardly be used as the acid test for the correctness of the neo-Fregean analysis of number.

Given the foregoing options, it would seem, then, that Parsons' criticism of Frege's original analysis does not undercut the neo-Fregean analysis.

36. Epistemological Issues

We come finally to the epistemological worries about logicism. Consider the principle of predication that was used to derive from logic the complete theory for $=$, NN, 0, and $'$:

> x has the property of being a natural number that is distinct from 0 only if x is the successor of some natural number *iff* x is a natural number that is distinct from 0 only if x is the successor of some natural number.

i.e.,

$$x \,\Delta\, [NNx \;\&\; (x \neq 0 \supset (\exists y)(NNy \;\&\; x = y'))]_x$$
$$\equiv (NNx \;\&\; (x \neq 0 \supset (\exists y)(NNy \;\&\; x = y'))).$$

How do we know this? The answer is that we know it in the same way that we know, e.g., that x has the property of being Socrates *iff* x is Socrates. We know it in the same way that we know elementary axioms of first-order quantifier logic. We know it in the same way that we know that the various instances of modus ponens are valid. We know it in the same way we know, e.g., the theorems of T1 or T2, say, the theorem that $x \,\Delta\, [NNx]_x \vee x \not\Delta\, [NNx]_x$. These truths of logic are just as obvious, trivial, absurd to doubt, etc., as the principle of predication mentioned above. And as far as I can tell the same thing goes for every single one of the principles of predication that is used in the derivation of classical mathematics from Peano's postulates (which are themselves derivable from the logic T2).

The Need for a Complete Epistemological Account

The logicist has often been criticized for not having kept his promise to provide an account of mathematical knowledge. But this is wrong. The logicist does provide an account of mathematical knowledge; it is just not complete. The account, as far as it goes, is this. Elementary or complex mathematical truths are identical to complex logical validities. Thus, knowledge of mathematical truths is knowledge of complex validities. Hence, mathematical knowledge has the very same explanation—whatever it is— as does knowledge of complex validities. As far as I am concerned this is all the logicist needs to say, for the logicist is not required to give an account of how we come to have knowledge of complex validities. That is a general topic in epistemology, not philosophy of mathematics. It is a topic about which the logicist *qua* logicist should say nothing in particular except that knowledge of complex validities is not dependent on some further special kind of mathematical knowledge such as non-logical, pure mathematical intuition. The demand that the logicist provide a complete epistemological account is based on a confusion about the relationship between the special hybrid areas in philosophy such as philosophy of mathematics, philosophy of science, philosophy of law, etc., and the primary general areas of philosophy, namely, epistemology, metaphysics, and value theory. A successful theory in one of the special hybrid areas is one that can be integrated with successful general theories in the primary areas.

An Integrated Epistemological Account

My last remark brings me to a further epistemological worry about logicism, namely, the worry that the logicist account of mathematical truth is such that it cannot be integrated into any successful epistemology. This worry is voiced by Paul Benacerraf in his paper 'Mathematical Truth':

> ... two quite distinct kinds of concerns have separately motivated accounts of the nature of mathematical truth: (1) the concern for having a homogeneous semantical theory in which semantics for the propositions of mathematics parallel the semantics for the rest of language, and (2) the concern that an account of mathematical truth mesh with a reasonable epistemology. It will be my general thesis that all accounts of the concept of mathematical truth can be identified with serving one or another of these masters *at the expense of the other*. (p. 661)

Now the logicist theory of mathematical truth that I have given above yields the kind of homogeneous semantics mentioned in (1). So does this logicist theory mesh with a reasonable epistemology? I will try to show that it does. But I will not try to show this categorically; rather, I will attempt to establish my point relative to the plausible standard that Benacerraf suggests in his paper. Specifically, Hilbert's theory is identified there as the paradigm of a theory of mathematical truth that meshes with a reasonable epistemology.[18] Surely no one will object to this standard.

My argument is really very simple. On Hilbert's view it is a precondition of our having knowledge of complex logical and mathematical truths that we should have knowledge of the axioms and rules of elementary extensional logic with identity. Hilbert, however, provides no theory of how we come to have the latter kind of knowledge. Still, his theory presumably meshes with a reasonable epistemology. Now consider the axioms and rules of the logic for L_ω and consider the modest principles of predication that suffice for the derivation of classical mathematics from these axioms and rules. On the face of it, we seem to know these elementary axioms, principles, and rules in exactly the same way that we know the axioms and rules of elementary extensional logic with identity. As far as a reasonable epistemological account is concerned, then, knowledge of these elementary truths of intensional logic with predication would on the face of it appear to be quite on a par with knowledge of elementary truths of extensional logic with identity. A reasonable account of one would on the face of it appear to be easily adapted so as to provide an account of the other. Therefore, those who are skeptical are obliged to produce a reason for doubting that this is indeed the case. Without such reason we may safely conclude that, if there is a reasonable epistemological account of our knowledge of elementary extensional logic with identity, then there is a reasonable epistemological account of our knowledge of the above elementary principles of intensional logic with predication. Given this conclusion the rest of the argument is easy. For on Hilbert's view (like Frege's view before it), if there is a reasonable epistemological account of our knowledge of the premises and rules that we use in giving a proof, then there is a reasonable epistemological account of our knowledge of what we have proved. However, given the neo-Fregean analysis of number, we can prove the theorems of classical mathematics using known premises and rules. And given the previous conclusion, our knowledge of these premises and rules is assured of having a reasonable epistemological account. It follows, therefore, that relative to the plausible standard of reasonableness indicated earlier, our knowledge of classical mathematics is assured of having a reasonable epistemological account that meshes with the logicist theory of mathematical truth.

Now someone might object that, whereas the logicist is faced with the question of how he knows that his premises and rules are consistent, the follower of Hilbert is faced with no analogous question, for elementary extensional logic with identity has been proved consistent. In addition, someone might object that, whereas the logicist is faced with the question of how to account for the acquisition of new knowledge of the infinite not derivable from his present premises and rules (e.g., knowledge of the axiom of choice, knowledge of the continuum hypothesis or its negation, etc.), the follower of Hilbert is not faced with analogous problems. However, both of these objections, I would suggest, result from a failure to appreciate the force of Gödel's two incompleteness theorems.

Consider the first objection. The proof of the consistency of elementary extensional logic with identity is given within a background theory that is stronger than the original theory. In fact, the background theory includes elementary extensional logic with identity and concatenation. Moreover, this concatenation theory (as normally formulated) is equivalent to first-order arithmetic. But Gödel's second incompleteness theorem states that the consistency of first-order arithmetic cannot be proved without appealing to a still stronger background theory. And so it goes. Thus, if one is really in doubt about the consistency of elementary extensional logic with identity, it is difficult to understand why these consistency proofs should resolve the doubt. (Kleene reports that Tarski, when asked whether he felt more secure about classical mathematics from Gentzen's proof, replied, 'Yes, by an epsilon.') With regard to the question of how consistency can ever be really known, the follower of Hilbert is in no better position than the logicist. To be sure, the stronger a theory the greater its risk of inconsistency. But differences here are only in degree, not in kind. So regarding actual epistemological security, the logic espoused by the follower of Hilbert differs from the one espoused by the logicist only in degree.

Next consider the second objection. Gödel's first incompleteness theorem is that there is no complete axiomatization of first-order arithmetic. Therefore, regardless of how rich one's axiomatization of arithmetic is, it will always be possible to discover new elementary truths about the natural numbers that cannot be proved from those axioms. How is such new knowledge of the finite acquired? The follower of Hilbert would seem to be obliged to answer this question. However, with regard to the possibility of finding an answer that is compatible with a reasonable epistemology, this question seems to be analogous to the question facing the logicist. Thus, against these two objections, it still appears that, concerning its compatibility with a reasonable epistemology, logicism is not essentially worse off than Hilbert's theory.

But how do we know elementary logical validities? This question, like the question of how we know complex validities, is a general question in epistemology. It has no special dependence upon the problems in philosophy of mathematics that logicism is designed to solve. On this question the logicist and his competitors, including the followers of Hilbert, are all in the same boat. My own inclination is to think that some kind of rationalistic answer to this question is the only reasonable one. My reason, put bluntly, is that if someone is not endowed with powers of reason sufficient for *a priori* knowledge of elementary validities, he will alas not be intelligent enough to learn them *a posteriori* either.

Reduction and Degrees of Certainty

There are other epistemological criticisms of logicism, e.g., those of Poincaré, Wittgenstein, and Charles Parsons. In *Mathematical Knowledge* Mark Steiner cogently rebuts these criticisms. He goes on to defend the popular anti-logicist position that natural numbers are irreducible objects. The defense is itself epistemological, and it proceeds as follows. Arithmetic unreduced is more certain than arithmetic reduced *à la* logicism; at the same time, logicist reductions of arithmetic do not make any improvements on arithmetic unreduced; therefore, the best unified mathematical theory is one that keeps arithmetic in its unreduced form, rejecting all logicist reductions.

I find this argument dubious. But before I state my objection, let us take note of the view of reduction that is at work here. For if this view were correct, then my objection would be unfounded:

> One might wonder whether reduction is then ever possible, since all reductions seem to reduce a weak but more certain theory (Boyle's and Charles' laws) to a stronger but less certain theory (molecular theory). Such an objection would overlook a cardinal difference: bona-fide reductions effect changes that improve the original theories. They explain why the originals fail to be universally true. (p. 86, Steiner, *Mathematical Knowledge*)

I find this standard for what it takes to be a justified reduction far too high, for its leaves out the mundane cases of justified reduction. Consider the following mini-reduction:

This pencil weighs more than 1 gram. ⇒
The spatially present rigid collection of molecules that is producing these tactile sensations weighs more than 1 gram.

This reduction does not meet the above standard for two reasons. First, it is not clear that the mundane theory in any relevant sense fails to be universally true, and moreover, even if it should fail to be universally true, the explanation would be quite independent of the reduction. Secondly, the reduction does not *improve* upon the original mundane theory. There is nothing in the original theory that needs improving; it is just fine as it stands. And yet the reduction is as justified (this is not to say that it is as interesting) as any known reduction. The identification of this pencil with the spatially present rigid collection of molecules that is producing these tactile sensations is easily justified as follows. The molecular theory of perception provides us with good reasons for holding that space is populated widely by molecules and, in particular, that a spatially present rigid collection of molecules is producing these tactile sensations. Now consider the "dualist" theory that everyday material bodies inhabit the very same places as rigid collections of molecules. This theory is not defective in its observational content—that is not its flaw. What is wrong with it is that it is very uneconomical and, at the same time, it leaves unexplained the fact that so many properties of any given material body are the same as the properties of the rigid collection of molecules that inhabits the same place (e.g., it leaves unexplained why so many properties of this pencil are the same as the properties of the spatially present rigid collection of molecules that is producing these sensations). The reduction of everyday material bodies to rigid collections of molecules removes both of these defects. This alone is enough to justify the reduction. And this reduction is justified despite the fact that the reduced theory that this molecular collection weighs more than 1 gram makes no improvement on the unreduced theory and despite the fact that the unreduced theory (i.e., the everyday-body theory) is more certain than the reduced theory (i.e., the molecular-collection theory).

But look at the parallelism between the reduction of everyday bodies to rigid collections of molecules and the logicist reduction of natural numbers to properties. We have good logico-linguistic reasons (chapter 1) for holding that property theory is built into the logical syntax of natural language. And we have good logico-linguistic reasons (§33) for holding that a certain purely logical property is what corresponds semantically to a numerical adjective (e.g., '12') in such natural language sentences as 'There are 12 apostles'. Consider the "dualist" theory that what corresponds semantically to such numerical adjectives are different from the entities (i.e., the natural numbers) that correspond semantically to the typographically identical numerical expressions in the language of the science of arithmetic. What is wrong with this dualist theory is that it is uneconomical and, at the same time, it leaves unexplained the fact that so many properties of the two kinds of entities are the same. Reducing natural numbers to the indicated purely logical properties removes both of these defects. Now this justification parallels perfectly the one given for the everyday-body/molecular-collection reduction. Therefore, since the one is justified, so is the other. And this is so despite the fact that the reduction makes no improvement on any particular laws of arithmetic. Further, this would be so even if the unreduced theory (i.e., the laws of arithmetic) were more certain than the reduced theory (i.e., the laws of intensional logic with predication that are used in the reduction). And this concludes my argument.

Is the unreduced theory really more certain than the reduced theory? Many philosophers of mathematics today assume that it is, and they then use this assumption to upbraid the logicist for not providing mathematics with a new, epistemologically more secure footing than it had previously. I want to make a few comments against this popular position. To my mind the basic laws of arithmetic are not more certain than the associated laws of intensional logic with predication. After all, the latter laws are highly compelling. However, there is a deeper point here that many of the epistemological critics of logicism appear to have forgotten. Granted, many of us feel quite certain about the basic laws of arithmetic. Yet in reply to, say, someone who is seriously in doubt about the objective existence of natural numbers (e.g., a classical nominalist, a conventionalist, or an ontological relativist such as Carnap), what do we really have to say by way of justification for these laws?

First, there is the naive realist's commonsense reply that arithmetic is just one of the special sciences (i.e., one of the several epistemologically justified special disciplines), that its subject matter is the natural numbers, and that these laws are its first principles. But this reply hardly suffices, for it merely asserts the very sort of thing that is being challenged. Secondly, there is the intuitionistic reply that we know the basic laws of arithmetic by some special faculty of mathematical intuition.[19] But this will not do, for the existence of such a special faculty is as much in doubt as arithmetic itself. Thirdly, there is

Quine's reply that arithmetic is needed in the empirical sciences and, therefore, that it is justified by the empirical sciences taken as a whole. Quine's reply might be enough to overcome doubts about the objective existence of natural numbers. However, I think that we can do much better. For given Quine's view, arithmetic cannot be more certain than the empirical sciences. But this just seems false. Fourthly, there is Gödel's reply, which is an amalgamation of the intuitionistic reply and Quine's reply. Although Gödel's reply does not posit a faculty of mathematical intuition as an ultimate authority, it takes mathematical intuitions, collectively, to be a special body of empirical data to be explained (in part) by the associated special science of mathematics. According to Gödel, the best explanation of a (sound) mathematical intuition is that it is a special kind of perception of mathematical reality; our mathematical theories are just the best known systematization of these perceptions in much the same way that our physical theories are the best known systematization of our sense perceptions. Thus, mathematical theories are justified in much the same way that physical theories are justified. Although Gödel's reply, like Quine's reply, has certain virtues, it is not decisive. For it is not clear that Gödel's explanation of mathematical intuition is the best one.[20]

Finally, we come to the logicist's reply to doubts about the objective existence of natural numbers. In contrast to Gödel, the logicist need not appeal to mathematical intuition. Instead, his reply is that, upon analysis, the natural numbers are seen to be purely logical objects whose existence is independently justified by logic. Specifically, natural numbers are properties, and the ontology of properties, relations, and propositions is required in the best formulation of logic. In the same vein, the basic laws of arithmetic are, upon analysis, seen to be laws of logic. As such, they are justified in the way that laws of logic are usually justified. (As before, the logicist *qua* logicist is under no special obligation to say how our knowledge of laws of logic is justified.) Previously, skeptics in the philosophy of mathematics felt free to doubt such things as the existence of numbers. But now we see that such doubts are tantamount to doubting laws of logic. For this reason, if those who are skeptical about the foundations of mathematics are not careful, their doubts might compromise their commitment to logic.

By virtue of his ability to reply to doubts regarding the *foundations* of arithmetic (doubts about the existence of natural numbers are merely representative of a wide range of doubts that can be met in this way), the logicist provides a kind of epistemological justification not available under the competing philosophies of mathematics. And this justification does not depend on any epistemological resources not already employed in these competing philosophies. All that is required in addition is the neo-Fregean analysis of natural number.

How does one know that the neo-Fregean analysis of number is right? In the same way that one usually comes to know complex, informative definitions, namely, by having a theoretical justification. In the case of the neo-Fregean definitions, the justification is highly theoretical. In fact, it is pretty much the one laid out in the course of this chapter.

There is one more way to attempt to refute logicism epistemologically; it goes as follows. There are numerous arithmetic truths that seem obvious to the naive eye. Yet their neo-Fregean counterparts not only fail to be obvious to the naive eye but in fact are so complex that they almost defy understanding. This shows that the unreduced arithmetic truths are more certain than their neo-Fregean counterparts. And this, in turn, shows that the neo-Fregean analysis must be mistaken. For if it were correct, the unreduced arithmetic truths and their neo-Fregean counterparts would have to have the same degree of certainty.

To argue in this way, however, is to show that one has forgotten the paradox of analysis. In fact, the intensional logic developed for resolving the paradox of analysis (§20) nips this last little argument in the bud.

In this part of the work we have seen that the predication relation is what thrusts logic into incompleteness and indeed threatens it with paradox. And yet the predication relation also gives logic a great deal of power: the predication relation can claim responsibility for bringing all of classical mathematics into the domain of logic. This, however, is not the only way in which our conception of logic must be expanded. By the end of part III, it will have been expanded in several other surprising ways.