The Scope of Sequential Screening with Ex-Post Participation Constraints

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Joint work with D. Bergemann (Yale) and G. Weintraub (Stanford)

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Problem: Sequential Screening

- When and how to sell when a buyer learns her valuation over time?
- Classic example: Airline tickets
- Initial purchase is based on an imperfect estimate: buyer's type could be leisure/business travelers (**Period 1**)
- Buyer knows true willingness-to-pay only at date of travel(**Period 2**)

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- Classic paper of Courty and Li (2000); also Akan et.al. (2015)
- Menu of upfront fees/refund contracts

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- Ex.1: in online shopping buyers can return purchases at low or no cost (Krähmer and Strausz 2015).
- Ex. 2: online display advertising markets: auction based and typical business constraint.

Online Display Advertising Motivation

Online Display Advertising: Waterfall Auction

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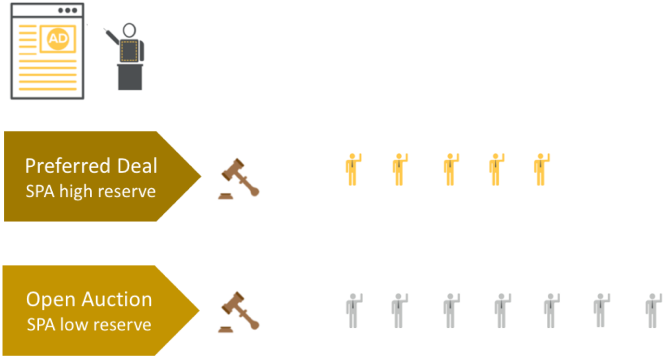

Think of period 1

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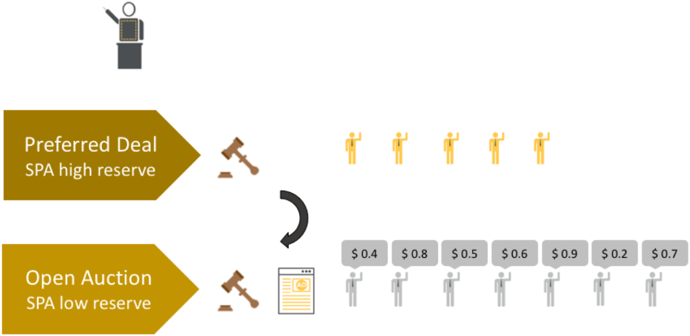

Think of period 2

This Paper

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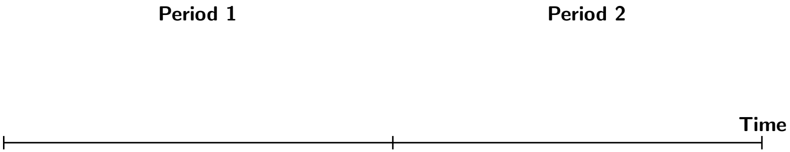
  - Cai et. al (2016) and Devanur & Weinberg (2017) dual approach also applies
- (Partially) Shed light on practical mechanisms as effective price discrimination devices such as Waterfall Auctions

Model: Mechanism Design Formulation

Seller: single item
Single Buyer



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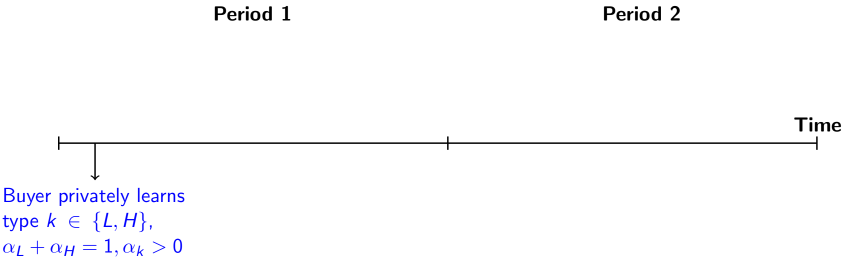

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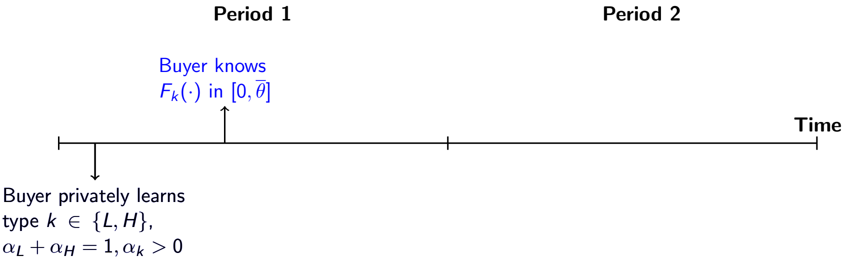

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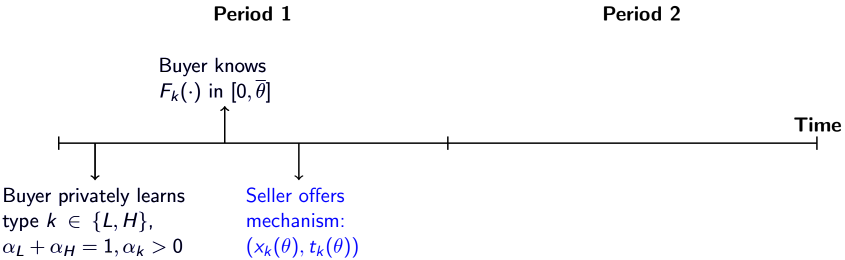

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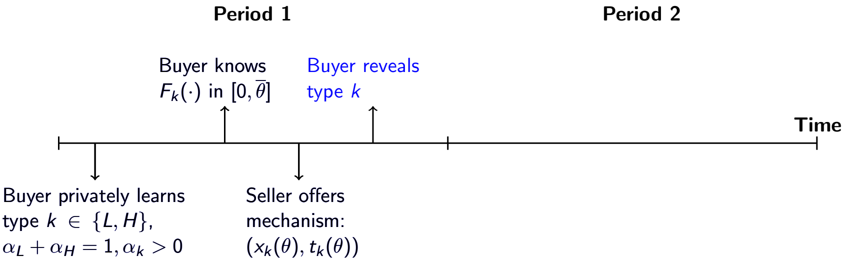

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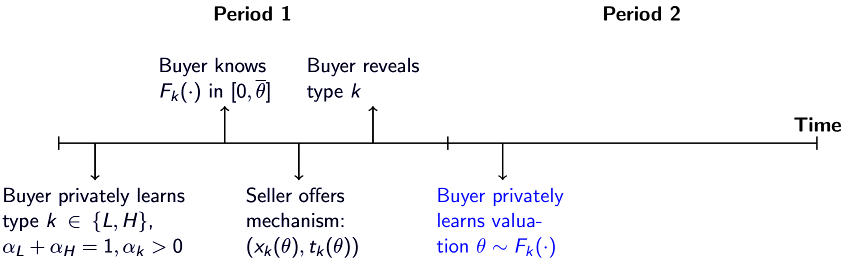

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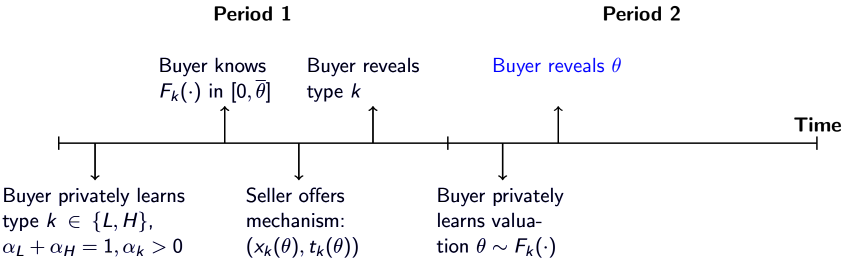

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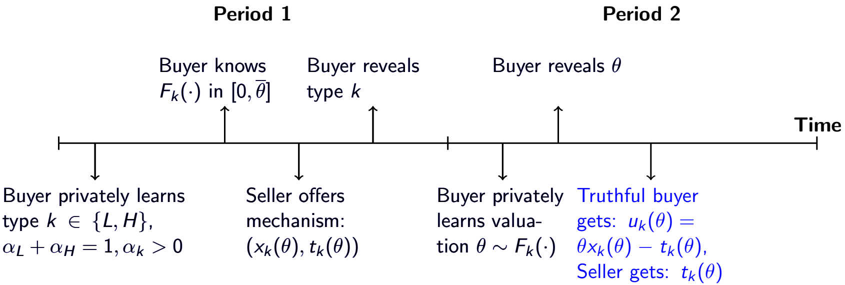

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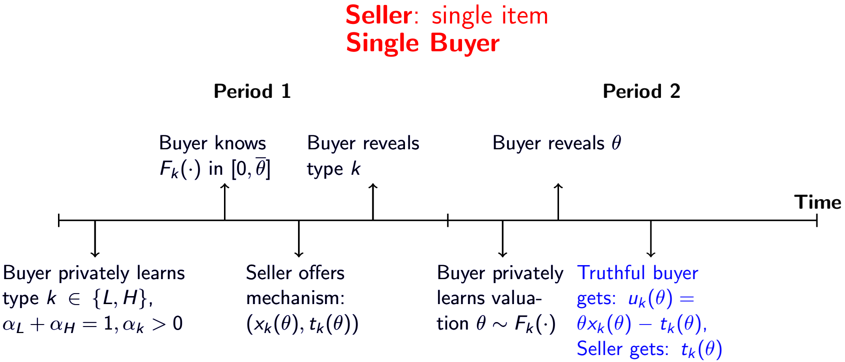
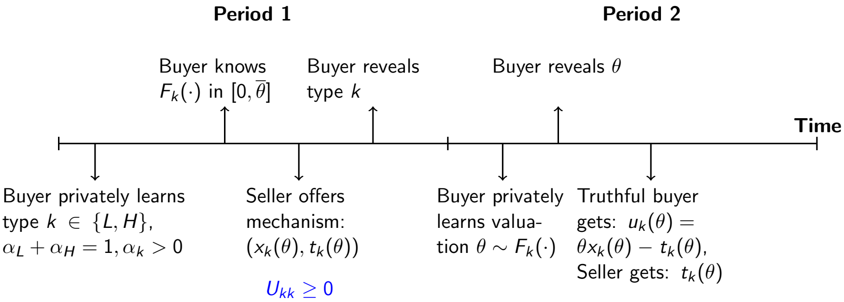


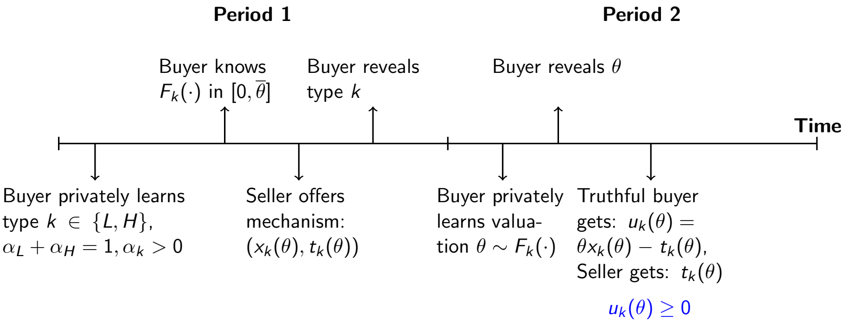
- Model primitives are common knowledge
- Parties are risk-neutral
- Non-increasing hazard rates. WLOG $\hat{\theta}_L \leq \hat{\theta}_H$

Revenue Maximizing Mechanisms



- **Courty and Li:** What is the revenue maximizing sequential screening mechanism under interim participation constraints?

Revenue Maximizing Mechanisms



- **Our Question:** What is the revenue maximizing sequential screening mechanism under ex-post participation constraints?

$$\text{[Ex-post PC]} \quad u_k(\theta) \geq 0, \ \forall k \in \{L, H\}, \ \forall \theta$$

Seller's Problem

The seller's problem is

$$(\mathcal{P}^d) \quad \max_{0\leq \mathbf{x}\leq 1, t} \quad \sum_{k\in\{L,H\}} \alpha_k \cdot \int_0^{\bar{\theta}} t_k(z)\cdot f_k(z)dz$$

$$\begin{aligned}
\text{s.t.} \quad u_k(\theta) &\geq \theta\cdot x_k(\theta') - t_k(\theta') \quad \forall k,\theta \text{ [Ex-post IC]} \\
\int_0^{\bar{\theta}} u_k(z)f_k(z)dz &\geq \int_0^{\bar{\theta}} u_{k'}(z)f_k(z)dz, \forall k, k \text{ [Interim IC]} \\
u_k(\theta) &\geq 0, \quad \forall k,\theta \text{ [Ex-post PC]}
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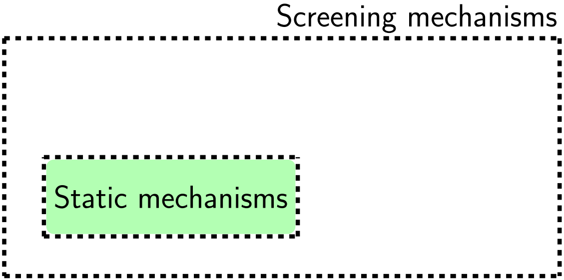
$$u_k(\theta) \geq 0, \quad \forall k,\theta \text{ [Ex-post PC]}$$

- Ex-post IC: By the envelope theorem it is enough to solve for non-decreasing allocations $x_k(\cdot)$ and the utility of the lowest ex-post types $u_k(0)$
- Interim IC: More challenging (together with ex-post PC)

Optimal Mechanisms

Screening mechanisms

Optimal Mechanisms



Optimal Mechanisms

Screening mechanisms

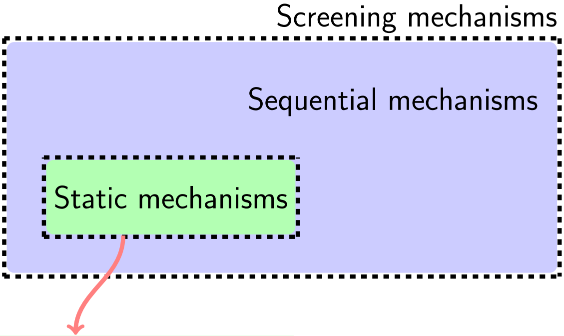
Static mechanisms

– A contract such that $x_k(\cdot) \equiv x(\cdot)$ and $t_k(\cdot) \equiv t(\cdot)$ for all k in $\{L, H\}$

– Pooling of interim types

– Myerson for the mixture distribution: **posted price** θ^s

Optimal Mechanisms



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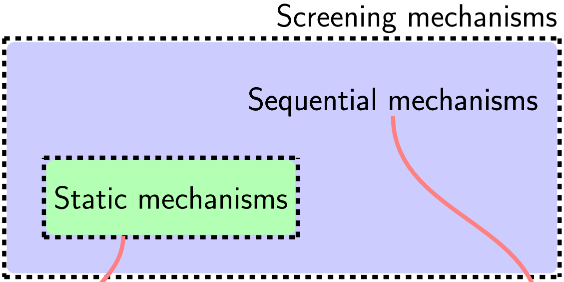

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–A contract such that $x_L(\cdot) \neq x_H(\cdot)$ and $t_L(\cdot) \neq t_H(\cdot)$

– Separate interim types

– Contract can be arbitrarily complex

Research Questions/Contributions

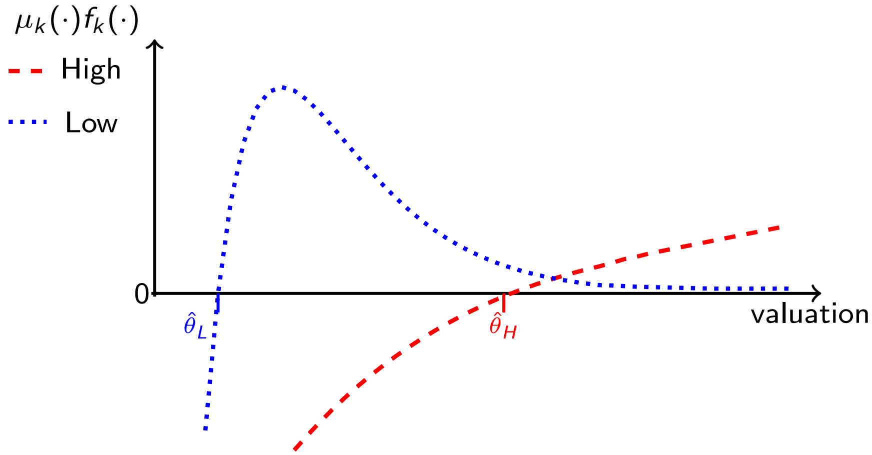
1. **When is a static contract optimal? When it is not?**
   - Krähmer and Strausz 2015: Sufficient condition
   - Us: Necessary and sufficient condition

Research Questions/Contributions

1. **When is a static contract optimal? When it is not?**
   - Krähmer and Strausz 2015: Sufficient condition
   - Us: Necessary and sufficient condition
2. **If a sequential contract is optimal, what does the optimal mechanism look like?**
   - Us: Full characterization
   - Very different to Courty and Li
   - Significant revenue improvement over static contract

The "Simple Economics" of Optimal Sequential Contracts

Let's look at weighted virtual values ("marginal revenues");
$\mu_k(\theta) = \theta - \frac{1-F_k(\theta)}{f_k(\theta)}$



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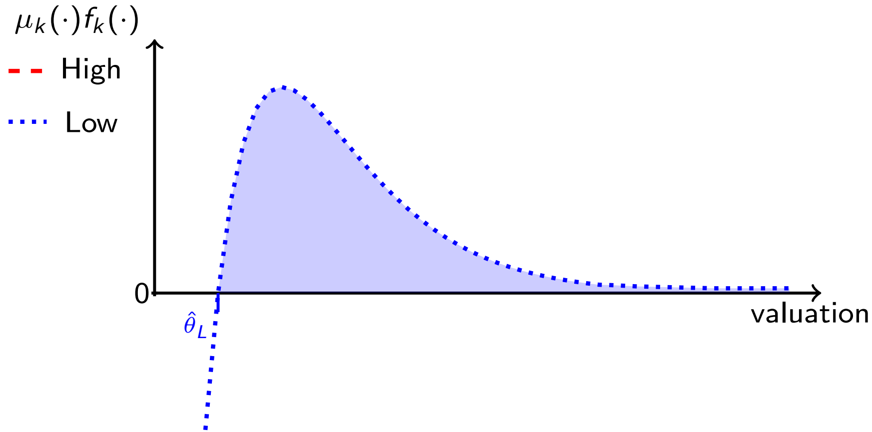

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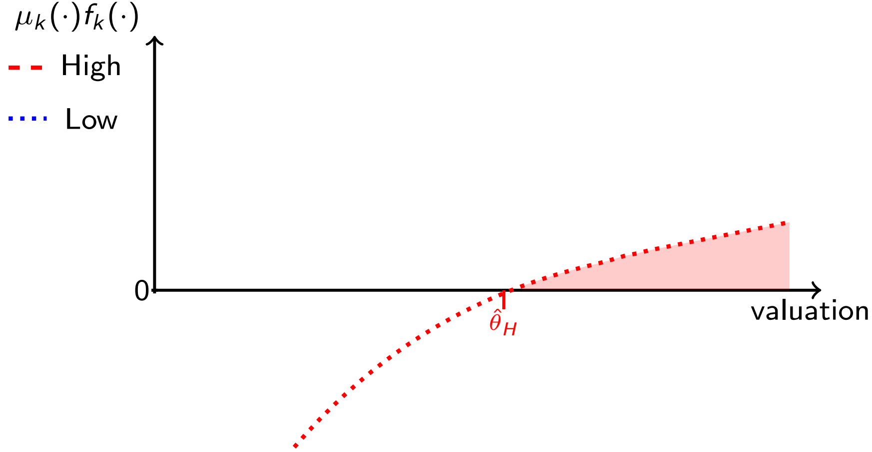

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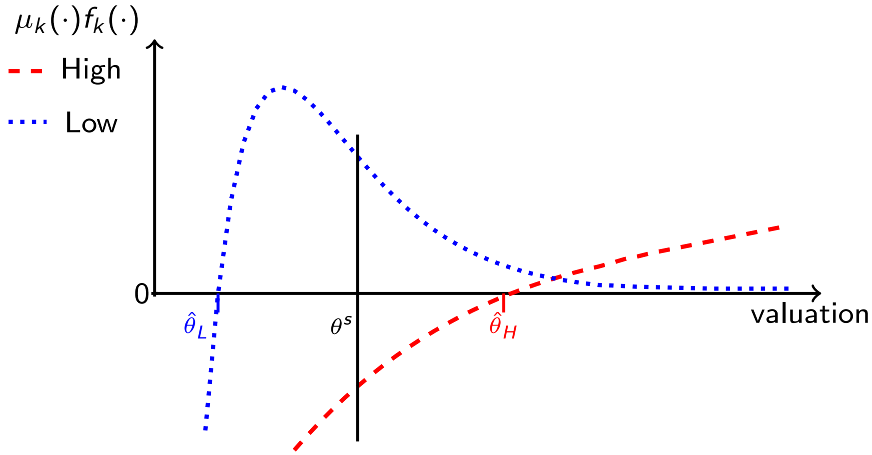

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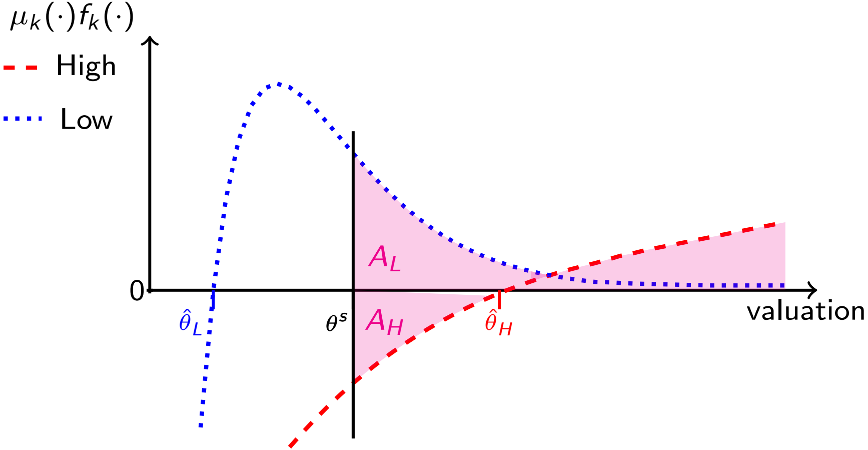
Rev (static)$= \alpha_L \cdot A_L + \alpha_H \cdot A_H$



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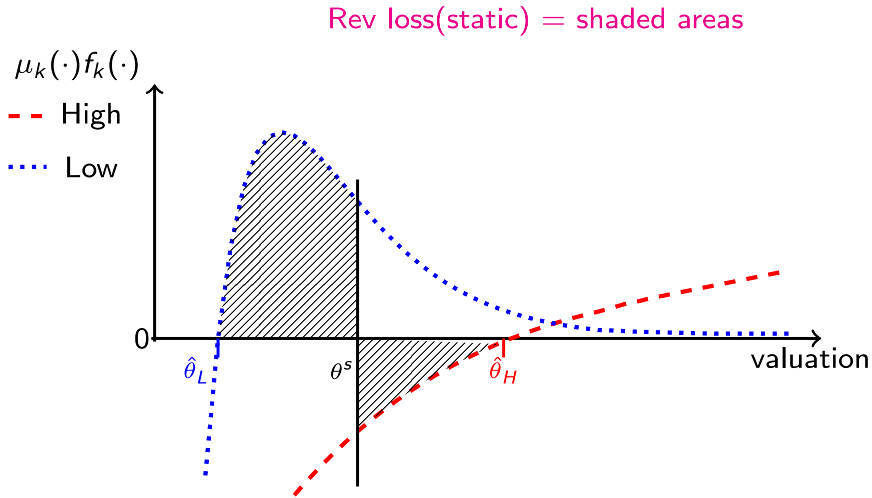

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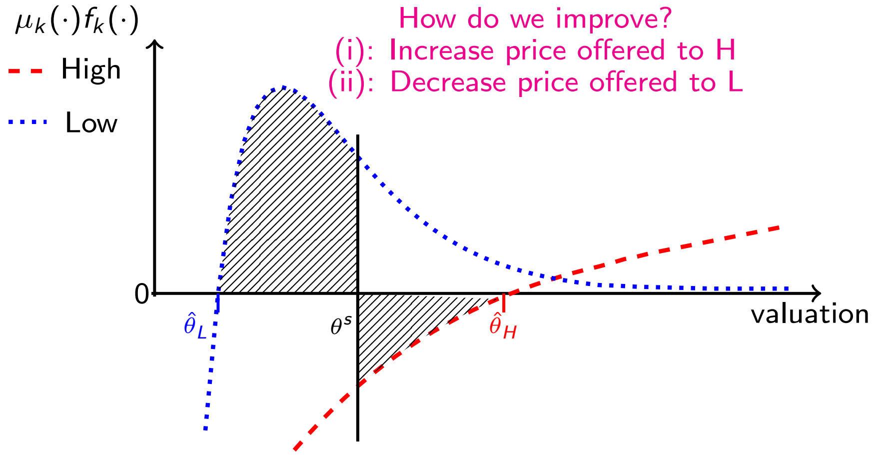
Rev loss(static) = shaded areas

How do we improve?
(i): Increase price offered to H
(ii): Decrease price offered to L



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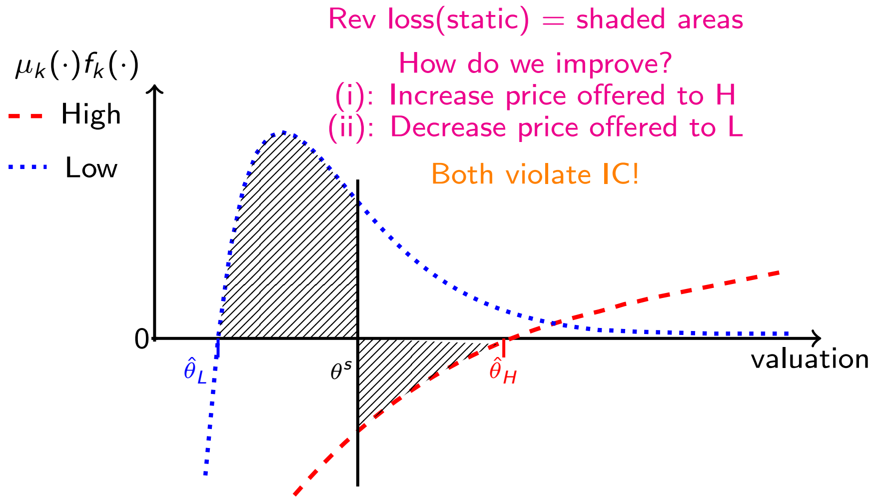

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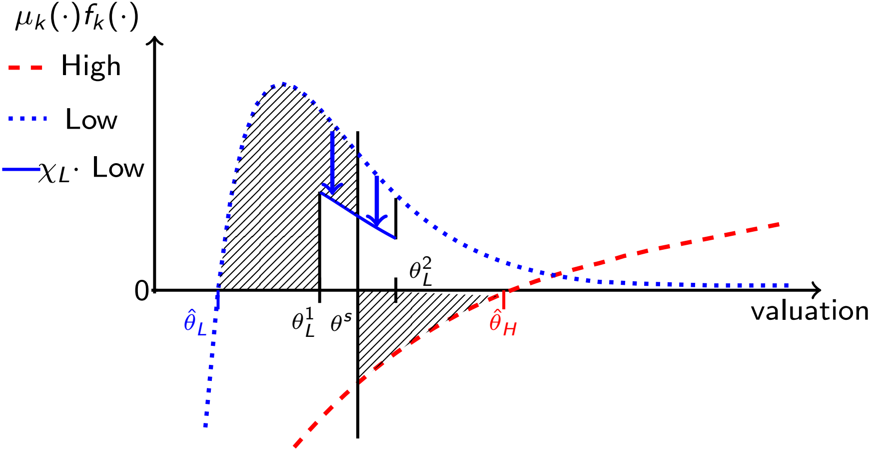
We can improve by randomizing L



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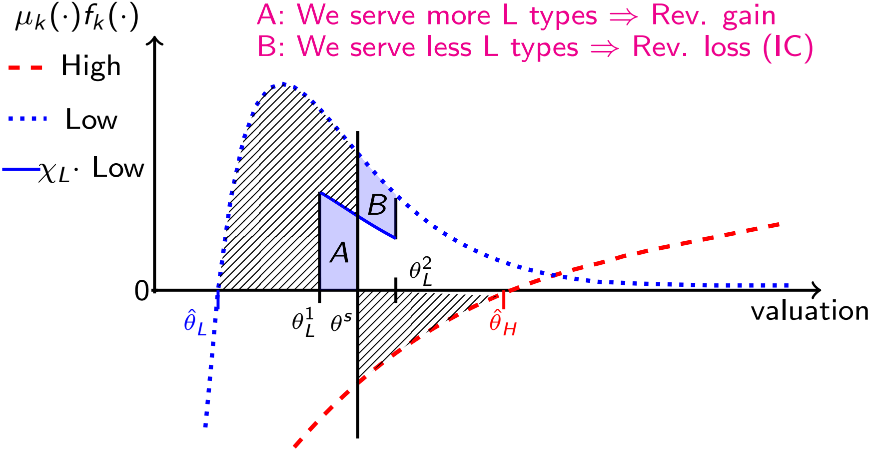
A: We serve more L types $\Rightarrow$ Rev. gain

B: We serve less L types $\Rightarrow$ Rev. loss (IC)



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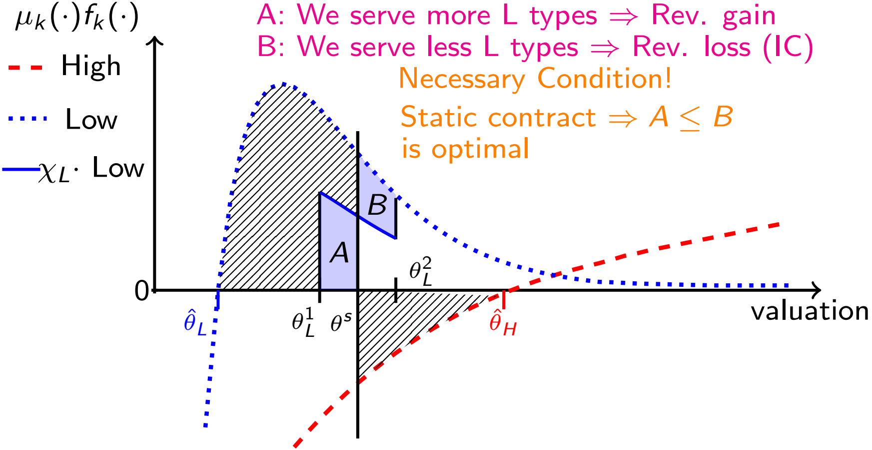
Necessary Condition!

Static contract $\Rightarrow A \leq B$ is optimal



General Necessity and Sufficiency

Theorem

The static contract is optimal if and only if

$$\max\left\{ \underbrace{\textit{Region } A}_{\textbf{revenue gain}} \right\} \leq \min\left\{ \underbrace{\textit{Region } B}_{\textbf{revenue loss}} \right\}$$

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1. Condition can be rigorously stated in terms of primitives:

$$\max_{\theta \leq \theta^s} \frac{\int_\theta^{\theta^s} \mu_L(\theta) f_L(\theta) d\theta}{\int_\theta^{\theta^s} (1 - F_H(\theta)) d\theta} \leq \min_{\theta^s \leq \theta} \frac{\int_{\theta^s}^{\theta} \mu_L(\theta) f_L(\theta) d\theta}{\int_{\theta^s}^{\theta} (1 - F_H(\theta)) d\theta}$$

2. **Sharp intuitive characterization for optimality of static contract!**
3. Necessity formalizes picture above; sufficiency relaxes L to H IC and applies Lagrangian duality.

Exponential Valuations

$$f_k(\theta) = \lambda_k e^{-\lambda_k \theta}, \quad k = \{L, H\} \quad \theta \geq 0, \quad \lambda_L > \lambda_H.$$

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Proposition

The static contract is optimal if and only if

$$\lambda_L - \lambda_H \leq \frac{1}{\theta^s}$$

- θ^s: optimal Myerson price for mixture distribution
- λ_L and λ_H close then screening is not optimal
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Corollary

Assume $\lambda_L \in (\lambda_H, 2\lambda_H]$, *then the static contract is optimal.*

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Corollary

Assume $\lambda_L > 2\lambda_H$*, then there exists* $\bar{\alpha} \in (0, 1)$ *such that the sequential contract is optimal iff* $\alpha_L \in (0, \bar{\alpha})$*.*

General Necessity and Sufficiency

Krähmer and Strausz 2015 sufficient condition:

$$\frac{\mu_\ell(\theta) f_\ell(\theta)}{\bar{F}_k(\theta)}$$

is increasing for all ℓ, k

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is increasing for all ℓ, k

- Stronger pointwise condition
- It implies our condition
- Ex.: does not necc. hold for exponential valuations.

General Characterization

Full characterization for optimal sequential contract!

Theorem

Consider problem ($\mathcal{P}^d$) and assume profit-to-rent cond. does not hold, the optimal solution has allocations

$$x_L^\star(\theta) = \begin{cases} 0 & \text{if } \theta < \theta_L^1 \\ \chi_L \in [0,1] & \text{if } \theta_L^1 \leq \theta \leq \theta_L^2 \\ 1 & \text{if } \theta_L^2 < \theta, \end{cases} \qquad x_H^\star(\theta) = \begin{cases} 0 & \text{if } \theta < \theta_H \\ 1 & \text{if } \theta_H \leq \theta, \end{cases}$$

for some values $\theta_L^1, \theta_H, \theta_L^2$ with $\theta_L^1 \leq \theta_H \leq \theta_L^2$, and $u_L(0) = u_H(0) = 0$

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- Optimal contract is simple
- Optimal contract departs from bang-bang contract with one buyer and one item
- $\hat{\theta}_L \leq \theta_L^1, \ \theta_H \leq \hat{\theta}_H$

Proof Sequential Contract

- Relax L to H interim IC constraint (check feasibility at the end)

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  - H type allocation has one or more intermediate steps

The Value of Sequential Screening: Optimal Revenues

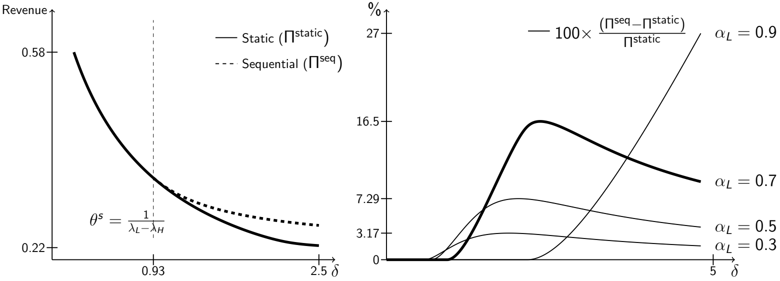

Figure: **Left: Optimal expected revenue** for static and sequential. **Right: Percentage improvement** of the sequential over the static contract. In both figures we set set $\lambda_L = \lambda_H + \delta$ with $\lambda_H = 0.5$.

Back to Waterfall Auctions

- In Waterfall Auctions **low type buyers are randomized**: can only bid when high-reserve auction does not clear
- "high-reserve auction does not clear" $\Leftrightarrow$ high type value $\leq \theta_H$
- Seller revenue:

$$\max_{\theta_H \geq \theta_L \geq 0, (IC)} \quad \alpha_L \underbrace{F_H(\theta_H)}_{\textbf{randomization}} \bar{F}_L(\theta_L)\theta_L + \alpha_H \bar{F}_H(\theta_H)\theta_H$$

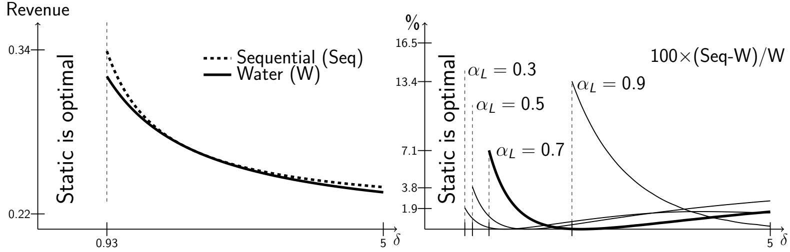

Figure: *Left:* Optimal expected revenue for Waterfall and Sequential. *Right:* Percentage improvement of the Sequential over the Waterfall contract. In both figures we set set $\lambda_L = \lambda_H + \delta$ with $\lambda_H = 0.5$.

Multiple Interim Types

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- Multiple-type analysis is more complex because there is not an obvious relaxation of the math program.

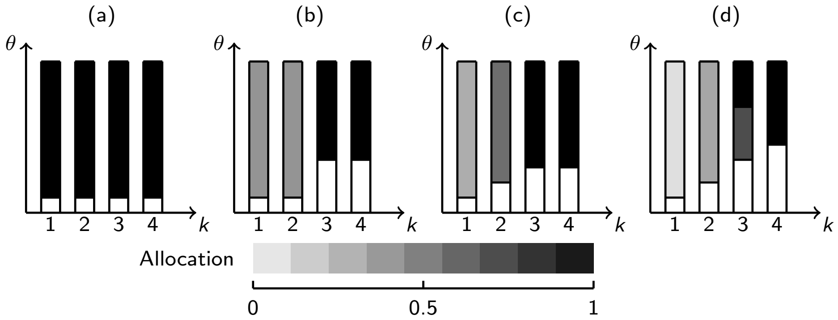

Figure: Optimal allocations for 4 interim types with exponential valuations. In each panel the vertical axis corresponds to buyers' valuations and the horizontal axis corresponds to the type. Each bar represents the allocation for each type, lighter grey indicates lower probability of allocation while darker grey indicates higher probability of allocation. Different distributions of interim types across instances.

Conclusions and Future Work

Summary

- Complete characterization for the optimal mechanism for two interim types and one buyer
- Both static and sequential contracts can be optimal
- When the sequential contract is optimal, the seller has to randomize the low-type and give a deterministic allocation to the high-type
- Some extensions to multiple types

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Current and Future work

- Study multiple buyers: may need ironing
- Connections with practical real-world mechanisms, such as waterfall auctions (randomization)
- Analyze performance guarantees of “simple mechanisms”