Online Appendix to

Counterfactuals with Latent Information

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Abstract

This appendix is organized as follows. Section A presents general results on single-player counterfactuals and contains additional examples based on the Roy model of Section 4, including informationally-robust rankings as described in Section 7. Section B provides a more formal and complete analysis of the entry game of Section 4.2. Section C studies counterfactual predictions in two-player zero-sum games. Section D studies counterfactuals in a first-price auction with reserve price. Section E discusses how various nominal assumptions in the model of Section 2 are in fact normalizations and are without loss of generality.

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A Single Player Example: Further Analysis

In Section 4.1, we explained how in single-player games, minimum counterfactual welfare is obtained with the minimally informative information structure, in which a player's signal is their observed action. We now give a general statement of this result:

Proposition 1 (Minimum single-player counterfactual welfare).
Suppose $N = 1$, and fix an observed decision problem $\mathcal{G} = (A, u)$ and moment restriction $M = \{\phi\}$. Define an information structure $\mathcal{I} = (S, \pi)$ by $S = A$ and such that $\pi(a, \theta) = \phi(a, \theta)$ for all a and θ. Then the obedient strategies are an equilibrium of $(\mathcal{G}, \mathcal{I})$, and $(\mathcal{I}, \sigma)$ induce the observed outcome. Moreover, for every counterfactual decision problem $\widehat{\mathcal{G}} = \left(\widehat{A}, \widehat{u}\right)$, the minimum expected counterfactual welfare across all counterfactual predictions is attained when the information structure is $\mathcal{I}$, and minimum counterfactual welfare is

$$\sum_{a \in A} \max_{\widehat{a} \in \widehat{A}} \sum_{\theta \in \Theta} \phi(a, \theta) \, \widehat{u}(\widehat{a}, \theta).$$

The proof is elementary, and follows the argument given in the text.

We next give a general statement of the result that with binary states, there is a maximally informative information structure which attains maximum counterfactual welfare. When $\Theta = \{\theta_1, \theta_2\}$, we can represent the player's belief conditional on their signal as the probability that the state is θ_1. For each observed action $a \in A$, there is an interval of beliefs for which that action is optimal, which we can denote by $[x_L(a), x_H(a)]$. Conditional on taking the action a, every realized belief must be in this interval. The Blackwell-most informative belief distribution consistent with the data must have all of the mass concentrated on the end points of this interval. Any information structure that generates this distribution of beliefs will maximize the player's welfare in all counterfactual decision problems. One such information structure is as follows: Let $\mathcal{I} = (S, \pi)$ where $S = A \times \{L, H\}$, and define the conditional probabilities $\pi(a, H, \theta)$ and $\pi(a, L, \theta)$ to solve the following system of equations:

$$\begin{aligned}
\pi(a, H, \theta_1) + \pi(a, L, \theta_1) &= \phi(a, \theta_1);\\
\pi(a, H, \theta_2) + \pi(a, L, \theta_2) &= \phi(a, \theta_2);\\
\frac{\pi(a, H, \theta_1)}{\pi(a, H, \theta_1) + \pi(a, H, \theta_2)} &= x_H(a);\\
\frac{\pi(a, L, \theta_1)}{\pi(a, L, \theta_1) + \pi(a, L, \theta_2)} &= x_L(a).
\end{aligned}$$

When $x_L(a) < x_H(a)$, there is a unique solution:

$$\begin{aligned}
\pi(a, H, \theta_2) &= \frac{1 - x_H(a)}{x_H(a)} \frac{\phi(a, \theta_1) - \frac{x_L(a)}{1 - x_L(a)}\phi(a, \theta_2)}{1 - \frac{x_L(a)}{1 - x_L(a)}\frac{1 - x_H(a)}{x_H(a)}};\\
\pi(a, H, \theta_1) &= \frac{\phi(a, \theta_1) - \frac{x_L(a)}{1 - x_L(a)}\phi(a, \theta_2)}{1 - \frac{x_L(a)}{1 - x_L(a)}\frac{1 - x_H(a)}{x_H(a)}},
\end{aligned}$$

and $\pi(a, L, \theta) = \phi(a, \theta) - \pi(a, H, \theta)$. Otherwise, if $x^L(a) = x^H(a)$ (so that there is a unique belief at which a is a best response, which must be the belief conditional on being recommended a) then there is a continuum of solutions to this system, where $\pi(a, H, \theta_1) = \pi(a, H, \theta_2)$. Thus, we can just take $\pi(a, H, \theta_1) = \phi(a, \theta_1)$ and $\pi(a, H, \theta_2) = \phi(a, H, \theta_2)$. With this information structure, the player has an optimal strategy to choose a after the signals (a, H) and (a, L). Moreover, $\mathcal{I}$ is Blackwell-more informative than any other information structure that rationalizes the data. We have proven the following proposition:

Proposition 2.

Suppose that $N = 1$ and $\Theta = \{\theta_1, \theta_2\}$, and fix an observed game $\mathcal{G} = (A, u)$ and moment restriction $M = \{\phi\}$. Let the information structure $\mathcal{I}$ be as constructed in the preceding paragraph. Then the obedient strategies are an equilibrium of $(\mathcal{G}, \mathcal{I})$. Moreover, for every counterfactual decision problem $\widehat{\mathcal{G}} = \left(\widehat{A}, \widehat{u}\right)$, the maximum expected counterfactual welfare across all counterfactual predictions is attained when the information structure is $\mathcal{I}$, and *maximum counterfactual welfare is*

$$\sum_{(a,k)\in A\times\{L,H\}} \max_{\widehat{a}\in\widehat{A}} \sum_{\theta\in\Theta} \widehat{u}\left(\widehat{a},\theta\right)\pi\left(a,k,\theta\right).$$

At a high level, this result depends on the fact that the set of distributions over beliefs partially ordered by mean-preserving spreads is a lattice when $|\Theta| = 2$. When $|\Theta| > 2$, this partially ordered set is no longer a lattice, and in particular, there need not be a most informative distribution of beliefs that rationalizes the data.

Finally, we argue that there is always a unique local counterfactual in single-player games:

Proposition 3.

Suppose that $N = 1$ and $M = \{\phi\}$. If the counterfactual game $\widehat{\mathcal{G}}$ is equal to $\mathcal{G}$, then there is a unique counterfactual welfare in all counterfactual predictions, which is welfare under ϕ:

$$\sum_{a\in A}\sum_{\theta\in\Theta} u\left(a,\theta\right)\phi\left(a,\theta\right).$$

The argument is that given in the text: Fix an information structure $\mathcal{I}$ and observed and counterfactual equilibrium strategies σ and $\widehat{\sigma}$ (that is, σ and $\widehat{\sigma}$ are optimal decision rules). Since the two games are the same, the payoffs in the games are respectively

$$U = \sum_{\theta\in\Theta}\int_{s\in S}\sum_{a\in A} u\left(a,\theta\right)\sigma\left(a|s\right)\pi\left(ds,\theta\right) \text{ and } \widehat{U} = \sum_{\theta\in\Theta}\int_{s\in S}\sum_{a\in A} u\left(a,\theta\right)\widehat{\sigma}\left(a|s\right)\pi\left(ds,\theta\right).$$

But $\widehat{\sigma}$ is a feasible strategy in the observed game, so the fact that σ is an equilibrium must be $U \geq \widehat{U}$. By an analogous argument, $\widehat{U} \geq U$, so in fact they are equal. Finally, by the definition of a counterfactual prediction, we must have that σ and $\mathcal{I}$ induce ϕ, so that U is equal to welfare under ϕ.

Ranking two counterfactuals We will next use the Roy model to illustrate the methodology for ranking games discussed in Section 7. Specifically, we ask for which pairs of counterfactual parameters z^1 and z^2 does the agent always attain higher welfare under z^1 than under z^2, when we restrict attention to those information structures which are consistent with the observed outcome. Figure 1 plots the set of pairs $\left(\widehat{U}^1, \widehat{U}^2\right)$ of agent welfare obtained for the pairs $(z^1, z^2) = (0.5, 0.75)$ when the observed outcome corresponds to $\alpha = 0.375$, both under the assumption of fully-observable outcomes (the blue set) and partially-observable outcomes (the red set).

The picture clearly shows that agent is unambiguously better off under z^2. This can be seen from the fact that all of the sets lie above the 45 degree line. Indeed, this conclusion is theoretically trivial: the counterfactual with z^2 has payoffs that are pointwise higher, so that the agent could achieve a higher payoff with z^2 than with z^1 simply by using whatever strategy was optimal for z^1. Note that while this conclusion is theoretically obvious, it is not apparent in Figure 1: For many pairs $z^2 > z^1$, the set of possible welfare outcomes for the agent overlap. It is only by plotting agent welfare resulting from joint counterfactual predictions that we can see that higher values of z dominate.

Nonetheless, this example illustrates the power of fixing information when computing informationally-robust rankings: Without holding information fixed, there would be no dominance ranking between z^1 and z^2, whenever the two are sufficiently close together.

Welfare versus behavior In Section 4.1, we primarily focused on the player's welfare. This is not the only counterfactual outcome of interest. More broadly, we may ask how the player's *behavior* could change in the counterfactual, i.e., the probability of opting in for each state. While we do not analyze this question in detail, we can say that there are generally much weaker restrictions on behavior than on welfare. This is illustrated in Figure 2, which depicts the total probability that the player opts in as we vary z, for the cases considered above.

The left panel describes the counterfactual probability of opting in when we observe the entire outcome, including the state distribution when the player opts out. When the observed

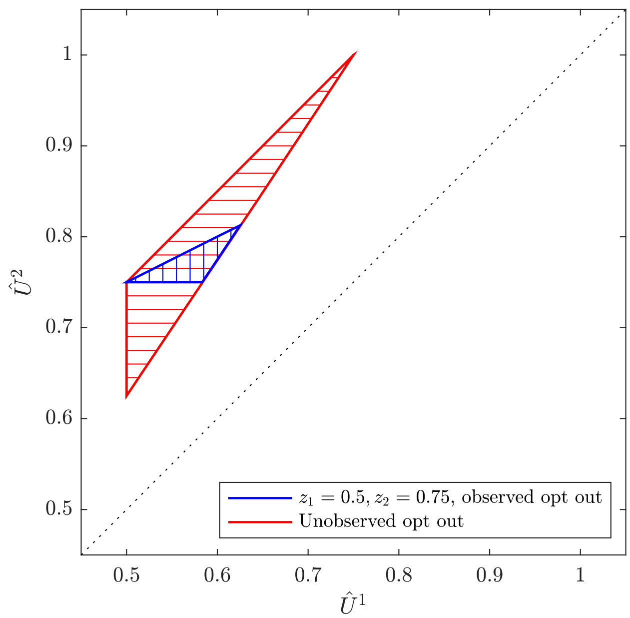

Figure 1: Ranking counterfactuals in the Roy model.

outcome is consistent with either either no information (the green curve) or full information (the blue curve), there is generically a point prediction for counterfactual behavior. However, for no information and $z = 0$, there are counterfactual predictions consistent with any opt-in probability between zero and one. This is true even though there is a point prediction for counterfactual welfare, simply because when $z = 0$, the player is indifferent between actions. For the intermediate case of partial information, there is always a fat set of counterfactual opt-in probabilities. Again, this is true even when $z = 0$, when there is a point prediction for welfare.

The counterfactual prediction for behavior when we do not observe the state after opting out is depicted in the right panel of Figure 2. The prediction is even more permissive in this case: For every $z > 0$, any opt-in probability between $1/2$ and 1 is consistent with all three cases considered. For, in each of these examples, the player must always opt in when the state is good, and there is a state distribution that rationalizes the player's observed decision to opt out when $z = 0$ but such that they would strictly prefer to enter if $z > 0$.

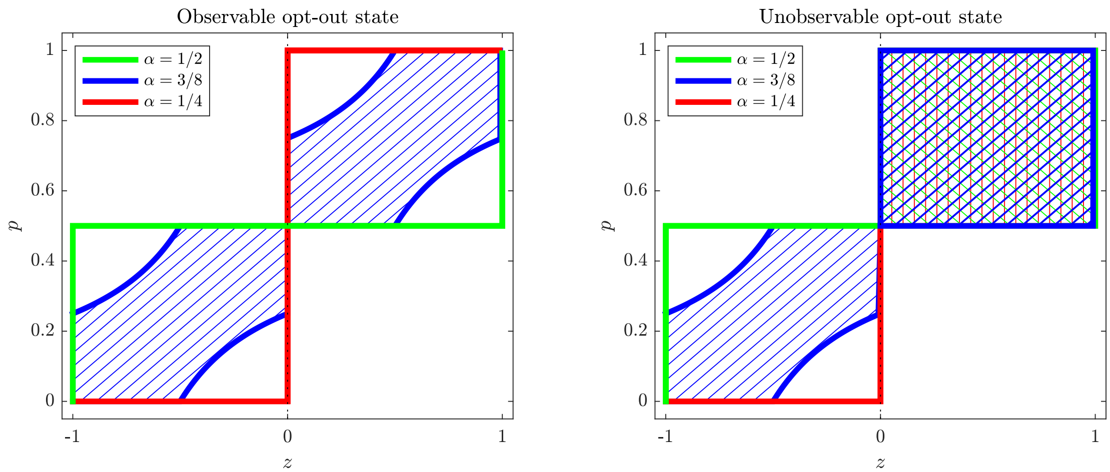

Figure 2: Counterfactual probability p of choosing the action $a = 1$.

B Entry Game: Further Analysis

Linear program The linear program for maximum counterfactual producer surplus is

$$\max_{\overline{\phi} \geq 0} \sum_{(a,\widehat{a},c)} \left(\overline{\phi}\left(a,(E,N),c\right)(3-c_1) + \overline{\phi}\left(a,(N,E),c\right)(3-c_2) + \overline{\phi}\left(a,(E,E),c\right)(2-c_1-c_2)\right)$$

$$\text{s.t.} \sum_{\widehat{a}} \overline{\phi}(a,\widehat{a},c) = \begin{cases} \frac{1}{4} & \text{if } (a_i,c_i) \in \{(E,0),(N,2)\} \ \forall i; \\ 0 & \text{otherwise;} \end{cases}$$

$$\sum_{(\widehat{a}_2,c_2)} \left[\overline{\phi}(N,E,\widehat{a}_1,\widehat{a}_2,c_1,c_2)(3-c_1) + \overline{\phi}(N,N,\widehat{a}_1,\widehat{a}_2,c_1,c_2)(1-c_1)\right] \leq 0 \ \forall (\widehat{a}_1,c_1);$$

$$\sum_{(\widehat{a}_2,c_2)} \left[\overline{\phi}(E,E,\widehat{a}_1,\widehat{a}_2,c_1,c_2)(3-c_1) + \overline{\phi}(E,N,\widehat{a}_1,\widehat{a}_2,c_1,c_2)(1-c_1)\right] \geq 0 \ \forall (\widehat{a}_1,c_1);$$

$$\sum_{(\widehat{a}_1,c_1)} \left[\overline{\phi}(E,N,\widehat{a}_1,\widehat{a}_2,c_1,c_2)(3-c_2) + \overline{\phi}(N,N,\widehat{a}_1,\widehat{a}_2,c_1,c_2)(1-c_2)\right] \leq 0 \ \forall (\widehat{a}_2,c_2);$$

$$\sum_{(\widehat{a}_1,c_1)} \left[\overline{\phi}(E,E,\widehat{a}_1,\widehat{a}_2,c_1,c_2)(3-c_2) + \overline{\phi}(N,E,\widehat{a}_1,\widehat{a}_2,c_1,c_2)(1-c_2)\right] \geq 0 \ \forall (\widehat{a}_2,c_2);$$

$$\sum_{(a_2,c_2)} \left[\overline{\phi}(a_1,a_2,N,E,c_1,c_2)(3+z-c_1) + \overline{\phi}(a_1,a_2,N,N,c_1,c_2)(1+z-c_1)\right] \leq 0 \ \forall (a_1,c_1);$$

$$\sum_{(a_2,c_2)} \left[\overline{\phi}(a_1,a_2,E,E,c_1,c_2)(3+z-c_1) + \overline{\phi}(a_1,a_2,E,N,c_1,c_2)(1+z-c_1)\right] \geq 0 \ \forall (a_1,c_1);$$

$$\sum_{(a_1,c_1)} \left[\overline{\phi}(a_1,a_2,N,E,c_1,c_2)(3+z-c_2) + \overline{\phi}(a_1,a_2,N,N,c_1,c_2)(1+z-c_2)\right] \leq 0 \ \forall (a_2,c_2);$$

$$\sum_{(a_1,c_1)} \left[\overline{\phi}(a_1,a_2,E,E,c_1,c_2)(3+z-c_2) + 7\overline{\phi}(a_1,a_2,N,E,c_1,c_2)(1+z-c_2)\right] \geq 0 \ \forall (a_2,c_2).$$

The program for minimizing counterfactual producer surplus is the same, except that we change the maximization to minimization.

Detailed calculations for entry counterfactuals We analytically construct the equilibria that attain the boundaries of the numerically computed counterfactual prediction in Figure 2. We do not give a proof that these bounds are optimal.

Both firms always entering is an equilibrium if $z \geq 1$, and the resulting payoff is $2(1+z) - 2 = 2(3+z) - 6$. This is the unique counterfactual prediction when $z > 1$, when entering becomes strictly dominant.

When $z < 1$, always entering is not an equilibrium. As long as $z \geq 0$, there is a mixed strategy equilibrium in which low-cost firms always enter and a firm with high cost enters with probability α, to make the other firm indifferent between entering and not entering:

$$3 + z - (1+\alpha)/2 - 2 = 0 \iff \alpha = z.$$

Thus, these strategies are an equilibrium for $z \in [0,1]$. Since this equilibrium makes high-cost firms indifferent between entering and not entering, the payoff of the high-cost firm is zero, and the payoff when the cost is low is just the high cost, which is 2, so that the overall payoff in this equilibrium is 2.

We now construct equilibria for $z \in [0,1]$ that attain the upper and lower bounds of the counterfactual welfare. Firms observe the outcome of a correlation device that produces signals (s_1, s_2) that are independent of the firms' costs and has the following probabilities:

s_1/s_2	0	1
0	$1-\beta-2\gamma$	γ
1	γ	β

where $\gamma \in [0, 1/2]$ and $\beta \in [0, (1-\gamma)/2]$. In the equilibria we now construct, low-cost firms ignore this signal and always enter, but a high-cost firm i enters if and only if $s_i = 1$.

The obedience constraints are as follows: Conditional on $s_i = 1$, the likelihood of the other firm entering is $(\gamma + 2\beta)/(2\gamma + 2\beta)$. The reason is that the other firm will enter regardless of their signal if their cost is low, but will only enter if they get the high signal when their cost is high. Conditional on this signal, the payoff from entering must be non-negative:

$$1 + z - 2\frac{\gamma + 2\beta}{2(\gamma + \beta)} \geq 0.$$

Similarly, conditional on being told to not enter and having a high cost, the payoff from entering must be non-positive:

$$1 + z - 2\frac{1 - \beta - 2\gamma + 2\gamma}{2(1 - \beta - \gamma)} \leq 0.$$

The equilibrium payoffs are

$$\frac{1}{2}\left(3 + z - 2\frac{1 + \gamma + \beta}{2}\right) + \frac{1}{2}\left[(\gamma + \beta)(1 + z) - 2\frac{\gamma + 2\beta}{2}\right].$$

To obtain minimum counterfactual welfare, we set $\beta = 1 - 2\gamma$ and make the obedience constraint for entering hold as an equality. Intuitively, we are pushing down welfare by having firms enter with high probability. Solving for β, we obtain

$$\beta = 1 - 2\gamma = \frac{z}{2 - z}.$$

It is straightforward to verify that the obedience constraint for entering is always satisfied with these values for β and γ and $z \in [0, 1]$. The resulting aggregate payoff is

$$2 + z - \frac{1}{2 - z}.$$

which coincides with the simulated minimum counterfactual welfare.

For maximum counterfactual welfare, we set $\beta = 0$ and make the obedience constraint for not entering hold as an equality. Intuitively, we increase welfare by having firms enter less often, so as to avoid the low-payoff from duopoly. Solving for γ, we obtain

$$\gamma = 1 - \frac{1}{1+z} = \frac{z}{1+z}.$$

So γ goes from 0 to 1/2 as z goes from 0 to 1. Note that when $\beta = 0$, the obedience constraint for entering is unambiguously satisfied, since the left-hand side reduces to 1/2, and the right hand side is always at least 1/2. The resulting payoff is

$$2(3+z) - 4 - 2\frac{z}{1-z},$$

which coincides with the simulation.

We next consider the equilibrium to enter if and only if $c_i = 0$. The payoff from entering with a low cost is clearly positive. The payoff from entering with the high cost is just z, and the payoff from entering with a low cost is $2+z$, so this is an equilibrium if $z \in [-2, 0]$. The resulting ex ante sum of payoffs is

$$\frac{3+z}{2} + \frac{1+z}{2} = 1 + z.$$

This is the unique counterfactual prediction when $z \in (-1, 0)$, and it is the lower boundary of the counterfactual prediction when $z \in [-2, -1)$.

If $z \in [-3, -2]$, there is an equilibrium in which low-cost firms mix over whether they enter, which results in a payoff of zero. This attains the lower boundary of the counterfactual prediction for $z \in [-3, -2]$.

Next we construct the producer surplus maximizing BCE when $z \in [-3, -1]$. Using a correlation device as we did above for $z \in [0, 1]$, we can coordinate the low firms' behavior so that firms enter only if they have low cost, a firm enters with probability one if they are the only low-cost firm, and when both firms have low-cost, and exactly one firm enters when both firms have low cost. This is obviously an equilibrium: Entering is strictly dominated for high signals, and if a firm with low cost does not enter in equilibrium, then the other low-cost firm must be entering, so the payoff from deviating would be $1 + z \leq 0$. The resulting aggregate payoff would be $3(3+z)/4$ (that is, 3/4 of the time exactly one firm enters, and it is a firm with low cost). This coincides with the upper boundary of the simulation.

Finally, we construct an equilibrium that attains the low payoff at $z = -1$. First, there is a correlation device as above when $\gamma = 1/2$. In addition, we assume that low-cost firms can observe the cost of the other firm. Consider the following strategies: A high-cost firm enters if and only if $s_i = 1$. A low cost firm enters with probability 1 if the other firm's cost is low or if the other firm's cost is high and $s_i = 1$. Otherwise, when the other firm's cost is high and $s_i = 0$, the low-cost firm does not enter. The high-cost firm gets zero surplus from entering. Relative to the equilibrium where firms enter if and only if the cost is low, producer surplus has dropped by 1/2, since 1/4 of the time it is a high cost firm entering as a monopolist rather than a low-cost firm. This equilibrium is knife edge: First, it depends on the fact that the loss from duopoly is the same as the high entry cost, so that the low-cost firm is indifferent to entering as a duopolist, and the high-cost firm is indifferent to entering as a monopolist. Second, if z is a little bigger than 1, low-cost firms would strictly prefer to enter when the high-cost firm enters, and if z is a little smaller than 1, the high-cost firm would be unwilling to enter.

Informationally-Robust Rankings in the Entry Game In this appendix, we conduct version of the joint counterfactual prediction analysis described in Section 7. In this case, we ask whether higher z are necessarily associated with higher payoffs for the firms. We conducted six versions of this counterfactual, which are depicted in Figure 3.

We computed joint predictions for counterfactual producer surplus for two different counterfactual games: $z = -0.6$ and $z = -0.4$. Three versions of this computation under different informational assumptions are depicted in Figure 3.

First, we computed a counterfactual prediction when we restrict attention to information structures that can rationalize the data used in Section 4.2, namely that when $z = 0$, firms enter if and only if their cost is low. Thus, in this example, we are actually computing joint predictions for three games, where $z \in \{-0.6, -0.4, 0\}$, and we impose a data restriction on the $z = 0$ game and plot the set of pairs of counterfactual producer surplus for the games with $z = -0.6$ and $z = -0.4$. In fact, for this case, the set of joint counterfactual predictions can immediately be read from Figure 2: For $z \in (-1, 0)$, there is a unique counterfactual prediction for aggregate payoffs under fixed information, and this prediction is increasing in z. Indeed, we see in Figure 3 that the joint counterfactual prediction when we have the data restriction is the single blue point. This point is above the 45 degree, meaning that the firms are unambiguously better off in the aggregate when $z = -0.4$ than they are when $z = -0.6$.

Second, for these same parameters, we computed a joint prediction when we impose that the same information structure is used for both values of z, but we allow all private-cost information structures (depicted in red). In this case, the joint prediction spans both sides of the 45 degree line, so that $z = -0.4$ does not dominate $z = -0.6$ with fixed information, when we do not have a restriction from the data. A fortiori, $z = -0.4$ does not dominate $z = -0.6$ under unrestricted information, even when we restrict to private-cost information structures.

Third, we computed the joint prediction when we allow all information structures. This most permissive joint prediction for producer surplus is in green. Again, it is clear that neither game dominates the other.

This example illustrates the potential benefit of combining methodologies: When we use only joint predictions for informationally-robust rankings, without a data-based restriction,

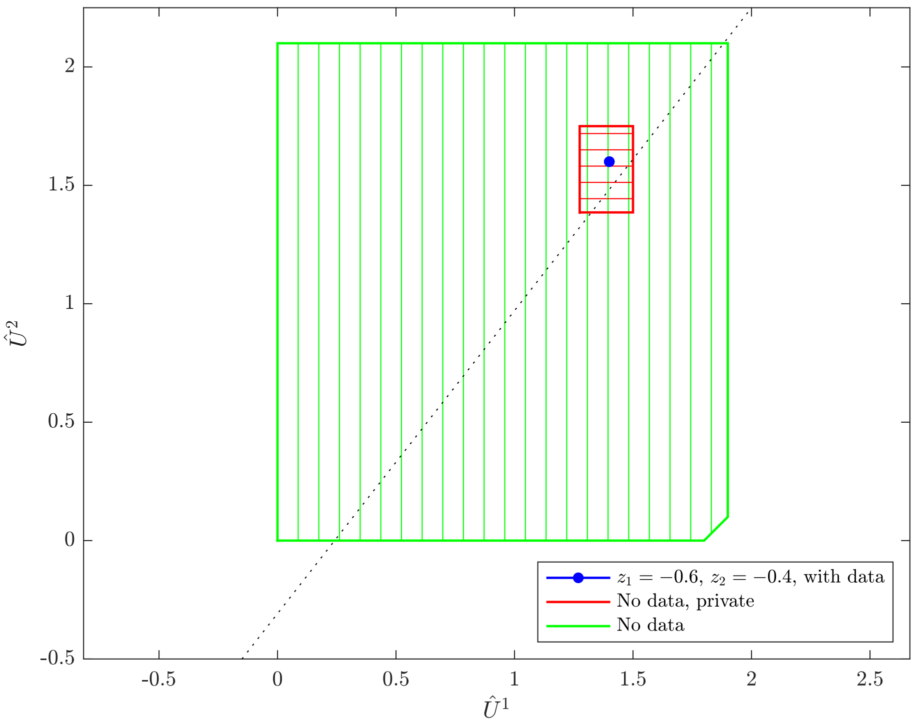

Figure 3: Joint counterfactuals in the entry game.

it is not possible to rank $z = -0.6$ and $z = -0.4$. But when we use data to further refine the joint counterfactual prediction, we do obtain an unambiguous ranking.

C Two-Player Zero-Sum Game

We now consider a setting with two players, binary actions, and binary states. The observed game is the following:

$\theta = 0$			$\theta = 1$		
a_1/a_2	0	1	a_1/a_2	0	1
0	$(2, -2)$	$(-1, 1)$	0	$(0, 0)$	$(-1, 1)$
1	$(-1, 1)$	$(0, 0)$	1	$(-1, 1)$	$(2, -2)$

In each state, the game has the form of an asymmetric matching pennies. Both states are equally likely, so that in expectation the game is symmetric. Thus, if the players have no information about the state, there is a unique equilibrium in which they both randomize with equal probabilities, and both players' payoffs are zero. If they have full information about the state, then there is again a unique (and symmetric) equilibrium in which they play $a = 0$ with probability 1/4 in state $\theta = 0$, and they play $a = 0$ with probability 3/4 in state $\theta = 1$. In both states, player 1's payoff is $-1/4$.

We assume that we have observed ϕ exactly, and $\phi(a, \theta) = 1/8$ for all (a, θ). This is the joint distribution of states and actions that arises under no information. In the counterfactual, we multiply all of the payoffs by a factor $2 - z$ in state 0 and by z in state 1, for some $z \in [0, 2]$. This is equivalent to varying the relative likelihoods of the two states. The observed game corresponds to $z = 1$. The counterfactual outcome of interest is player 1's payoff.

We numerically computed maximum and minimum payoffs for player 1 for a fine grid of z values. The range of counterfactual outcomes under variable and fixed information are depicted in Figure 4 as a function of z. When information is variable, then again, the only thing we learn from the data is that both states are equally likely. The gray lines represent upper and lower bounds on welfare. The range of possible outcomes is largest at $z = 1$, when the counterfactual game is a copy of the observed game. In this case, any payoff in $[-1/2, 1/2]$ can be attained with some information structure. The highest payoff of 1/2 can be achieved by letting player 1 observe the state and player 2 receiving no information. Under that information, there is an equilibrium where $\widehat{a}_1 = \theta$ and player 2 mixes with equal probabilities. Similarly, the payoff of $-1/2$ can be achieved by giving no information to player 1 and full information to player 2. In fact, it is a result of Peski (2008) that these are the information structures that achieve extreme welfare outcomes in any two-player zero-sum game, and it is not particular to our example.[1]

[1] Here is a sketch of the proof. Player 1's payoff in $(\mathcal{G}, \mathcal{I})$ is at least their maxmin payoff, where the max and min are taken over player 1 and player 2's strategies, respectively. Player 2 has the option to use a strategy that does not depend on their private information t_2, so player 1's maxmin payoff would increase if we restricted player 2 to use only those constant strategies. This is what happens if player 2 has no information. Next, if we look at information structures where only player 1 gets information, then it must be that player 1's payoff is maximized by having as much information is possible. For, any strategy under

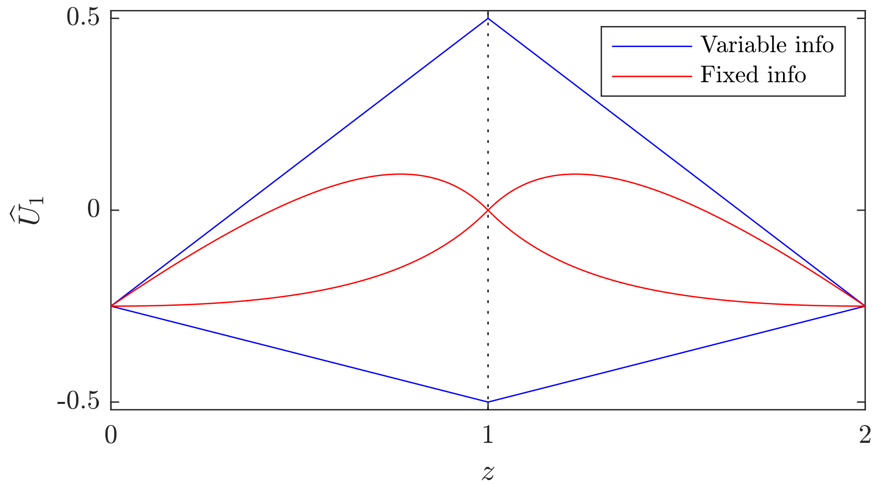

Figure 4: Counterfactual payoffs for player 1 in the zero-sum game.

Note that when $z = 0$ or $z = 1$, then payoffs are zero in one state, so that it is effectively a game with a single state, and thus the value of the game is uniquely pinned down independent of the information.

When we fix information, the range of counterfactual outcomes is tighter. Indeed, when $z = 1$, there is a unique counterfactual prediction when the counterfactual game coincides with the observed game. Once again, this is a general insight that is not particular to our example. In any two-player zero-sum game, if there is an information structure $\mathcal{I}$ and equilibrium σ that rationalizes the observed actions and in which player 1's payoff is u_1, then it must be that the zero-sum game $(\mathcal{G}, \mathcal{I})$ has a value which is u_1, and hence all equilibria have the same payoffs. This observation completes an analogue of Proposition 3 for zero-sum games:

Proposition 4 (Two-Player Zero-Sum Counterfactuals)**.**
Consider a two-player zero-sum game in which players' observed payoffs are $(u_1, -u_1)$*. If the counterfactual and observed games are the same, then under fixed information, there is point identification of the players counterfactual payoffs, which must be* $(u_1, -u_1)$*. Under*

partial information can be replicated under full information simply by "simulating" the noisy signal, so the effective strategy space is largest under full information. Finally, in the extreme case of full information/no information, the game is finite so the minimax theorem holds, and the maxmin payoff is player 1's equilibrium payoff.

unrestricted information, then a tight upper bound on player 1's payoff is given by what is attained when player 1 has full information and player 2 has no information, and a tight lower bound is what is attained when player 1 has no information and player 2 has full information.

Thus, it is a general phenomenon that there are point predictions for local counterfactuals in two-player zero-sum games under fixed information, although there is generally a fat set of counterfactual predictions under unrestricted information.

Returning now to the particular example, as z moves away from 1, the range of counterfactual payoffs expands, before contracting again as we approach the complete information extremes. Thus, the predictive power of fixed information is large when the counterfactual is close to the observed game, and it degrades as the counterfactual environment diverges from that which generated the data.

The broad economic conclusion is that player 1 prefers moderate z, while player 2 prefers extreme values. Specifically, when information is fixed and $|z-1| > 0.58$, then we can unambiguously say that player 1 is worse off and player 2 is better off in the counterfactual than in the observed game. When $|z-1| \leq 0.58$, then the change in welfare is ambiguous: player 1 may be better off or worse off, depending on the true information structure. A similar statement applies when information is variable, but the conditions for player 1 to be better off are more stringent, and we can unambiguously sign the change in welfare only when $|z-1| > 2/3$.

D First-Price Auction

Our final example is a private-values first-price auction (cf. Section 5). This setting is similar to the one initially studied by Syrgkanis et al. (2021), except that we consider counterfactuals with fixed information, whereas they allow unrestricted information that there are two bidders with values in $V = \{0, 1/9, \ldots, 8/9, 1\}$. We also restrict bids to be in the value grid, and we also assume that bidders do not bid more than their values. There is no reserve price in the auction. Bidders learn at least their own value, but may learn more. We assume that the values are iid uniform, and the econometrician observes either the BCE that minimizes the auction's revenue or the BCE that maximizes revenue (both of which are computed numerically). The counterfactual of interest is revenue as we vary the reserve price. In particular, does there exist a reserve price under which revenue unambiguously increases, relative to the observed game without a reserve price?

Let us first consider the case where the observed outcome was the revenue minimizing BCE. Figure 5 shows how the counterfactual prediction for revenue varies with the reserve price. In particular, the solid red curves represent maximum and minimum counterfactual revenue. There are two features to notice: First, even if the reserve price stays at zero, there is a fat set of counterfactual revenue levels. This indicates that there exist information structures that could induce the revenue minimizing BCE for which there are multiple equilibria, and that revenue varies across these equilibria. So, even if the reserve price does not change, revenue could in principle increase if the bidders coordinated on a different equilibrium. This multiplicity persists at higher reserve prices. However, for moderate reserve prices, the lower bound on revenue increases above the observed level. This lower bound is maximized at 5/9. At this reserve price, we can unambiguously say that regardless of the information and equilibrium, revenue would necessarily be higher than in the observed outcome. Note that since the lowest value is zero, it is necessarily the case that minimum revenue increases when the reserve price changes from 0 to 1/9, although it is not obvious that revenue should continue to increase in the reserve price beyond this point.

Figure 5 also shows how the counterfactual prediction if we allowed information vary, but held fixed the value distribution. For the lower bound, the predictions are not substantively different, although the upper bound on revenue is considerably more permissive. This is not surprising: The simulated data came from the revenue-minimizing information structure, so

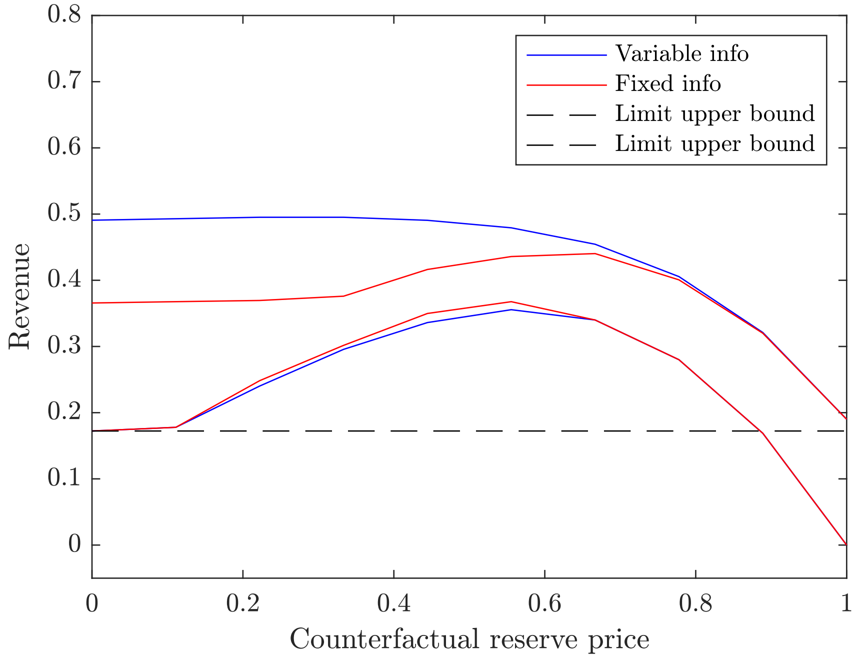

Figure 5: Counterfactual when observed outcome is the revenue minimizing BCE.

the fact that the lower red and blue curves nearly coincide is a reflection of the fact that the revenue-minimizing information does not vary significantly with the reserve price.

We next consider the case where the observed outcome is the revenue maximizing BCE. The corresponding counterfactual prediction is depicted in Figure 6. In this case, adding a reserve price cannot lead to a significant increase in revenue, and will necessarily cause revenue to decrease if it the reserve price is sufficiently high. Again, this prediction is substantively the same as what we would obtain with unrestricted information, although in this case it is the lower bound on revenue that is more permissive with unrestricted information. In fact, we can give an analytical justification for both the fact that maximum revenue is (nearly) decreasing in the reserve price, and also the fact that the fixed- and unrestricted-information bounds coincide. As discussed in Bergemann, Brooks, and Morris (2017, Section 5.4), under the hypothesis that bidders do not bid more than their values, there is an elementary lower bound on bidder surplus, which is the maximum payoff a bidder could obtain if others were bidding their values. With two bidders whose values are exactly uniformly distributed on $[0,1]$, and when the reserve price is r, the lower bound for a bidder with value $v \geq r$ is the maximum of

$$\max_{b\in[r,v]} (v-b)\,b = \begin{cases} \frac{v^2}{4} & \text{if } v \geq 2r; \\ (v-r)\,r & \text{if } r \leq v < 2r. \end{cases}$$

(If $v < r$, the lower bound on bidder surplus is zero.) The lower bound on ex ante bidder surplus when $r \leq 1/2$ is therefore

$$2\left[\int_{v=r}^{2r} (v-r)\,r dv + \int_{v=2r}^{1} \frac{v^2}{4} dv\right] = \frac{1}{6} - \frac{r^3}{3},$$

and when $1/2 \leq r \leq 1$, the lower bound is

$$2\int_{v=r}^{1} (v-r)\,r dv = \left.\left(v^2 - 2rv\right) r\right]_{v=r}^{1} = r - 2r^2 + r^3.$$

At the same time, total surplus when the reserve price is r is at most the expected highest value times an indicator for the highest value being above r, which is

$$\int_{v=r}^{1} vd\left(v^2\right) = \frac{2}{3}\left(1-r^3\right).$$

Thus, an upper bound on revenue with a reserve price r is

$$\overline{R}\,(r) = \frac{2}{3}\left(1-r^3\right) + \begin{cases} \frac{r^3}{3} - \frac{1}{6} & \text{if } r < \frac{1}{2}; \\ 2r^2 - r - r^3 & \text{if } \frac{1}{2} \leq r \leq 1. \end{cases}$$

We have plotted $\overline{R}$ in green in Figure 6. It is straightforward to verify that this function is decreasing. Moreover, Bergemann, Brooks, and Morris (2017) show that the bound is tight when $r = 0$, meaning that there exists an information structure and equilibrium in which revenue is $\overline{R}\,(0)$, so that in the limit when the value and bid grids fill in all of $[0,1]$, maximum

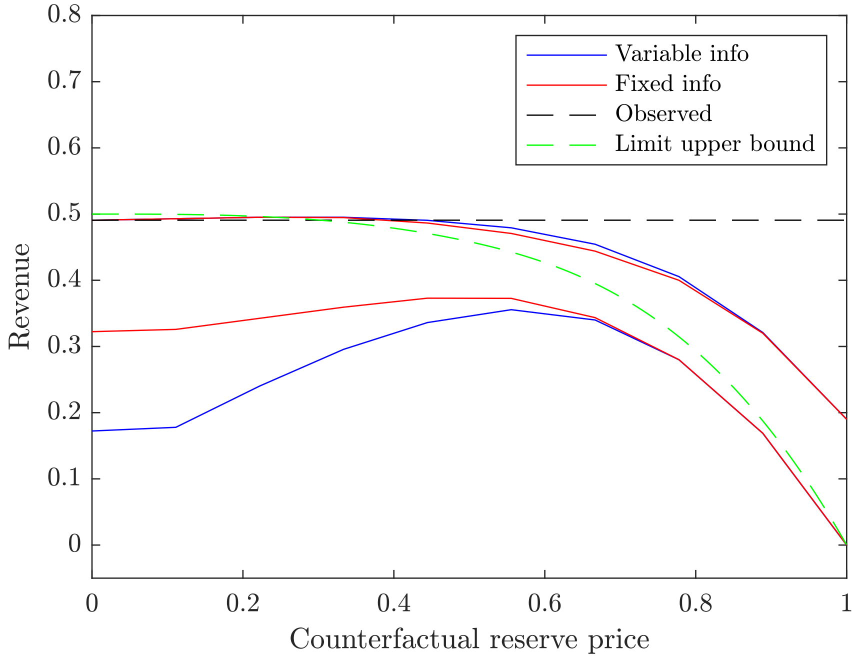

Figure 6: Counterfactual when observed outcome is the revenue-maximizing BCE.

revenue must be decreasing in the reserve price. We conjecture that this construction can be generalized to $r > 0$, so that in fact $\overline{R}(r)$ is maximum revenue across all BCE.

As a final note, this counterfactual exercise extracts as much from the data as possible about players' information, as it pertains to this particular counterfactual prediction. We may contrast this approach with one suggested by us in our analysis of BCE of interdependent value first-price auctions (Bergemann, Brooks, and Morris, 2017). In that paper, we identified a tight lower bound on the winning bid distribution across all BCE consistent with a given ex post value distribution. We suggested using that bound to partially identify the value distributions that can rationalize observed winning bids. This partially identified set could then be used to generate counterfactual predictions. Such an exercise would allow information to vary between the observed and counterfactual auctions. In contrast, the methodology in the present paper holds information fixed between observation and counterfactual. Our focus is also less on the identification of values than on the identification of information, although we could have also treated the value distribution as a latent variable to be identified from the BCE, in which case we would have been using the entire observed bid distribution to implicitly restrict the value distribution, rather than just the distribution of the winning bid.

E Innocuous Assumptions

Our model imposes a great deal of structure on the environment. In particular, we have assumed that information is described by a single information structure, utilities are known, the prior over the state is held fixed, and there is a single equilibrium that is played in the observed game and a single equilibrium in the counterfactual. At first glance, this structure seems restrictive for empirical applications in which the data is generated by many different instances from the observed game, and where conditions may vary from one instance to another. But, as we will now explain, these assumptions are without loss of generality and could be relaxed at the expense of a richer model.

1. All players receive signals from the same information structure. In practice, players with different characteristics, in different locations, or different points in time may receive qualitatively different forms of information. We may, however, consider these to be special cases of global description of players' information, where the heterogeneity in information is encoded as an extra dimension of signal. For example, suppose that for each $k = 1, \ldots, K$, a fraction $\beta_k \in [0, 1]$ of the data is generated when the players have common knowledge that the information structure is $\mathcal{I}^k = \left\{S_1^k, \ldots, S_n^k, \pi^k\right\}$. We could equivalently represent this economy with a new information structure in which $S_i = \sqcup_{k=1}^{K} \{k\} \times S_i^k$, i.e., each player's set of signals is a disjoint union of the k information structures, and

$$\pi(X, \theta) = \begin{cases} \beta_k \pi^k(Y, \theta) & \text{if } X = \{k\} \times Y \text{ for some } k; \\ 0 & \text{otherwise.} \end{cases}$$

In words, with probability one, all players get signals in the same S^k, and each k has probability β_k. Our counterfactual prediction implicitly allows for information structures of this form.

2. The utility functions $u_i(a,\theta)$ are known to the analyst. Uncertainty about preferences can be incorporated by expanding the state space. For example, suppose we start with a state space Θ, a moment restriction $M = \{\phi(a,\theta)\}$, and two possible utility functions u^1 and u^2. Then we can expand the state space to $\widetilde{\Theta} = \{1,2\} \times \Theta$, utility function $u(a,(k,\theta)) = u^k(a,\theta)$, and the moment restriction is

$$M = \left\{ \widetilde{\phi} \in \Delta\left(A \times \widetilde{\Theta}\right) \middle| \sum_{k=1,2} \widetilde{\phi}(a,(k,\theta)) = \phi(a,\theta) \right\}.$$

Thus, the prevalence of u^1 and u^2 in the population is a free variable, and is partially identified from the data.

3. The distribution over states μ is held fixed in the counterfactual. In fact, we can allow a different distribution $\widehat{\mu}$ in the counterfactual, as long as it is absolutely continuous with respect to μ, meaning that it can be written as $\widehat{\mu}(\theta) = \eta(\theta)\mu(\theta)$ for some $\eta : \Theta \to \mathbb{R}_+$, and the conditional distribution of signals remains the same, meaning that the joint distribution of signals and states in the counterfactual is $\hat{\pi}(ds,\theta) = \eta(\theta)\pi(ds,\theta)$. In particular, when we are only interested in varying the prior and the absolute continuity hypothesis is satisfied, then we can set the counterfactual utility to $\widehat{u}_i(a,\theta) = \eta(\theta) u_i(a,\theta)$, in which case equilibrium utility is simply

$$\begin{aligned}\sum_{\theta\in\Theta}\int_{s\in S}\sum_{a\in A}\widehat{u}_i(a,\theta)\sigma(a|s)\pi(ds,\theta) &= \sum_{\theta\in\Theta}\int_{s\in S}\sum_{a\in A}\eta(\theta)u_i(a,\theta)\sigma(a|s)\pi(ds,\theta)\\ &= \sum_{\theta\in\Theta}\int_{s\in S}\sum_{a\in A}u_i(a,\theta)\sigma(a|s)\hat{\pi}(ds,\theta),\end{aligned}$$

and the represented payoffs are equivalent to those that would obtain with the different prior. This is merely a reflection of the well-known indeterminacy of probabilities versus utilities in the subjective expected utility model, when utilities are state dependent (Savage, 1954; Anscombe and Aumann, 1963). Indeed, this transformation was being used in the single-player analysis of Section 4.1, which can be reinterpreted as variations of the prior.

4. All players play the same equilibria of the observed and counterfactual games. This is also without loss of generality. Suppose that the information structure is $\mathcal{I}$, and a share β_k of the data is generated from players who play strategies σ^k for $k = 1, \ldots, K$. The same outcome can be induced with a single information structure $\widetilde{\mathcal{I}}$, in which $\widetilde{S}_i = \{1, \ldots, K\} \times S_i$, $\widetilde{\pi}(\{k\} \times X, \theta) = \beta_k \pi(X, \theta)$, and strategies are $\tilde{\sigma}_i(a|(k,t)) = \sigma_i^k(a|s)$. In effect, the first coordinate of the new signal $\widetilde{s}_i$ is a public randomization device which is equal to k with probability β_k. Strategies on the larger space say to play σ^k when $X = k$.

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