Dynamic Matching – Part 2

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Dynamic Matching Processes with Changing Participants

Common to many matching processes:

- **Child Adoption:**
  - About 1.6 million, or 2.5%, of children in the U.S. are adopted
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- **Organ Donation:**
  - Organ Donation and Transplantation Statistics – a new patient added to the kidney transplant list every 14 minutes and about 3000 patients added each month
  - In 2013, about a third of ~$17,000$ kidney transplants involved living donors

Two Questions

- What outcomes do decentralized interactions generate?
- How to optimally match individuals in a dynamic market?

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- How to optimally match individuals in a dynamic market?
- $\Longrightarrow$ What are environments in which centralized intervention would be particularly beneficial?

Today

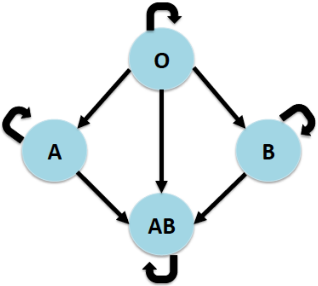
- (In brief) Kidney exchange
- (In detail) Optimal dynamic mechanisms in a particular setting

Kidney Exchange

- **Original References:** Roth, Sonmez, and Ünver (2004, 2005, 2007), *Ünver (2010)*
- **Underlying Trade-off:** Compatibility between pairs and waiting time

Compatibility between Pairs

- Blood compatibility



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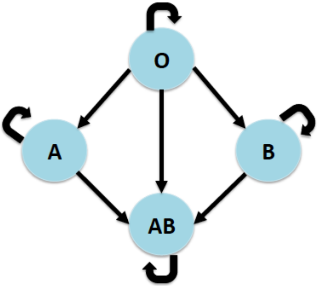

- Tissue type compatibility: Percentage Reactive Antibodies (PRA)

A. Patient ABO Blood Type	Frequency
O	48.14%
A	33.73%
B	14.28%
AB	3.85%
B. Patient Gender	**Frequency**
Female	40.90%
Male	59.10%
C. Unrelated Living Donors	**Frequency**
Spouse	48.97%
Other	51.03%
D. PRA Distribution	**Frequency**
Low PRA	70.19%
Medium PRA	20.00%
High PRA	9.81%

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- If waiting were costless, could we get everyone matched?
- Discard of all compatible pairs
- Consider n remaining pairs, where each two pairs are compatible with probability p

Large Random Networks

- Consider a graph with n nodes and each link occurring with probability p
- Divide the n pairs randomly into two groups of k pairs, call them S and R

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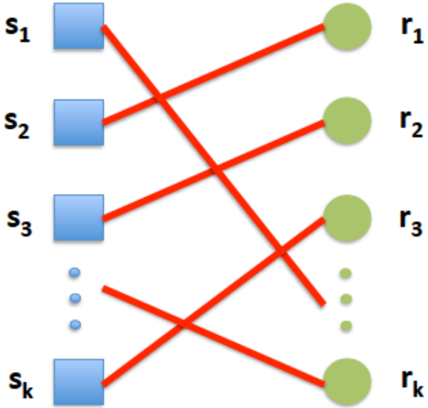
- Each node in S is connected to (compatible with) each node in R with probability p

Large Random Networks (2)



Large Random Networks (3)

Erdos and Renyi (1964): *As long as the graph is connected enough, namely as long as* p *approaches* 0 *slower than* $\frac{\log n}{n}$, *there is a perfect matching with probability approaching 1.*

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- In fact, p is bounded above 0, even when we consider different blood types
- So if we wait long enough, we can match everyone!

Market Design – Two Problems

- In reality, waiting is costly. So, how should we create matches?
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  - Keep useful donors (say, those with O blood type) for the future

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- Overall maximization of matched pairs may not maximize a particular hospital's number of matched pairs (Ashlagi and Roth, 2013)

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- Akbarpour, Li, and Oveis Gharan (2015) – Holding on to some pairs could be useful
- No characterization of the optimal mechanism

Optimal Dynamic Matching

- **Reference:** Baccara, Li, and Yariv (2015)
- **Our (Broad) Goal:** In a simple setting,
  - Analyze the optimal matching protocol
  - Compare it with a decentralized setting as well as with a few simple interventions

Underlying Trade-off

- **Quality of matches varies, thick markets are beneficial**
  $\Rightarrow$ incentive to accumulate participants
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  - Labor markets – foregone wages for the unemployed, lacking workforce for employers, etc.

Main Questions

- **What is the optimal dynamic mechanism to balance this trade-off?**
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  - Hold on to a stock of certain agents up to a threshold, match others immediately
- **What is the welfare gain relative to decentralization?**
  - Decentralization leads to inefficient long queues
- **What are simple, and useful, interventions?**
  - Budget-balanced tax/subsidy schemes
  - Matching in fixed time intervals

A Model of Dynamic Matching

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- **Note:** Symmetry makes things simpler (but not crucial), could assume random arrival of pairs

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- Convenient to denote:

$$U_{xy} \equiv U_x(y) + U_y(x) \text{ for } x = A, B, \; y = \alpha, \beta$$

- **Super-modularity assumption:**

$$U \equiv U_{A\alpha} + U_{B\beta} - U_{A\beta} - U_{B\alpha} > 0$$

$\Longrightarrow$ The efficient matching entails the maximal number of (A, α) and (B, β) pairs

A Model of Dynamic Matching (3)

- In all the matching protocols we consider, we assume agents leave the market by matching
- Bounds on utilities from remaining unmatched assure this is individually rational

The Optimal Mechanism

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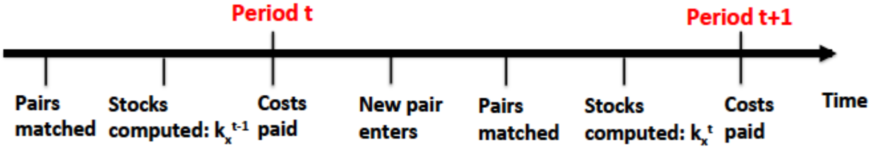
- At any time t, before a new square-round pair arrives, a queue is represented by $(k_A, k_\alpha, k_B, k_\beta)$
    - Length of queues of squares: k_A, k_B
    - Length of queues of rounds: k_α, k_β

Summary Timeline



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- At time t, *after* a new square-round pair enters the market, there is a queue $\mathbf{s}^t = (s_A^t, s_B^t, s_\alpha^t, s_\beta^t)$
- A mechanism is characterized by a mapping $\mu : \mathbb{Z}_+^4 \to \mathbb{Z}_+^4$ such that for every $\mathbf{s} \in \mathbb{Z}^4$, $\mu(\mathbf{s}) = \mathbf{m} = (m_{A\alpha}, m_{A\beta}, m_{B\alpha}, m_{B\beta})$ is a feasible profile of matches:
  - $m_{x\alpha} + m_{x\beta} \leq s_x$ for $x \in \{A, B\}$
  - $m_{Ay} + m_{By} \leq s_y$ for $y \in \{\alpha, \beta\}$

- New queue $\mathbf{k} = (k_A, k_B, k_\alpha, k_\beta)$ contains remaining agents:

$$\begin{aligned} k_x &= s_x - (m_{x\alpha} + m_{x\beta}) \quad \text{for } x \in \{A, B\} \\ k_y &= s_y - (m_{Ay} + m_{By}) \quad \text{for } y \in \{\alpha, \beta\} \end{aligned}$$

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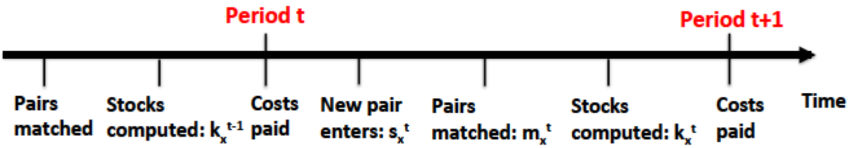

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- The welfare generated at time t is:

$$w(\mathbf{s}^t,\mathbf{m}^t) \equiv S(\mathbf{m}^t) - C(\mathbf{s}^t,\mathbf{m}^t)$$

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- The value of each mechanism is assessed using the *average welfare*, defined as:

$$W(\mu) \equiv \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} w(\mathbf{s}_t, \mu(\mathbf{s}_t))\right]$$

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- An *optimal mechanism* is a mechanism achieving the maximal average welfare
- An optimal mechanism exists since there is only a finite number of stationary and deterministic mechanisms leading to a bounded stock of agents in each period

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- May as well match (A, α) immediately
- Same for a (B, β) pair

Proposition 1 *An optimal dynamic mechanism is identified by a pair of thresholds* $(\bar{k}_A, \bar{k}_\alpha) \in Z_+$ *such that*

1. *whenever more than* $\bar{k}_A$ *A-squares are present,* $s_A - \bar{k}_A$ *pairs of type* (A, β) *are matched immediately, and*
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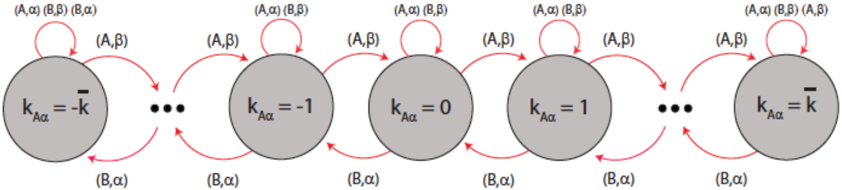
Symmetry $\Rightarrow \bar{k}_A = \bar{k}_\alpha = \bar{k}$.

Structure of the Optimal Mechanism

- Denote by $k_{A\alpha} = k_A - k_\alpha$
- $k_{A\alpha}$ follows a Markov process, where states correspond to values $-\bar{k} \leq k_{A\alpha} \leq \bar{k}$

Structure of the Optimal Mechanism



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- When $-\overline{k} < k_{A\alpha} < \overline{k}$,
  - Stock does not change if (A, α) or (B, β) arrive $\Longrightarrow$ probability $p^2 + (1-p)^2$
  - Stock changes (up or down) if (A, β) or (B, α) arrive $\Longrightarrow$ probability for either $p(1-p)$

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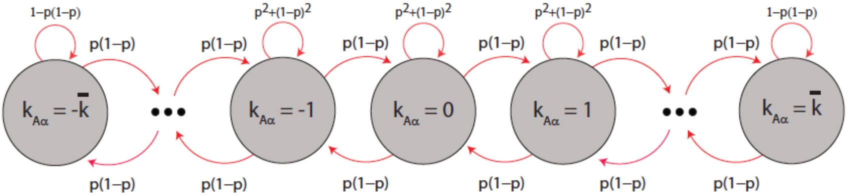

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- $\mathbf{x}^{t+1} = \mathbf{T}_{\bar{k}}\mathbf{x}^t$ (a Markov process with $2\overline{k}+1$ states)

$$T_{\overline{k}} = \begin{bmatrix} 1-p(1-p) & p(1-p) & 0 & & 0 \\ p(1-p) & p^2+(1-p)^2 & & & \\ 0 & & \ddots & & \\ & & & p^2+(1-p)^2 & p(1-p) \\ 0 & & & p(1-p) & 1-p(1-p) \end{bmatrix}$$

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- This process is ergodic $\Longrightarrow$ There is a unique steady state distribution, which is the uniform distribution over states $\Longrightarrow$ each state has probability $\frac{1}{2\overline{k}+1}$

Optimal Mechanism – Costs

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- We use the steady state distribution to identify costs and benefits for any $\bar{k}$
- In state r, $2|r|$ participants are present on the market
- Expected total waiting costs:

$$C(\overline{k}) = \frac{1}{2\overline{k}+1}\left(\sum_{l=-\overline{k}}^{\overline{k}} 2|r|\right)c = \frac{\overline{k}(\overline{k}+1)c}{2\overline{k}+1}$$

Optimal Mechanism – Benefits

- Suppose an (A, β) pair has arrived at the market (similarly for a (B, α) pair) when the queue is r:
  - If $-\overline{k} \leq r < 0$, the optimal mechanism creates one (A, α) match and one (B, β) match $\longrightarrow U_{A\alpha} + U_{B\beta}$
  - If $0 \leq r < \overline{k}$, the mechanism creates no matches $\longrightarrow 0$
  - If $r = \overline{k}$, the mechanism creates an (A, β) match $\longrightarrow U_{A\beta}$

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  - If $r = \overline{k}$, the mechanism creates an (A, β) match $\rightarrow U_{A\beta}$
- Expected per-period total match surplus (after some algebra):

$$B(\overline{k}) = pU_{A\alpha} + (1-p)U_{B\beta} - \frac{p(1-p)U}{2\overline{k}+1}$$

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- $\overline{k}^{opt}$ **balances market thickness and cost of waiting**
- $\overline{k}^{opt}$ decreases in c and positive only when $c \leq \frac{p(1-p)U}{2}$
- $\overline{k}^{opt}$ increases as $p(1-p)$ (probability of arrival of a mismatched pair) increases, and maximized at $p = 1/2$

Optimal Mechanism – Welfare

- Welfare = Match Surplus - Waiting Costs

$$W^{opt}(c) = pU_{A\alpha} + (1-p)U_{B\beta} - \frac{p(1-p)U}{2\overline{k}^{opt}+1} - \frac{\overline{k}^{opt}(\overline{k}^{opt}+1)c}{2\overline{k}^{opt}+1}$$

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- From super-modularity, maximal conceivable welfare (absent costs):

$$S_\infty = pU_{A\alpha} + (1-p)U_{B\beta}$$

Optimal Mechanism – Welfare

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- Same matching surplus, lower costs
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- $\overline{k}^{opt}$ decreases in c
- As c increases, fewer individuals wait and effect of marginal cost increase is smaller
- $\Longrightarrow$ **Welfare loss is concave in** c

Optimal Mechanism – Welfare

Corollary (Optimal Welfare)

The welfare under the optimal mechanism is given by $W^{opt}(c) = S_\infty - \Theta(c)$, *where* $\Theta(c)$ *is continuous, increasing, and concave in* c, $\lim_{c \to 0} \Theta(c) = 0$, *and* $\Theta(c) = p(1-p)U$ *for all* $c \geq \frac{p(1-p)U}{2}$.

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$\Longrightarrow$ For small waiting costs, the optimal mechanism achieves approximately the maximal conceivable welfare

Decentralized Markets

- Many dynamic markets are decentralized (e.g., adoption in the U.S.)

Decentralized Markets

- Many dynamic markets are decentralized (e.g., adoption in the U.S.)
- Identify the performance of decentralized processes in order to understand when centralized intervention might be beneficial

A Stark Model of Decentralized Dynamic Markets

- We assume that at each period:
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- Focus on trembling-hand equilibrium

Example 1

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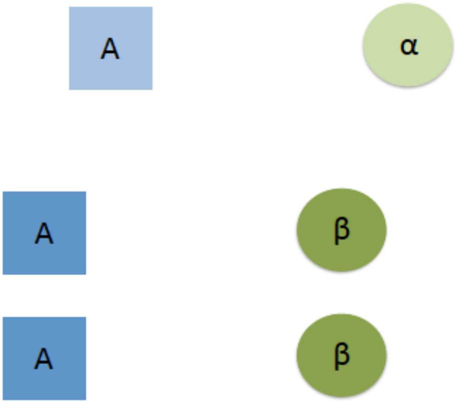

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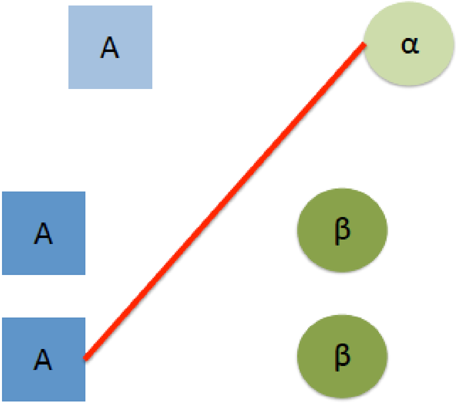

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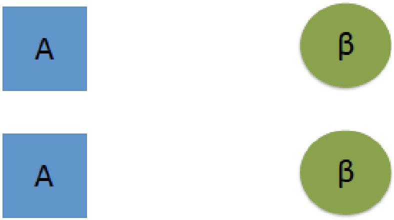
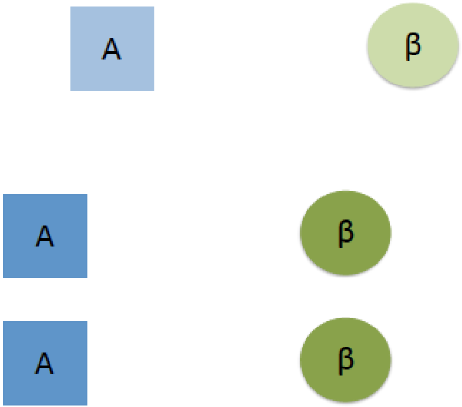
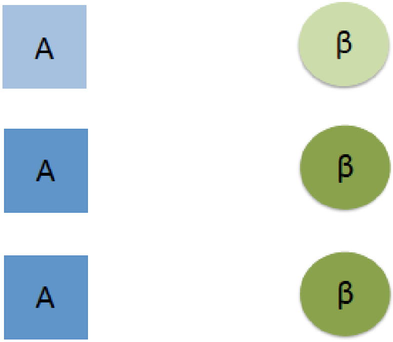
Example 2



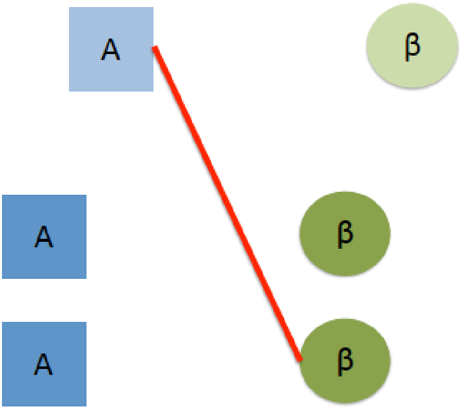
Example 2



Example 2 – First Option



Example 2 – Second Option



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  - When first in queue $\longrightarrow$ time to wait for an α-round is $1/p$
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  - ... When k-th in queue $\longrightarrow$ time to wait for an α-round is k/p
- **Cost of waiting when k-th in line**: $\frac{kc}{p}$
- **Benefit of waiting**: $U_A(\alpha) - U_A(\beta)$ (note that a β-round is always willing to match)

Equilibrium Characterization (2)

We get bounds on $k_{A\alpha} = k_A - k_\alpha$:

Lemma 2 $-\overline{k}^{dec} \leq k_{A\alpha} \leq \overline{k}^{dec}$, *where*

$$\begin{aligned} \overline{k}^{dec} &\equiv \max\left\{k \in \mathbb{Z}_+ \mid \frac{kc}{p} < U_A(\alpha) - U_A(\beta)\right\} \\ &= \max\left\{k \in \mathbb{Z}_+ \mid \frac{kc}{p} < U_\alpha(A) - U_\alpha(B)\right\}. \end{aligned}$$

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- **Regular environment:** Agents are never indifferent between waiting and leaving
- Note that if an A-square decides to wait for a superior match, she will wait in subsequent periods as well, as her position can only improve; Similarly for an α-round

Equilibrium Characterization (3)

- Consider the decision of β-rounds to wait in line
- A β-round may decide to wait for an A-square, who becomes available for β-rounds every time the $A - \alpha$ queue exceeds $\overline{k}^{dec}$

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- A β-round may decide to wait for an A-square, who becomes available for β-rounds every time the $A - \alpha$ queue exceeds $\overline{k}^{dec}$
- Two conflicting forces:
  - the longer the line of β-rounds, the weaker the incentive to wait
  - the closer $k_{A\alpha}$ is to the threshold, the stronger the incentive to wait

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- Suppose the β-round arrives with a B-square
- To match with an A-square, she needs to wait for at least $\overline{k}^{dec} + 1$ of A-squares to arrive (first $\overline{k}^{dec}$ wait in queue)

Equilibrium Characterization (5)

Intuition (continued):

- **Cost of waiting:** at least $(\overline{k}^{dec} + 1)c/p$, greater than the maximal waiting cost $\overline{k}^{dec}c/p$ that A-squares experience

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- β-rounds who arrive later in the process face similar calculus (Lemma 3)

Equilibrium Characterization – Summary (Proposition 3)

- (A, α) or (B, β) pairs match immediately
- There is a stock of (A, β) or (B, α) pairs up to a threshold

$$\bar{k}^{dec} = \left\lfloor \frac{p\left(U_A(\alpha) - U_A(\beta)\right)}{c} \right\rfloor = \left\lfloor \frac{p\left(U_\alpha(A) - U_\alpha(B)\right)}{c} \right\rfloor$$

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- Unique steady state distribution, which is uniform over values of $k_{A\alpha} = -\bar{k}^{dec}, \dots, \bar{k}^{dec}$

Comparing Optimal and Decentralized Processes

- There is a **negative externality of waiting** – imposes longer waits (and possibly inferior matches) on those that follow, which dominates the positive externality of waiting
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- Formally: $\bar{k}^{dec} \geq \bar{k}^{opt}$ (Corollary 2)

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Corollary (Decentralized Welfare)

The welfare under the decentralized process is given by $W^{dec}(c) = S_\infty - \Psi(c)$, *where* $\lim\limits_{c\to 0}\Psi(c) = p\left(U_A(\alpha) - U_A(\beta)\right)$, *and* $\Psi(c) = p(1-p)U$ *for all* $c \geq p\left(U_A(\alpha) - U_A(\beta)\right)$.

Welfare Comparison

- By definition, $W^{opt}(c) \geq W^{dec}(c)$
- When might centralization be particularly beneficial?
  - What are the comparative statics of the welfare wedge $W^{opt}(c) - W^{dec}(c)$ with respect to c, p, and U?

Welfare Comparison – Waiting Costs

- An increase in costs has two effects on the welfare gap:
  - **Direct Effect:** Since $\bar{k}^{dec} \geq \bar{k}^{opt}$, greater costs $\longrightarrow$ greater welfare gap
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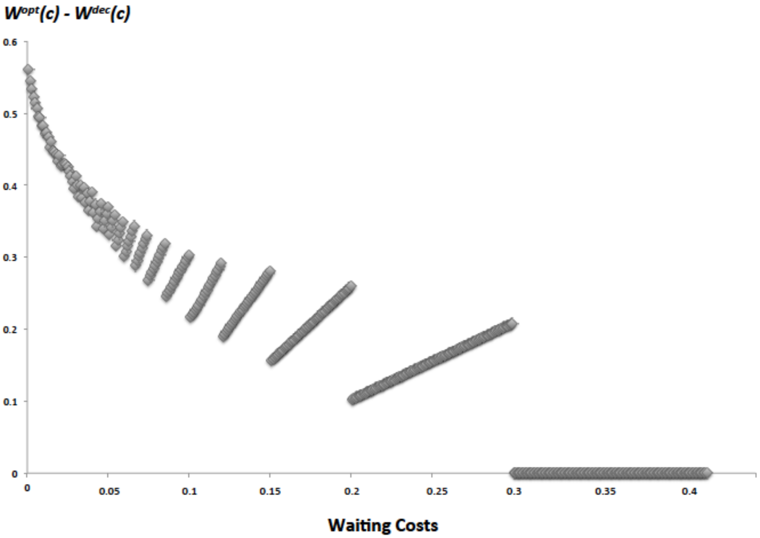
- As costs become prohibitively high, both processes lead to instantaneous matches and identical welfare levels

Welfare Comparison – Waiting Costs



Welfare Comparison – Type Distribution and Preferences

Proposition 4

1. *For any interval* $[\underline{c}, \overline{c})$, *where* $\underline{c} > 0$, *there is a partition* $\{[c_i, c_{i+1})\}_{i=1}^{M-1}$, *where* $\underline{c} = c_1 < c_2 < \ldots < c_M = \overline{c}$, *such that* $W^{opt}(c) - W^{dec}(c)$ *is continuous and increasing over* (c_i, c_{i+1}) *and*

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for all $i = 1, ..., M - 1$.

2. *As* c *becomes vanishingly small, the welfare gap* $W^{opt}(c) - W^{dec}(c)$ *converges to a value that is increasing in* $p \in [0, 1)$ *and in* $U_\alpha(A) - U_\alpha(B)$.

Welfare Comparison – Type Distribution

Intuition:

- Consider p_1 and p_2 such that $p_1 < p_2 = mp_1$, $m > 1$
- Suppose costs are very low $\longrightarrow$ match surplus $\approx S_\infty$ under p_1

- Individual incentives to wait are higher under p_2 than under p_1
- In the decentralized setting, under p_2, roughly $1 - 1/m$ of the steady-state probability mass allocated to queue lengths larger than those seen under p_1

- Large steady state queue lengths $\longrightarrow$ increased waiting costs
- Marginal effect on match surplus minuscule

- In contrast, the optimal mechanism internalizes externalities $\longrightarrow$ impacts of type distribution on induced welfare levels are weak for low c

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The optimal mechanism may be difficult to implement in certain environments:

- Imposes matches on participants
  - Linear balanced-budget tax schemes that lead to optimal welfare
- Requires continuous monitoring of the market
  - Fixed-window mechanisms potentially provide a substantial improvement over decentralized ones

Extensions

- More than two types
- Independent arrivals
- Incomplete information
- Transfers
- Non-linear waiting costs

THE END

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- As long as utility from remaining unmatched $< U^{\min}$, IR satisfied for B-squares

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$\Longrightarrow$ Greater welfare wedge between decentralized and optimal or fixed-window protocols under sub-modularity

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- $\implies$ Greater welfare gap between decentralized and optimal or fixed-window protocols under asymmetric costs

Asymmetric Distributions

- $\mathbf{x}^{t+1} = \mathbf{T}_{\bar{k}_A,\bar{k}_\alpha}\mathbf{x}^t$

$$\mathbf{T}_{\bar{\mathbf{k}}_\mathbf{A},\bar{\mathbf{k}}_\alpha} = \begin{pmatrix} 1-(1-p_A)p_\alpha & p_A(1-p_\alpha) & \dots & 0 & 0 \\ (1-p_A)p_\alpha & p_A p_\alpha + (1-p_A)(1-p_\alpha) & \dots & 0 & 0 \\ 0 & (1-p_A)p_\alpha & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & p_A(1-p_\alpha) & 0 \\ 0 & 0 & \dots & p_A p_\alpha + (1-p_A)(1-p_\alpha) & p_A(1-p_\alpha) \\ 0 & 0 & \dots & (1-p_A)p_\alpha & 1-p_A(1-p_\alpha) \end{pmatrix}$$

Asymmetric Distributions

- If $p_\alpha < p_A$, for small waiting costs, can approximate the optimal mechanism with a one-threshold mechanism
  - α-rounds wait without bound
  - A-squares wait until their stock reaches $\bar{k}_A$
- When $c \longrightarrow 0$, the optimal mechanism generates welfare approaching S_∞

Simple Interventions – Tax Schemes

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- Tax of τ on waiting each period $\Longrightarrow$ effective cost of waiting is $c + \tau$
- Choose τ^* such that

$$\tilde{k}^{dec} = \left\lfloor \frac{p\left(U_A(\alpha) - U_A(\beta)\right)}{c + \tau^*} \right\rfloor = \bar{k}^{opt}$$

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  - But that distorts incentives to enter the market
- Can find linear balanced-budget tax schemes that lead to optimal welfare and have expected returns of 0 to a new entrant

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- "Simpler" than monitoring the market every period
- Consider a window of size n at the end of which k_x agents of type x are present
- The only efficient (and stable) matching is:
  - $min\left\{k_A, k_\alpha\right\}$ A-squares matched with the same number of α-rounds
  - $max\left\{k_A, k_\alpha\right\} - min\left\{k_A, k_\alpha\right\}$ mismatched pairs
  - $min\left\{k_B, k_\beta\right\}$ B-squares matched with β-rounds

Fixed-Window Mechanisms

- Can find the optimal window size n^o that balances market thickness and waiting time
- We show $n^o \leq \bar{k}^{dec}$

Welfare Comparisons

Corollary (Performance of Fixed-Window Mechanisms)

$$\lim_{c\to 0} \frac{W^{opt}(c) - W^{fix}(c)}{W^{opt}(c) - W^{dec}(c)} = \frac{(1-p)U}{2\left(U_A(\alpha) - U_A(\beta)\right)} < 1.$$

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$\Longrightarrow$ Fixed-window mechanisms could potentially provide a substantial improvement over decentralized ones

Linear Taxation

- τ can be set so that

$$\frac{p\left(U_A(\alpha) - U_A(\beta)\right)}{c + \tau} = \sqrt{\frac{p(1-p)U}{2c}}$$

- A linear tax scheme – an agent who is k-th in line pays $\tau^* k$ (whether square or round)

Linear Taxation

- When the length of queue is $\hat{k}$, charge net tax for an agent who is k-th in line:

$$\tau^* k - \frac{2 \cdot \sum_{k=1}^{\hat{k}} \tau^* k}{2\hat{k}} = \tau^* \left(k - \frac{\hat{k}+1}{2} \right)$$

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- Choose

$$\frac{\tau^*(\bar{k}^{opt} - 1)}{2} = \tau \quad \Longleftrightarrow \quad \tau^* = \frac{2\tau}{\bar{k}^{opt} - 1}$$

Fixed-Window Mechanisms

- The total matching surplus:

$$S(k_A, k_\alpha) \equiv \begin{cases} k_\alpha U_{A\alpha} + (k_A - k_\alpha) U_{A\beta} + (n - k_A) U_{B\beta} & \text{if } k_A \geq k_\alpha, \\ k_A U_{A\alpha} + (k_\alpha - k_A) U_{B\alpha} + (n - k_\alpha) U_{B\beta} & \text{otherwise} \end{cases}$$

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- The first square and round to arrive wait for $n-1$ periods, the second square and round to arrive wait for $n-2$ periods, etc. Thus, the total waiting cost is

$$2 \cdot c \cdot ((n-1) + (n-2) + \cdots + 0) = cn(n-1)$$

Fixed-Window Mechanisms

- Therefore, the expected per-period welfare for each square-round pair generated by a window size n is

$$\frac{1}{n}\sum_{0\leq k_\alpha, k_A\leq n}\binom{n}{k_A}\binom{n}{k_\alpha}p_A^{k_A}(1-p_A)^{n-k_A}p_\alpha^{k_\alpha}(1-p_\alpha)^{n-k_\alpha}S(k_A, k_\alpha$$
$$-c(n-1).$$

- Easy to show there is a finite solution (first term bounded above)
- Hard to find closed-form solution

Fixed-Window Mechanisms

Proposition 5

1. *An upper bound for the optimal window size is given by*

$$n^o \leq \frac{p(1-p)U}{4c}.$$

2. *For any* $\varepsilon > 0$*, there exists* c^* *such that if* $c < c^*$*, a lower bound for the optimal window size is given by*

$$n^o \geq \frac{p(1-p)U}{(4+\varepsilon)c}.$$