LO1: Multidimensional Mechanism Design & Optimization

Christos Tzamos
Optimal Mechanism Design Problem

One or Many items to sell

Which *auction* maximizes seller’s revenue?

- Focus on the case of a *single* buyer
[Myerson 81]: The optimal auction for a single item is a take-it-or-leave it offer at price $p^* \in \arg \max \{z \cdot (1 - F(z))\}$

Buyer’s utility:
Multiple Items

- 1 seller with n items
- 1 additive buyer with value $z \sim F$
- $F$ known to the seller

Challenge already with 2 items
Example 1

Obvious Approach:
- Run Myerson separately for each item
- Price each at $1 \rightarrow E[\text{revenue}] = $2

Optimal? Sell bundle at $3

\[ \text{Expected Revenue} = 75\% \cdot \$3 = \$2.25 \]

\[ \therefore \text{Bundling items increases revenue} \]
Example 2

$4 \quad \$2.50$

Expected Revenue = $50\% \cdot \$4 + 25\% \cdot \$2.5 = \$2.625$

∴ Optimal mechanisms may not only bundle items, but also use randomization.
Example 3

\[ \text{Beta}(1, 2) \quad \text{Beta}(1, 2) \]

\[ \therefore \text{Optimal mechanisms may offer uncountably many random bundles.} \]
Example 4

$F_1 \quad F_1 \quad \text{vs} \quad F_2 \quad F_2$

with $F_2 >_1 F_1$

[Hart, Reny’12] example with $\text{Rev}(F_1 \times F_1) > \text{Rev}(F_2 \times F_2)$

$F_1 = \begin{cases} 10 \text{ with probability } \frac{4}{15} \\ 46 \text{ with probability } \frac{1}{90} \\ 47 \text{ with probability } \frac{1}{3} \\ 80 \text{ with probability } \frac{7}{30} \\ 100 \text{ with probability } \frac{7}{45} \end{cases}$

$F_2 = \begin{cases} 10 \text{ with probability } \frac{2399}{9000} \\ 13 \text{ with probability } \frac{1}{9000} \\ 46 \text{ with probability } \frac{1}{90} \\ 47 \text{ with probability } \frac{1}{3} \\ 80 \text{ with probability } \frac{7}{30} \\ 100 \text{ with probability } \frac{7}{45}. \end{cases}$

$\therefore$ Optimal mechanisms may not be monotone w.r.t the prior distributions
Summary

Optimal mechanism for multiple items:
• May involve bundling items together.
• May require randomization.
• May offer uncountably many choices
• May satisfy unnatural properties (non-monotonicity)

How can we find the optimal auction?

Given mechanism, can we certify optimality?
Certifying Optimality of Solutions

Two Generic Approaches:

- Concave Optimization, first order conditions
- Duality, complementary slackness

Single-item auctions [Myerson’81]

Multi-item auctions
Optimizing over auctions

**Approach 1:** Search over options given to the buyer

Option \((x,p)\) costs \(p\) and allocates items with probabilities \(x\)

- **Problem 1:** there can be infinitely many options in the optimal mechanism
- **Problem 2:** finding which option the buyer prefers given his values for the items is very difficult
Optimizing over auctions

Approach 2: Search over allocation and pricing function

Keep functions $x(z)$, $p(z)$ that give directly what option a buyer should buy according to his values for the items.

Goal: maximize $E[p(z)]$

- **Problem 1**: must make sure, the function gives him the best option
  
  Add Constraint: $x(z) \cdot z - p(z) \geq x(z') \cdot z - p(z')$

- **Problem 2**: too many values to keep track of
  
  Myerson removes the dependence on $p(z)$ by relating it to $x(z)$
  
  Not possible to do directly in multidimensional settings
Optimizing over auctions

Optimize over utility functions!

- Given a mechanism, every type \( z \in \mathbb{R}^n \) enjoys some utility \( u(z) \)
- **[Rochet’85]**: If \( u \) is induced by a mechanism then and only then:
  \[ u(z): \text{ 1-Lipschitz w.r.t } L_1, \text{ convex, non-decreasing, non-negative} \]

- Moreover: \( u \) is differentiable almost everywhere and
  \[
  \nabla u(z): \text{ allocation probabilities to type } z \text{ (exists a.e.)} \\
  \nabla u(z) \cdot z - u(z): \text{ price paid by type } z
  \]
Single item example

\[ u(z) = (z - p^*)_+ \]

- \( x(z) = u'(z) \)
  - 0 if \( z < p^* \)
  - 1 if \( z > p^* \)

- \( p(z) = u'(z) z - u(z) \)
  - 0 if \( z < p^* \)
  - \( p^* \) if \( z > p^* \)

- Notice that \( u \) is not differentiable at \( p^* \)
  (i.e. on a set of measure 0)
Proof of Rochet’s Theorem

On board
Application of Rochet’s Theorem

From Rochet’s Theorem, we obtain the following optimization program in terms of the utility function

Maximize expected revenue: \[ \int (\nabla u(z) \cdot z - u(z)) f(z) dz \]

Subject to

\[ u(z) : \text{ 1-Lipschitz w.r.t } L_1, \text{ convex, non-decreasing, non-negative} \]
Simplifying the objective

Applying the divergence theorem:

Expected Revenue = ∫_{Z} [∇u(z) \cdot z - u(z)] f(z) dz

We can rewrite this as ∫ udμ

where μ is a signed measure
Example: 2 items $U[0,1]$

The interior satisfies:
$$\nabla f(z) \cdot z + (n + 1) f(z) = -3$$

The boundary consists of 4 pieces:
- Top: Outer Normal $(0,1)$
- Right: Outer Normal $(1,0)$
- Bottom: Outer Normal $(0,-1)$
- Left: Outer Normal $(-1,0)$

The top and right have mass +1 each while the bottom and left boundaries have 0 mass.

Since $u(0) = 0$, we can set $\mu(\{0\}) = +1$ to make positive and negative masses equal.
Optimizing over auctions

Expected Revenue: \[ \int u \, d\mu \]

\[ \mu = \mu_+ - \mu_- \]

Want to pick \( u \) so that expected revenue is as high as possible under the constraints that

1. \( u \) is convex
2. \( u \) is continuous
3. \( \nabla u \in [0, 1]^n \) almost everywhere
4. \( u(\vec{0}) = 0 \)
A Dual Problem

$$\max \int ud\mu$$

s.t. (1) $u$ is convex
(2) $u$ is continuous
(3) $\nabla u \in [0,1]^n$ almost everywhere
(4) $u(\hat{0}) = 0$

Suppose it costs $\sum_i (x_i - y_i)_+$ to move a unit of mass from $x$ to $y$. What is the minimum-cost way of transforming $\mu_+$ into $\mu_-$?

Continuous analog of min-cost matching duality
Duality Proof &
Complementarity Conditions

On board
Uniform[0,3] example

Mechanism offers
• One item at price 2
• Both items at price $4 - \sqrt{2}$

Goal:
Show optimality by constructing a complementary dual solution
Next time

• How to certify optimality without explicitly constructing a dual solution?
• How to identify candidate mechanisms?
• What to do when approach fails?

Thank you!