Optimal Dynamic Auctions and Simple Index Rules*

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Abstract

A monopolist seller has multiple units of an indivisible good to sell over a discrete, finite time horizon. Potential buyers with unit demand arrive and depart over this period. Each buyer privately knows her arrival time, her value for a unit and the time by which she must make a purchase. We study the revenue maximizing Bayes-Nash incentive compatible allocation rule. The allocation rule can be characterized as an *index rule*: each buyer can be assigned an index, and the allocation rule allots the good to a buyer if her index exceeds some threshold. We show that 'simple' index policies are optimal under an increasing hazard rate condition on the distribution of valuations, and, roughly speaking, that less patient buyers have stochastically higher valuations than more patient buyers. By simple, we mean that the index of a buyer depends only on his own valuation and the distribution of values of buyers with the same arrival date and patience. The optimal allocation rule can therefore be described recursively, and solved for by backward induction. When the environment violates these conditions, the optimal allocation rule cannot be calculated recursively, and the index of a buyer may depend on allocation decisions made in the past.

Keywords: dynamic mechanism design, multidimensional signals, optimal auctions, virtual valuation, revelation principle

JEL classification numbers: D44, D82

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1 Introduction

The problem of optimally selling a finite number of indivisible goods to buyers arriving over time has a long pedigree (see for example Stokey [23] and Bulow [5]). One setting restricts the seller to using posted prices and capacity controls. Such mechanisms are widely used in the airline, hotel, and car-rental industries. With the rise in Internet commerce, many sellers have begun experimenting with alternative pricing mechanisms such as auctions. This has inspired a number of theoretical investigations into dynamic auctions, for example Vulcano et al [24] and Lavi and Nisan [13].

This paper assumes a risk neutral seller seeking to sell C identical and indivisible units over T discrete time periods, indexed by t. Buyers with unit demand arrive over time, and have private valuations for a unit of the good. In particular, buyer i has a valuation v_i for one unit of the good, excess units are worthless. Furthermore, each buyer has an arrival time t_i and a deadline $\bar{t}_i$. This means buyer i cannot participate in the auction before time t_i or after time $\bar{t}_i$. Goods assigned to agent i before time t_i or after time $\bar{t}_i$ have no value to him. No buyer observes how many units remain, how many buyers are present or what messages are sent. The 3-tuple $(v_i, t_i, \bar{t}_i)$, called buyers i's type, are assumed to be the private information of buyer i. Thus buyer i is thus in a position to claim a valuation smaller or larger than v_i, an arrival later than t_i and a deadline earlier than $\bar{t}_i$.

In this paper we investigate the structure of the allocation rule in the revenue maximizing mechanism in this setting. We are interested in describing the allocation rule as an index rule: by an index rule, we mean each buyer is assigned an index as a function of his type, and goods are allocated to the highest index subject to it being higher than a threshold.

Recall that in the standard static setting (Myerson [16]), the revenue maximizing auction can be described by an index rule. Assign to an agent with value v an index equal to the 'virtual valuation.' Allot the good to the highest non-negative index.[1]

We ask the corresponding question for the dynamic setting– when can the revenue maximizing mechanism be described by a 'simple' index rule? By simple, we mean that the index of a type is independent of the allocation rule in the past, so that the allocation rule can be recursively computed. We exhibit sufficient conditions for this. In addition to the standard increasing rate condition, loosely speaking it requires that conditioning on arrival time, impatient buyers have sufficiently stochastically higher values than patient buyers. When these conditions hold, we also fully describe the mechanism. A necessary and sufficient condition, albeit not in terms of primitives, is easy to describe: use the indices we propose and calculate a candidate optimal allocation rule. If the resulting mechanism is incentive compatible (easy to verify), it is also optimal.

When the 'simple' index policy we identify is not incentive compatible, a dynamic analogue of Myerson's ironing procedure is needed. We do not offer a characterization of such a procedure.[2] The allocation rule in this case cannot be written in a recursive fashion, or alternately, the index of a buyer depends on the allocation rule in the past.

[1] This is subject to the distribution of agents' values satisfying an 'increasing virtual value' property we describe below.

[2] See Section 1.2 for recent work that provides a partial characterization.

As a result the optimal allocation rule is both hard to compute, and hard to describe.

The airline industry is an example which corresponds closely to the model we analyze. The separation of buyers over time is typical to this industry. The no-early-entry assumption can be justified on the grounds that travelers will not seek tickets until they know that they actually need to travel. The no-late-exit assumption can be justified on the grounds that the buyer may need time to make other arrangements (hotel accommodations, take time off work etc.), and getting the ticket later may not leave the buyer with sufficient time to make these arrangements.

Before we proceed, a caveat: unlike much of the literature on mechanism design, we assume a discrete type space. This obviates the need for vector calculus to derive the results. Instead we use a network flow interpretation of the problem (see, for example Malakhov and Vohra [14]). In our view the resulting analysis is transparent and comprehensible.

1.1 Overview of the Main Results

Since the setting is one where types are multi-dimensional, a complete analysis of the problem is perhaps beyond reach (see for example Rochet and Choné [22]). Our characterization of the optimal allocation rule is best understood by contrasting with the classic result of Myerson [16], since that can also be thought of as a 1 period case of our model.

1.1.1 A Refresher to Myerson [16]

Myerson considers the sale of a single indivisible good to a finite set of risk neutral buyers. Buyers' valuations for the good is their private information, and are i.i.d. draws from a commonly known distribution with pdf f and cdf F.[3] An invocation of the revelation principle allows one to formulate the problem thus:

$$\max_{a,P} \quad \int_v P(v)f(v)dv$$

subject to:

Incentive Compatibility,

Individual Rationality,

Feasibility (at most 1 unit gets allotted).

Here a is the interim expected allocation probability and P the expected payment.

Myerson solves this problem and characterizes the optimal auction in the following manner:

1. Incentive compatibility implies that the allocation rule must be monotonic, i.e. a higher type gets allotted with a higher interim probability.
2. In conjunction with individual rationality, the revenue maximizing payment rule

[3] The results trivially extend to a seller with multiple identical units.

given any (monotone) allocation rule is:

$$P(v) = va(v) - \int_0^v a(x)dx.$$

3. Substituting into the objective function and reorganizing yields the classic 'virtual valuation,' i.e. the surplus from allotting a type less the incentive rents that have to be paid to higher types. The program becomes:

$$\begin{aligned} \max_a \quad & \int_v \varphi(v)f(v)a(v)dv \\ & \textit{subject to:} \\ & a(v) \text{ increasing in } v, \\ & \text{Feasibility (at most 1 unit gets allotted).} \end{aligned}$$

Here, $\varphi(v) = v - \frac{1-F(v)}{v}$ is the 'virtual valuation' of type v.

If $\varphi(v)$ is non-decreasing in v, the optimal allocation rule allots the good to the type with the highest non-negative virtual value is optimal. The monotonicity constraint is satisfied 'for free' in this case. The resulting mechanism can be implemented as a second price auction with (an appropriately chosen) reserve.

If $\varphi(v)$ is not non-decreasing, then the monotonicity constraint must be enforced using a procedure called 'ironing.' The optimal solution will pool some types- i.e. randomize over them. Therefore, in this setting, the increasing virtual-valuation condition is necessary *and* sufficient for the 'virtually-efficient' mechanism to be optimal.

1.1.2 Our Results

We now compare Myerson's results to ours. First, note that due to the dynamic nature of the problem, the possible misreports by a type are restricted by assumption. As a result the revelation principle cannot be applied directly. Standard proofs of the revelation principle are for settings where any type can send any feasible message: by contrast, in our setting the periods in which an agent is present (and hence the periods in which she can send messages) depends on her type.[4] In Observation 1, we show it suffices to consider mechanisms in which buyers never receive an object before their deadline. This allows us to invoke an appropriate version of the revelation principle to restrict attention to direct revelation mechanisms. See Appendix B for a formal discussion.

Next, we use the incentive compatibility constraints to pin down properties of the optimal allocation rule. We establish that the adjacent (i.e., local) incentive compatibility constraints suffice, and that the allocation rule must be monotone in valuation (holding entry and exit time fixed). Coupled with individual rationality, the revenue maximizing payment rule is similar to Myerson's.[5] Substituting this payment rule into the inequality, incentive compatibility also requires monotonicity of the allocation rule

[4] Green and Laffont [11] identify a sufficient condition for the revelation principle to hold in settings where the set of messages an agent can send is a function of her type. However, they consider a static setting.

[5] Subject to a minor caveat described in Section 4.

in entry and exit time (defined formally below). Further, substituting this payment rule into the objective function, we get a notion of virtual valuation similar to Myerson's. We use this 'virtual valuation' as the index of our type.

We can now define a dynamic program to compute the optimal allocation rule. It allots a unit to agents whose index (virtual value) exceeds a threshold (marginal virtual surplus of that unit if that unit was not allotted to them). It should be clear that this algorithm is the dynamic analogue of Myerson.[6]

It is here where Myerson's result diverge from ours. In his setting, increasing virtual valuations were necessary and sufficient for this 'greedy' algorithm to produce the optimal allocation rule. Our results are weaker:

1. If the deadlines, $\bar{t}_i$, are common knowledge, then, the distribution of types satisfying an appropriate monotone hazard rate condition is necessary and sufficient for our algorithm (a simple index policy) to characterize the optimal allocation rule.
2. If we drop the assumption of common knowledge of the deadlines, then the appropriate monotone hazard rate condition is *not sufficient*. If the allocation rule produced by our algorithm is not incentive compatible, then a form of ironing will be required in spite of the maintained assumption of a monotone hazard rate. Alternately, the optimal allocation rule cannot be recursively computed.

We are silent on *how* to 'iron' in dynamic settings and suspect a clean analytic result is not easily forthcoming. Recent work by Mierendorff [15] considers a simpler model (2 periods, 2 buyers, simpler type space). He completely characterizes the optimal mechanism in this setting, but the methods appear involved, and hard to generalize. There has been some computational work on implementing ironing in a related dynamic setting- we refer the interested reader to Parkes et al [20]. While both will not (and should not) satisfy those interested in a complete characterization, we believe they highlight the difficulties of ironing in a dynamic setting.

1.2 Related Literature

There is a growing literature on mechanism design in dynamic settings. Comprehensively listing them would be beyond the purview of this paper. We refer readers to the excellent survey by Bergemann and Said [2] for a broad overview.

From the point of view of index rules in dynamic mechanism design, we can view the existing literature from one of two lenses.

The more relevant for this paper are other settings in which one can show optimality of index rules. Vulcano *et al* [24] consider a special case of our model where each buyer is present for at most one period (in terms of our model $\bar{t}_i = t_i$ for all i). The only private information is the buyer's valuation for a single unit. Gallien [8] considers a monopolist selling a finite number of identical items over an infinite horizon to time sensitive buyers with unit demands, and private value and time of entry. Time sensitivity of buyers is modeled using a common, commonly known discount factor. He derives the expected revenue optimal incentive compatible mechanism. Gershkov and Moldovanu [9] consider a similar setting, except that the monopolist now sells heterogenous goods

[6] Consider the algorithm for a single period. The virtual surplus of withholding is 0. Therefore the algorithm will allot to the highest buyer provided his virtual surplus exceeds 0.

which are commonly ranked by all buyers, and each buyer's private information is one dimensional. Further their model considers a finite horizon. They are able to derive a revenue maximizing mechanism for their setting, using a calculus of variations approach. Gershkov and Moldovanu [10] derive the efficient mechanism in the same setting.

In all these papers the authors succeed by limiting the dimensionality of the buyers' type- in particular, the the buyers' patience is commonly known. In our model, by contrast, buyers have an additional dimension of private information.

There are other works which restrict attention to index rules and attempt to show good (revenue) properties of these rules. For instance Pavan, Segal and Toika [21] consider mechanism design in a setting where agents have a one dimensional type which changes every period. They design the profit maximizing mechanism, however they restrict attention to 'monotonic' allocation rules (since they consider general mechanism design settings, there is no natural analog to index rules). Lavi and Nisan [13] consider the same model as Vulcano *et al* but in a setting where no prior distribution over types is assumed. They propose a index rule and perform a worst case analysis of the revenue achieved. Ng, Parkes and Seltzer [17] investigate a closely related model where the seller has C perishable units of a good in each period. They exhibit a dominant strategy mechanism, and perform a worst case analysis of the revenue achieved. Hajiaghayi, Kleinberg, Mahdian and Parkes [12] consider the same model but achieve a better competitive ratio.

For complete solutions when the optimal mechanism is not a simple index rule, there is the closely related recent paper by Mierendorff [15]. He considers our setting restricted to 2 periods and 2 buyers. He completely characterizes the optimal mechanism in this setting. When the optimal mechanism is not a simple threshold rule, it is hard to describe. Parkes et al [20], in a related setting, note that ironing in dynamic settings can be computationally demanding. They study algorithms which restore the appropriate monotonicity of the allocation rule, and which are only approximately optimal.

Another 'twist' in our model compared to standard mechanism design is that due to the dynamics buyers have only certain misreports available to them as a function of their type– they can only pretend to be someone arriving later and leaving earlier. Therefore we do not impose 'all' IC constraints, only those for available misreports. There are several previous works that have considered similar restrictions on misreports in static settings, for example, Blackorby and Szalay [3], and Beaudry, Blackorby and Szalay [1], Celik [6], Che and Gale [7], Malakhov and Vohra [14], Pai and Vohra [18], among others.

1.3 Organization of this Paper

In Section 2 we introduce notation and describe our model, and formulate the seller's decision problem. Section 3 describes the incentive compatibility constraints that a mechanism must meet, and then simplifies them. In Section 4 we analyze the seller's decision problem, and derive the (expected) revenue maximizing auction. In this section, we also discuss the difficulties associated with guaranteeing truthful report of exit times. Section 5 discusses possible extensions to this model and concludes.

2 Model and Notation

A seller wishes to sell C units of a good (identical and indivisible) over T time periods, numbered $1, 2, \ldots, T$.

Type Space: Buyers arrive over time. A buyer's type is a 3-tuple $(v, t, \bar{t})$, interpreted as follows:

- $v \in V$ is the valuation of the agent for one unit of the good. V, the set of all possible valuations, is a finite set of reals, i.e. $V \subset \Re_+$. For economy of notation we assume that V is an evenly spaced grid, $V = \{0, \epsilon, 2\epsilon, \ldots, \bar{V}\epsilon\}$, $\epsilon > 0$, with $\epsilon = 1$.[7]
- $t \leq T$ is the time that this buyer learns of his demand for a unit. Call this the entry time of this buyer into the system.
- $\bar{t} \in [t, T]$ is his deadline for being allotted a ticket, or the last time period in which a ticket holds any value. We refer to this as the buyer's exit time from the system.

Therefore, there is a finite set of buyers' types $\mathbb{T} \subset V \times T^2$. Given a probability of allotment a, an agent of type $(v, t, \bar{t})$, enjoys a monetary value as: [8]

$$V(a|(v, t, \bar{t})) = va. \tag{1}$$

Distribution over Types: In each period t, N_t risk neutral potential buyers arrive. N_t is a non-negative, discrete valued random variable, distributed according to a known probability mass function $g(\cdot)$, with support $\{0, 1, \ldots, \infty\}$.[9]

There is a common prior (a probability distribution with p.d.f. f) on the space of types $\mathbb{T}$. For any entry time t and exit time $\bar{t} \geq t$, let $f_{(t,\bar{t})}(\cdot)$ and $F_{(t,\bar{t})}(\cdot)$ denote the pdf and cdf respectively of the distribution of a buyer's valuation conditional on his entry at time t and exit at $\bar{t}$.

Agents arriving at time t receive i.i.d. draws from the posterior of this distribution (i.e. given time of arrival is t). Formally, the probability that an agent arriving at time t has type $(v, t, \bar{t})$ is

$$\mathbb{P}\{\text{type} = (v, t, \bar{t}) | \text{ arrival time } = t\} = \frac{f(v, t, \bar{t})}{\sum_{v' \in V, \bar{t}' \geq t} f(v', t, \bar{t}')}. \tag{2}$$

Thus, we assume the valuation and exit time of a buyer arriving at time t is independent of both the number and type of other buyers.

Consider a partial order $\succeq$ on the space $\mathbb{T}$, defined as

$$(v', t', \bar{t}') \succeq (v, t, \bar{t}) \equiv (v' \geq v) \wedge (t' \leq t) \wedge (\bar{t}' \geq \bar{t}). \tag{3}$$

In other words, a type is said to be partially ordered above another according to $\succeq$ if it has a (weakly) higher valuation, arrives earlier and has a later deadline. We now define the appropriate version of a 'monotone hazard rate' condition for this setting.

[7] See [14] for a characterization for general finite V.

[8] Since the agent here is sensitive to when he gets allotted the object, strictly speaking we should point out that this is if he gets the allotment in any period t', $t \leq t' \leq \bar{t}$.

[9] Our reasons for requiring full support are purely technical and discussed in Appendix A, Footnote 18.

Definition 1 *A distribution with pdf f on the space of types $\mathbb{T}$ satisfies the monotone hazard rate condition if*

$$(v', t', \bar{t}') \succeq (v, t, \bar{t}) \Rightarrow \frac{f_{(t',\bar{t}')}(v')}{1 - F_{(t',\bar{t}')}(v')} \geq \frac{f_{(t,\bar{t})}(v)}{1 - F_{(t,\bar{t})}(v)}, \tag{4}$$

The monotone hazard rate condition imposes two restrictions. First, fixing entry and exit times, the distribution of valuations of types with that entry and exit time has an increasing hazard rate. Second: fix $(t, \bar{t})$ and $(t', \bar{t}')$ such that $[t, \bar{t}] \subseteq [t', \bar{t}']$. The condition implies that: [10]

$$\mathbb{E}[v|(t, \bar{t})] \geq \mathbb{E}[v|(t', \bar{t}')]$$

Returning to our airline motivation, thinking of impatient buyers as business travelers, and patient buyers as 'leisure' travelers, we are effectively assuming that the former are expected to have higher valuations than the latter. Alternately, our assumption says that fixing an arrival time high valuation travelers tend to be more impatient than low valuation travelers. Note that this is orthogonal to the standard assumption in this literature that leisure travelers arrive early, while business travelers arrive late– neither implies nor contradicts the other.

Information Structure and Solution Concept: We assume that in each period, a buyer knows only her type, the current clock time t, and the model. She is ignorant of the number of other buyers (past and present), their types and reports, or of previous allocation decisions by the mechanism.

Our solution concept is the standard Bayes-Nash equilibrium- each buyer will have (correct) beliefs over all past and future activity activity in the mechanism, and take expectations appropriately.

Our 'no information' assumption is partly for tractability. If the buyer had such information, we would have to ensure she had no incentives to misreport at each possible information set she could have, which would be rather unwieldy. That said, one can show that if exit times are common knowledge, our results continue to hold even if buyers have the extra information alluded to above. In fact, under this assumption it can be shown that our allocation rule can be supported as an ex-post equilibrium (with a different pricing rule).

Direct Revelation Mechanisms: Invoking the revelation principle, we can restrict attention to direct revelation mechanisms.[11] Buyers are asked to reveal their valuation and their exit time. The time at which a buyer makes a report is noted as her time of entry. In each period, the auctioneer, as a function of the reports received up to time t, decides on which agents to allot a unit to (if any), and the payment each agent is to make.

[10] It actually has a stronger implication, i.e. that $F_{(t,\bar{t})}$ first order stochastically dominates $F_{(t',\bar{t}')}$.

[11] A proof that an appropriate revelation principle applies is in Appendix B. The standard revelation principle applies to game forms where all types can send the same set of messages. This does not hold in our setting due to the dynamics. For example, a later arriving buyer cannot send the same set of messages an early arriving buyer can.

We define an inventory I as a list of reports of agent types that arrived over the T time periods. The set of all possible inventories is denoted $\mathbb{I}$. At time t, the auctioneer only knows $I^t \subseteq I$, the reports that arrived at time t or earlier. Similarly, let $I_t \subseteq I$ be the reports in I that expire at or after t. Let $n(v,t,\bar{t})(I)$ be the number of agents of type $(v,t,\bar{t})$ in inventory I.

An allocation rule is a sequence of functions, one for each time period $\tau \in T$, $a^\tau : \mathbb{T} \times \mathbb{I}^\tau \to [0,1]$. In other words, $a^\tau_{(v,t,\bar{t})}(I^\tau)$ is the allocation that an agent who announces type $(v,t,\bar{t})$ gets in time period τ, given that the inventory of bids in the system up to that point in time are I^τ. If a type $(v,t,\bar{t})$ is not present in the inventory I^τ, we take $a^\tau_{(v,t,\bar{t})}(I^\tau) = 0$.

Further, let $P : \mathbb{T} \times \mathbb{I} \to \Re_+$ be the payment function, i.e. an agent who announces type $(v,t,\bar{t})$ when the inventory turns out to be I makes a payment of $P(v,t,\bar{t})(I)$. Note that we implicitly assume that all payments are collected at time T, however it will be clear that our results will not change if the payment of a type $(v,t,\bar{t})$ can only be collected in the periods t to $\bar{t}$.

An agent derives utility only if he receives a unit between his entry and exit times. An agent of type $(v,t,\bar{t})$ cares for exactly 1 unit, allotted to him sometime between t and $\bar{t}$. Let $\mathcal{A}_{(v,t,\bar{t})}(I) = \max\{a^\tau_{(v,t,\bar{t})}(I^\tau) : t \le \tau \le \bar{t}\}$; we refer to this as his allotment.

We now define the interim allocation and payment rules:

$$A(v,t,\bar{t}) = \mathbb{E}_I\big[\mathcal{A}_{(v,t,\bar{t})}(I) | [(v,t,\bar{t}) \in I]\big],$$
$$P(v,t,\bar{t}) = \mathbb{E}_I\big[P(v,t,\bar{t})(I) | [(v,t,\bar{t}) \in I]\big].$$

The expected monetary gain of an agent whose type is $(v,t,\bar{t})$ is $vA(v,t,\bar{t}) - P(v,t,\bar{t})$.

We are now in a position to describe the constraints that a direct revelation mechanism must satisfy.

1. Incentive Compatibility (IC): (Interim) - Incentive Compatibility requires that no agent should have an incentive to misreport his type. In other words, for every type $(v,t,\bar{t})$, and any feasible misreport $(v',t',\bar{t}')$ (i.e. $t' \ge t$ and $\bar{t}' \le \bar{t}$):

$$V(A(v,t,\bar{t})|(v,t,\bar{t})) - P(v,t,\bar{t}) \ge V(A(v',t',\bar{t}')|(v,t,\bar{t})) - P(v',t',\bar{t}'). \tag{5}$$

2. Individual Rationality (IR): All types must have non-negative expected surplus from participating in the auction (it will be clear that the mechanism can be implemented to be ex-post individually rational).

$$V(A(v,t,\bar{t})|(v,t,\bar{t})) - P(v,t,\bar{t}) \ge 0 \tag{6}$$

3. Feasibility of allocation (FEAS): For any inventory of bids, the allocation rule allots at most C units over the T periods.

$$\forall I : \sum_{\tau \in T} \sum_{(v,t,\bar{t}) \in \mathbb{T}} n(v,t,\bar{t})(I^t) A(v,t,\bar{t})^\tau(I^\tau) \le C. \tag{7}$$

4. No Clairvoyance (NC): An allocation rule cannot base the allotment at time τ on the types of agents who enter the system after time τ. We have implicity assumed this by specifying the allocation at time τ as a function of the inventory up to time τ, I^τ.

Therefore a full characterization of the revenue maximizing Bayesian Incentive compatible mechanism would require a solution to:[12]

$$\max_{(a^\tau)_1^T, P} \sum_{t=1}^{T} \sum_{N_t=1}^{\infty} N_t g(N_t) \Big(\sum_{(v,t,\bar{t})} f((v,t,\bar{t})|t) P(v,t,\bar{t}) \Big) \tag{OPT1}$$

$$\text{s.t.} \quad (5), (6), (7).$$

Note that the objective function can be rewritten as:

$$\begin{aligned}
&\max_{(a^\tau)_1^T, P} \sum_{t=1}^{T} \Big(\sum_{(v,t,\bar{t})} f((v,t,\bar{t})|t) P(v,t,\bar{t}) \Big) \sum_{N_t=1}^{\infty} N_t g(N_t) \\
=& \max_{(a^\tau)_1^T, P} \sum_{t=1}^{T} \mathbb{E}_g[N] \Big(\sum_{(v,t,\bar{t})} f((v,t,\bar{t})|t) P(v,t,\bar{t}) \Big) \\
=& \mathbb{E}_g[N] \max_{(a^\tau)_1^T, P} \sum_{t=1}^{T} \Big(\sum_{(v,t,\bar{t})} f((v,t,\bar{t})|t) P(v,t,\bar{t}) \Big)
\end{aligned}$$

Finally, we need to define threshold rules. An allocation rule A is said to be a *threshold rule* if an agent who reported type $(v,t,\bar{t})$ gets the good implies that holding everything else fixed the agent would also have gotten the good if he had reported $(v',t,\bar{t})$ for $v' \geq v$. In our notation:

Definition 2 *An allocation rule a is said to be a* threshold rule *if for all inventories I, types $(v,t,\bar{t})$, values $v' \geq v$ and time τ such that $t \leq \tau \leq \bar{t}$:*

$$\begin{aligned}
&a^\tau_{(v,t,\bar{t})}(I^\tau) \in \{0,1\}, \\
&a^\tau_{(v,t,\bar{t})}(I^\tau) = 1 \implies a^\tau_{(v',t,\bar{t})}(I'^\tau) = 1.
\end{aligned}$$

3 Incentive Compatibility

An agent of type $(v,t,\bar{t})$ can only misreport his type as $(v',t',\bar{t}')$, where:

- $t' \geq t$: This captures the fact that the agent can announce his type only after he is in the system.[13]
- $\bar{t}' \leq \bar{t}$: This says that an agent cannot claim to be in the system any later than he actually is, alternately, receiving the object after the agent's true deadline is worthless to the agent.

The relevant (Bayesisan) incentive compatibility (IC) constraints are listed below.

[12] No Clairvoyance is implicitly assumed rather than explicitly imposed as a constraint.

[13] Alternately, an agent can announce his type only after he knows it himself.

1. Misreport Value: An agent can misreport his valuation.

$$V(A(v,t,\bar{t})|(v,t,\bar{t})) - P(v,t,\bar{t}) \geq V(A(v',t,\bar{t})|(v,t,\bar{t})) - P(v',t,\bar{t}). \tag{8}$$

2. Under-report Presence: An agent present in the system from t through $\bar{t}$ periods may choose to report he is in the system for some strict contiguous subset of that, i.e. $t' \geq t$, $\bar{t}' \leq \bar{t}$.

$$V(A(v,t,\bar{t})|(v,t,\bar{t})) - P(v,t,\bar{t}) \geq V(A(v,t',\bar{t})|(v,t,\bar{t})) - P(v,t',\bar{t}), \tag{9}$$

$$V(A(v,t,\bar{t})|(v,t,\bar{t})) - P(v,t,\bar{t}) \geq V(A(v,t,\bar{t}')|(v,t,\bar{t})) - P(v,t,\bar{t}'). \tag{10}$$

3. Misreport Value and Presence:

$$V(A(v,t,\bar{t})|(v,t,\bar{t})) - P(v,t,\bar{t}) \geq V(A(v',t',\bar{t}')|(v,t,\bar{t})) - P(v',t',\bar{t}'). \tag{11}$$

Therefore our optimization problem, OPT1 is equivalent to:

$$\max_{(a^\tau)_1^T, P} \sum_{t=1}^{T} \left(\sum_{(v,t,\bar{t})} f((v,t,\bar{t})|t) P(v,t,\bar{t}) \right) \tag{OPT2}$$
$$\text{s.t. } (8) - (11), (6), (7).$$

3.1 Implications of Incentive Compatibility

In this section we show that the only IC constraints that matter are the **adjacent** misreports of value and misreports of presence.

Recall that given an allocation of a, an agent of type $(v,t,\bar{t})$ derives utility from it only if this allotment is made in a period $t' \in [t,\bar{t}]$.

Observation 1 *Consider any mechanism (a,P) that is feasible in OPT2. There exists an allocation rule a' where the allocation rule a' allots each type $(v,t,\bar{t})$ at its exit time $\bar{t}$ if at all, such that (a',P) is feasible in OPT2 and produces the same expected revenue.*

Proof: Consider the allocation rule a' constructed as follows: For each type $(v,t,\bar{t})$, reallocate the maximum probability of getting a unit during any period t through $\bar{t}$ to getting exactly at time $\bar{t}$. As long as a is feasible (7), a' will also be, since each agent gets weakly fewer units. Also a' will be incentive compatible with the same pricing rule since each agent, by construction, will be indifferent between a and a' regardless of his type. The result follows. ■

We shall refer to allocation rules that allot agents only at their reported exit times as having an *allot on exit* property. By Observation 1, restricting attention to such allocation rules is without loss of generality. Therefore from hereon in we only consider allocation rules that satisfy the allot on exit property, and shall refer to allocation rules only by the interim probability of allocation A. [14]

[14] As an aside, allocation rules that satisfy the allot on exit property remain Incentive compatible even if reporting a later exit time were allowed. See Observation 5 in the Appendix for details.

The following observation is based on the case of one dimensional types. It says that holding entry and exit times fixed, incentive compatibility of the interim probability of allocation of a type must be increasing in its valuation. We refer to this as *valuation monotonicity* of the allocation rule.

Observation 2 *If a mechanism (A, P) is incentive compatible, then:*

$$\forall t, \bar{t} : v' > v \Rightarrow A(v', t, \bar{t}) \geq A(v, t, \bar{t}). \tag{12}$$

Proof: Suppose not, i.e. suppose for some $t, \bar{t}$ and $v' > v$, we have that $A(v', t, \bar{t}) < A(v', t, \bar{t})$. From the fact that $(v, t, \bar{t})$ has no incentive to report her type as $(v', t, \bar{t})$, we have that:

$$v(A(v, t, \bar{t}) - A(v', t, \bar{t})) \geq P(v, t, \bar{t}) - P(v', t, \bar{t}).$$

Since $v' > v$, we have that:

$$v' \cdot (A(v, t, \bar{t}) - A(v', t, \bar{t})) > P(v, t, \bar{t}) - P(v', t, \bar{t}).$$

Rewriting, we see:

$$v'(A(v', t, \bar{t})|(v', t, \bar{t})) - P(v', t, \bar{t}) < v'(A(v, t, \bar{t})|(v', t, \bar{t})) - P(v, t, \bar{t}).$$

This contradicts incentive compatibility for type $(v', t, \bar{t})$ misreporting as $(v, t, \bar{t})$. ■

Next, we identify a subset of these IC constraints that are redundant. As a notational shorthand we refer to the IC constraint corresponding to where a type $(v, t, \bar{t})$ misreports his type as $(v', t', \bar{t}')$ by the notation $(v, t, \bar{t}) \to (v', t', \bar{t}')$.

Lemma 1 *The IC constraint for misreport of value and presence (11) is implied by the misreport of value (8) and misreport of presence (9 and 10) IC constraints.*

Proof: Add the ICs $(v, t, \bar{t}) \to (v, t', \bar{t}')$ and $(v, t', \bar{t}') \to (v', t', \bar{t}')$. This yields:

$$\begin{aligned} P(v, t, \bar{t}) - P(v', t', \bar{t}') \leq\ & V(A(v, t, \bar{t})|(v, t, \bar{t})) + V(A(v, t', \bar{t}')|(v, t', \bar{t}')) \\ & - V(A(v, t', \bar{t}')|(v, t, \bar{t})) - V(A(v', t', \bar{t}')|(v, t', \bar{t}')). \end{aligned} \tag{13}$$

Recall that by our assumptions on the functional form of v;

$$V(A(v, t', \bar{t}')|(v, t, \bar{t})) = V(A(v, t', \bar{t}')|(v, t', \bar{t}')), \tag{14}$$

$$V(A(v', t', \bar{t}')|(v, t', \bar{t}')) = V(A(v', t', \bar{t}')|(v, t, \bar{t})). \tag{15}$$

Substituting from (14) and (15) into (13), we see that (13) implies the IC (11). ■

We now show that of the remaining IC constraints, the 'adjacent' ones suffice.

Lemma 2 *All IC constraints are implied by the following 'adjacent' IC constraints:*

1. $(v, t, \bar{t}) \to (v + 1, t, \bar{t})$
2. $(v, t, \bar{t}) \to (v - 1, t, \bar{t})$

3. $(v, t, \bar{t}) \to (v, t+1, \bar{t})$

4. $(v, t, \bar{t}) \to (v, t, \bar{t}-1)$

PROOF: From Observation 2, we have that $v' > v \Rightarrow A(v', t, \bar{t}) \geq A(v, t, \bar{t})$.

To prove item 1, we show that:

$$\begin{aligned}
&V(A(v,t,\bar{t})|(v,t,\bar{t})) - P(v,t,\bar{t}) \;\geq\; V(A(v-1,t,\bar{t})|(v,t,\bar{t})) - P(v-1,t,\bar{t}),\\
&V(A(v-1,t,\bar{t})|(v-1,t,\bar{t})) - P(v-1,t,\bar{t}) \;\geq\; V(A(v-2,t,\bar{t})|(v-1,t,\bar{t})) - P(v-2,t,\bar{t})\\
\implies &V(A(v,t,\bar{t})|(v,t,\bar{t})) - P(v,t,\bar{t}) \;\geq\; V(A(v-2,t,\bar{t})|(v,t,\bar{t})) - P(v-2,t,\bar{t}).
\end{aligned}$$

The rest follows by induction. Adding the two inequalities above, and adding and subtracting $V(A(v-2,t,\bar{t})|(v,t,\bar{t}))$; we get

$$\begin{aligned}
V(A(v,t,\bar{t})|(v,t,\bar{t})) - P(v,t,\bar{t}) \geq &\Big(V(A(v-1,t,\bar{t})|(v,t,\bar{t})) - V(A(v-2,t,\bar{t})|(v,t,\bar{t}))\Big)\\
&- \Big(V(A(v-1,t,\bar{t})|(v-1,t,\bar{t})) - V(A(v-2,t,\bar{t})|(v-1,t,\bar{t}))\Big)\\
&+ V(A(v-2,t,\bar{t})|(v,t,\bar{t})) - P(v-2,t,\bar{t}).
\end{aligned}$$

Recall that by increasing differences, and monotonicity of the allocation rule:

$$\begin{aligned}
0 \;\leq\; &\Big(V(A(v-1,t,\bar{t})|(v,t,\bar{t})) - V(A(v-2,t,\bar{t})|(v,t,\bar{t}))\Big)\\
&- \Big(V(A(v-1,t,\bar{t})|(v-1,t,\bar{t})) - V(A(v-2,t,\bar{t})|(v-1,t,\bar{t}))\Big).
\end{aligned}$$

Therefore, we have that

$$V(A(v,t,\bar{t})|(v,t,\bar{t})) - P(v,t,\bar{t}) \geq V(A(v-2,t,\bar{t})|(v,t,\bar{t})) - P(v-2,t,\bar{t}).$$

Item 2 can also be proven in a similar manner, and the proof is omitted. We now proceed to show item 3. To show this we prove that the following pair of inequalities:

$$\begin{aligned}
&V(A(v,t,\bar{t})|(v,t,\bar{t})) - P(v,t,\bar{t}) \;\geq\; V(A(v,t+1,\bar{t})|(v,t,\bar{t})) - P(v,t+1,\bar{t}),\\
&V(A(v,t+1,\bar{t})|(v,t+1,\bar{t})) - P(v,t+1,\bar{t}) \;\geq\; V(A(v,t+2,\bar{t})|(v,t+1,\bar{t})) - P(v-2,t+2,\bar{t})\\
\implies &V(A(v,t,\bar{t})|(v,t,\bar{t})) - P(v,t,\bar{t}) \;\geq\; V(A(v,t+2,\bar{t})|(v,t,\bar{t})) - P(v,t+2,\bar{t}).
\end{aligned}$$

To see this add the two ICs and recall that based on our assumptions about functional form of v.

$$V(A(v,t+1,\bar{t})|(v,t,\bar{t})) = V(A(v,t+1,\bar{t})|(v,t+1,\bar{t})) \tag{16}$$

$$V(A(v,t+2,\bar{t})|(v,t+1,\bar{t})) = V(A(v,t+2,\bar{t})|(v,t,\bar{t})) \tag{17}$$

Making the appropriate substitutions via (16) & (17), we get the desired inequality. The proof of item 4 is similar and omitted. ■

3.2 Monotonicity

When types are one dimensional, an allocation rule is said to be monotonic if higher types have a higher (interim) probability of getting allotted than lower types. We modify the definition to account for the fact that in this setting, types are multi-dimensional.

Definition 3 *An allocation rule A is said to be monotonic if it satisfies:*

1. Valuation Monotonicity*:* $\forall v, v', t, \bar{t} : (v \geq v') \Rightarrow (A(v, t, \bar{t}) \geq A(v', t, \bar{t}))$*, i.e. a higher valuation increases probability of allotment (all other things being equal).*
2. Entry Monotonicity*:* $\forall v, t, \bar{t}, t' : (t \geq t') \Rightarrow (A(v, t, \bar{t}) \leq A(v, t', \bar{t}))$*, i.e. an earlier entry into the system increases probability of allotment (all other things being equal).*
3. Exit Monotonicity*:* $\forall v, t, \bar{t}, \bar{t}' : (\bar{t} \leq \bar{t}') \Rightarrow (A(v, t, \bar{t}) \leq A(v, t, \bar{t}'))$*, i.e. a later exit from the system increases probability of allotment (all other things being equal).*

Note that this is equivalent to the allocation rule being monotonic with respect to the partial order we imposed on the space of types, i.e.

$$(v, t, \bar{t}) \succeq (v', t', \bar{t}') \Rightarrow A(v, t, \bar{t}) \geq A(v', t', \bar{t}').$$

In our setting, valuation monotonicity follows from Observation 2. However in contrast to the classical 1-D types case, entry and exit monotonicity are not implications of incentive compatibility.

4 Characterization

Due to the results of Section 3, the optimization problem OPT2 can be written as:

$$\max_{(a^\tau)_1^T, P} \sum_{t=1}^{T} \Big(\sum_{(v,t,\bar{t})} f((v, t, \bar{t})|t) P(v, t, \bar{t}) \Big) \tag{OPT3}$$

$$\begin{aligned}
\text{s.t. } & V(A(v, t, \bar{t})|(v, t, \bar{t})) - P(v, t, \bar{t}) \geq V(A(v-1, t, \bar{t})|(v, t, \bar{t})) - P(v-1, t, \bar{t}) \\
& V(A(v, t, \bar{t})|(v, t, \bar{t})) - P(v, t, \bar{t}) \geq V(A(v+1, t, \bar{t})|(v, t, \bar{t})) - P(v+1, t, \bar{t}) \\
& V(A(v, t, \bar{t})|(v, t, \bar{t})) - P(v, t, \bar{t}) \geq V(A(v, t+1, \bar{t})|(v, t, \bar{t})) - P(v, t+1, \bar{t}) \\
& V(A(v, t, \bar{t})|(v, t, \bar{t})) - P(v, t, \bar{t}) \geq V(A(v, t, \bar{t}-1)|(v, t, \bar{t})) - P(v, t, \bar{t}-1) \\
& \text{Individual Rationality (6)} \\
& \text{Feasibility (7).}
\end{aligned}$$

4.1 The Pricing Rule

Now suppose a (feasible, allot at exit) allocation rule a is given. We characterize the expected revenue maximizing payment rule P such that (A, P) is feasible in OPT3. Our methodology is based on Malakhov and Vohra [14]. They show in their model that the problem of finding the optimal payment rule given an allocation rule is the dual of finding the shortest path in a network. They also show that if the allocation rule is monotonic, then one can characterize these shortest paths.

Fix a (feasible, allot at exit) allotment rule a. The relaxed linear program to maximize expected revenue for this allocation rule is:

$$\max_P \sum_{t=1}^{T} \Big(\sum_{(v,t,\bar{t})} f((v,t,\bar{t})|t) P(v,t,\bar{t}) \Big) \tag{OPTPRICE}$$

$$\begin{aligned}
\text{s.t. } & V(A(v,t,\bar{t})|(v,t,\bar{t})) - P(v,t,\bar{t}) \geq V(A(v-1,t,\bar{t})|(v,t,\bar{t})) - P(v-1,t,\bar{t}) \\
& V(A(v,t,\bar{t})|(v,t,\bar{t})) - P(v,t,\bar{t}) \geq V(A(v+1,t,\bar{t})|(v,t,\bar{t})) - P(v+1,t,\bar{t}) \\
& V(A(v,t,\bar{t})|(v,t,\bar{t})) - P(v,t,\bar{t}) \geq V(A(v,t+1,\bar{t})|(v,t,\bar{t})) - P(v,t+1,\bar{t}) \\
& V(A(v,t,\bar{t})|(v,t,\bar{t})) - P(v,t,\bar{t}) \geq V(A(v,t,\bar{t}-1)|(v,t,\bar{t})) - P(v,t,\bar{t}-1) \\
& V(A(v,t,\bar{t})|(v,t,\bar{t})) - P(v,t,\bar{t}) \geq 0
\end{aligned}$$

We now describe a network representation of this linear program. Introduce a vertex for each type in $\mathbb{T}$. Introduce a dummy vertex that corresponds to a dummy type that is always allotted nothing, and pays 0. This allows us to represent the IR constraint as an extra IC constraint. We need to introduce 5 types of edges corresponding to four classes of adjacent ICs in this model, and the IR constraint.

1. IC $(v,t,\bar{t}) \to (v-1,t,\bar{t})$: Introduce an edge from $(v-1,t,\bar{t})$ to $(v,t,\bar{t})$, of length $V(A(v,t,\bar{t})|(v,t,\bar{t})) - V(A(v-1,t,\bar{t})|(v,t,\bar{t})) = v(A(v,t,\bar{t}) - A(v-1,t,\bar{t}))$.
2. IC $(v,t,\bar{t}) \to (v+1,t,\bar{t})$: Introduce an edge from $(v+1,t,\bar{t})$ to $(v,t,\bar{t})$, of length $V(A(v,t,\bar{t})|(v,t,\bar{t})) - V(A(v+1,t,\bar{t})|(v,t,\bar{t})) = v(A(v,t,\bar{t}) - A(v+1,t,\bar{t}))$.
3. IC $(v,t,\bar{t}) \to (v,t+1,\bar{t})$: Introduce an edge from $(v,t+1,\bar{t})$ to $(v,t,\bar{t})$, of length $V(A(v,t,\bar{t})|(v,t,\bar{t})) - V(A(v,t+1,\bar{t})|(v,t,\bar{t})) = v(A(v,t,\bar{t}) - A(v,t+1,\bar{t}))$.
4. IC $(v,t,\bar{t}) \to (v,t,\bar{t}-1)$: Introduce an edge from $(v,t,\bar{t}-1)$ to $(v,t,\bar{t})$, of length $V(A(v,t,\bar{t})|(v,t,\bar{t})) - V(A(v,t,\bar{t}-1)|(v,t,\bar{t})) = v(A(v,t,\bar{t}) - A(v,t,\bar{t}-1))$.
5. IR constraint for $(v,t,\bar{t})$: Introduce an edge from the dummy node to each vertex $(v,t,\bar{t})$ of length $vA(v,t,\bar{t})$

OPTPRICE is the dual of a min-cost flow problem on the graph specified above. For each type $(v,t,\bar{t})$ a flow of $\mathbb{E}[N_t]f((v,t,\bar{t})|t)$ needs to be sent to its corresponding vertex from the dummy node.[15] Let $P(v,t,\bar{t})$ be the cost of the relevant flow.

The following theorem shows that if the allocation rule is monotonic (as defined in Definition 3), the shortest path from type $(1,\tau,\tau)$ to a generic type $(v,t,\bar{t})$ (where $t \leq \tau \leq \bar{t}$) is of the form $(1,\tau,\tau) \to (1,t,\bar{t}) \to (2,t,\bar{t}) \ldots \to (v,t,\bar{t})$. As a result the payments of any type $(v,t,\bar{t})$ is a linear function of its own allocation, and the allocations of types $(v',t,\bar{t})$, $v' < v$.

Theorem 1 *For any monotonic allocation rule A, the revenue maximizing payment rule P such that (A,P) is incentive compatible and individually rational is:*

$$P(v,t,\bar{t}) = vA(v,t,\bar{t}) - \sum_{k=1}^{v-1} A(k,t,\bar{t}), \quad \forall (v,t,\bar{t}) \in \mathbb{T}. \tag{18}$$

Unlike the static case, Incentive Compatibility does not imply 'full' monotonicity, only value monotonicity. Critically the edge corresponding to the IC constraint

[15] For details, see for instance Section 5.4 of Papadimitriou and Steiglitz [19]).

$(v, t, \bar{t}) \to (v+1, t, \bar{t})$ has negative length. By an application of the fundamental theorem of Linear Programming, given an allocation rule A, there exists a payment rule P such that (A, P) is incentive compatible and individually rational if and only if the corresponding dual network has no negative cycles. This fact can be leveraged to bound the length of the shortest path.

THEOREM 2 *Let the space of valuations be $V = \{0, \epsilon, 2\epsilon, \ldots, \bar{V}\epsilon\}$. Consider any allocation rule A, such that A is valuation monotonic. If there exists P such that (A, P) is incentive compatible, then P must satisfy:*

$$P(v, t, \bar{t}) \leq vA(v, t, \bar{t}) - \epsilon \sum_{k=0}^{v-\epsilon} A(k, t, \bar{t}), \quad \forall (v, t, \bar{t}) \in \mathbb{T}. \tag{19}$$

$$P(v, t, \bar{t}) \geq (v - \epsilon)A(v, t, \bar{t}) - \epsilon \sum_{k=\epsilon}^{v-\epsilon} A(k, t, \bar{t}). \tag{20}$$

PROOF: The proof of the first inequality is easy. Theorem 1 showed that the right hand side of (19) is the length of a particular path from the dummy vertex to the vertex corresponding to type $(v, t, \bar{t})$ (this path is shortest if A is monotonic). Therefore, the shortest path in the graph must have a length less than the length of this particular path.

The proof of the second inequality follows from the 'no negative cycle' requirement. Let $P(v, t, \bar{t})$ be the length of the shortest path from the dummy vertex to the vertex corresponding to type $(v, t, \bar{t})$. The upward IC constraints provide a (negative length) path from the vertex $(v, t, \bar{t})$ back to the dummy vertex. It must be that the length of this cycle is non-negative, i.e

$$\begin{aligned} 0 &\leq P(v, t, \bar{t}) + \sum_{k=\epsilon}^{v-\epsilon} k(A(k, t, \bar{t}) - A(k+\epsilon, t, \bar{t})) \\ &= P(v, t, \bar{t}) - \left((v - \epsilon)A(v, t, \bar{t}) - \epsilon \sum_{k=\epsilon}^{v-\epsilon} A(k, t, \bar{t}) \right). \end{aligned}$$

■

Observe that the gap between the upper and lower inequalities is at most 2ϵ. Therefore, if ϵ is small, the upper bound and lower bound are close. In particular, there is small loss in requiring that that P be as identified in (18).[16]

Hence, from hereonin we simply assume this, at the risk of a small approximation loss.

$$P(v, t, \bar{t}) = vA(v, t, \bar{t}) - \sum_{k=1}^{v-1} A(k, t, \bar{t}), \quad \forall (v, t, \bar{t}) \in \mathbb{T}.$$

[16] An alternate way to prove this would have been to assume that valuations V were drawn from a continuum. The standard argument would have pinned down the payment rule as

$$P(v, t, \bar{t}) = vA(v, t, \bar{t}) - \int_0^v a(v, t, \bar{t}).$$

The following observation follows from (IC).

Observation 3 *Let A be a valuation monotonic allocation rule. Consider the pricing rule P identified in (18). The mechanism (A, P) is incentive compatible if and only if:*

$$\forall (v, t, \bar{t}) \succeq (v, t', \bar{t}') : \sum_{k=1}^{v-1} A(k, t, \bar{t}) \geq \sum_{k=1}^{v-1} A(k, t', \bar{t}'). \tag{21}$$

The proof of this follows by noting that for (A, P) to be incentive compatible, the shortest paths in the dual graph must be the ones identified in the proof of Theorem 1. Note that monotonicity as defined in Definition 3 is sufficient (but not necessary) for (21).

Consider the program OPT3. By Theorem 2, and the subsequent approximation, we can substitute in the pricing rule as identified in (18). By Observation 3, the program can be rewritten as:

$$\max_{(a^\tau)_1^T} \sum_1^T \left(\sum_{(v,t,\bar{t})} f((v, t, \bar{t})|t) \left(vA(v, t, \bar{t}) - \sum_{k=1}^{v-1} A(k, t, \bar{t}) \right) \right) \tag{OPT:MONO}$$

s.t. Shortest Paths as Identified (21),

Value Monotonicity (12),

Feasibility of Allocation (7).

We can re-write the objective function of OPT:MONO as:

$$\sum_1^T \left(\sum_{(v,t,\bar{t}) \in \mathbb{T}} f(v, t, \bar{t}) A(v, t, \bar{t}) \varphi(v, t, \bar{t}) \right),$$

where:

$$\varphi(v, t, \bar{t}) = \left(v - \frac{1 - F_{(t,\bar{t})}(v)}{f_{(t,\bar{t})(v)}} \right) \tag{22}$$

is the type $(v, t, \bar{t})$'s virtual valuation (in the sense of Myerson). By our monotone hazard rate assumption on the distribution (recall Definition 1), the virtual valuation will be increasing in type (according to our partial order).

4.2 A Virtual Efficiency Maximizing Algorithm

Consider a relaxation of OPT:MONO, with the first two classes of constraints relaxed:

$$\max_{(a^\tau)_1^T} \sum_1^T \left(\sum_{(v,t,\bar{t})} f((v, t, \bar{t})|t) \left(vA(v, t, \bar{t}) - \sum_{k=1}^{v-1} A(k, t, \bar{t}) \right) \right) \tag{ROPT:MONO}$$

s.t. Feasibility of allocation (7).

We propose an algorithm to solve ROPT:MONO.

Recall that in Myerson's setting, the highest virtual valuation (and therefore the highest type) was allotted the unit. If there were multiple units, say k, the highest k

virtual valuations were allotted. Our algorithm will be more complicated since it has a dynamic element, but intuitively it is similar— technically, it is a 'dynamic knapsack' algorithm, whereas Myerson's was a static knapsack. In each period t, the state variable that the algorithm will use is:

1. The current time period t.
2. The reports that are expiring, $L_t(I)$ in our notation.
3. The reports it already has that are not expiring, $I_t^t \backslash L_t(I)$.
4. The number of units not yet allotted.

We now define the algorithm inductively:

- In period T: If there are k units left, allot to the k buyers with the highest (non-negative) virtual valuations present.
- In period $t < T$, 1 unit left to allot: in this case if the virtual valuation of the highest expiring report is larger than the expected 'virtual surplus' from having 1 unit in period $t+1$, then allot the unit to this expiring report.[17]
- In period $t < T$, and $k > 1$ units left to allot: For each $k_1, 0 \leq k_1 \leq k$, the algorithm calculates the expected virtual surplus if k_1 units are allotted to the highest k_1 (non-negative) virtual valuations present, and $k - k_1$ units are left for (optimal) allotment in period $t+1$ or later. It then chooses the k_1 that maximizes this expected virtual surplus, and allots accordingly.

It is intuitive that this algorithm solves ROPT:MONO.

4.3 The Allocation Rule

If the solution to ROPT:MONO satisfies the extra constraints in OPT:MONO, i.e. valuation monotonicity (12) and shortest paths as identified (21), we are done. We show that the monotone hazard rate assumption implies that the allocation rule computed by this algorithm will be value monotonic and entry monotonic. However exit monotonicity may be violated.[18] The following theorem summarizes these results.

Theorem 3 *Consider a relaxation of (OPT3) with the IC constraint corresponding to misreport of exit (10) relaxed. Suppose the distribution of types satisfies the monotone hazard rate (4).*[19] *Then there exists a solution of this relaxed program with an allocation scheme that satisfies valuation monotonicity and entry monotonicity. In particular, the allocation rule computed by the algorithm coupled with pricing rule (18) is optimal in this relaxed program.* [20]

Further, if this allocation rule satisfies (21), then it is optimal in OPT3.

[17] Note that this expected virtual surplus $t+1$ will be conditional on already having reports $I_t^t \backslash L_t(I)$ that expire in $t+1$ or later.

[18] See Example 2 for a case where the allocation rule violates exit monotonicity, but the implied shortest paths are the 'correct' ones, and therefore the allocation rule is optimal.

[19] The monotone hazard rate condition assumed here is slightly stronger than the increasing virtual valuation condition required by Myerson. It will be clear from the proofs that we also require only the latter. We impose the former because it admits an economic interpretation.

[20] Daniel Garrett (private communication) notes that even if our monotone hazard condition is not satisfied, one can use a Cremer-Mclean style payment rule to ensure it is incentive compatible for each agent to truthfully report their entry time. Essentially- the insight is this- each arriving buyer is forced to participate in a lottery which pays off according to the number of buyers who arrive in the same period. This lottery

If the allocation rule produced by our algorithm does not satisfy (21), the (expected) revenue maximizing mechanism is hard to specify. Clearly the allocation rule will need additional pooling relative to the 'virtually efficient' rule that solves ROPT:MONO. The essence of the required pooling is that since only exit monotonicity fails, the optimal mechanism may sometimes need to withhold a unit where the algorithm would have allotted that unit to an expiring report.

The other tack one could take would be to distort the virtual values , so that the allocation rule produced by our algorithm with these ironed virtual values maintains the required shortest paths in the dual graph (21). While this is conceptually 'simple': simply add the *correct* Lagrangian multipliers for these constraints to the virtual values, we have no analytical characterization of what such ironing might entail.

We close instead with an observation: if the virtual valuations are *sufficiently* increasing, the solution to ROPT:MONO will be monotonic in the sense of Definition 3.

A sufficient condition for allocations to be monotonic in exit time is that types with lower exit times ceteris paribus have a substantially lower hazard rate, i.e.

Definition 4 *A distribution is said to have sufficiently increasing hazard rate if, for any two types* $(v, t, \bar{t})$ *and* $(v, t, \bar{t}+1)$*, we have that:*

$$\frac{f_{(t,\bar{t})}(v)}{1 - F_{(t,\bar{t})}(v)} + c_{C,T,F,n}(v, t, \bar{t}) < \frac{f_{(t,\bar{t}+1)}(v)}{1 - F_{(t,\bar{t}+1)}(v)} \tag{23}$$

where $c_{C,T,F,n}(.)$ *is a non-negative function which depends on the model primitives,* T*,* C*,* F *and* n*.*

If the distribution over buyer types is 'sufficiently' monotone, the solution to ROPT:MONO will satisfy exit monotonicity. The next proposition summarizes this.

Proposition 1 *Suppose the distribution of types has a sufficiently increasing hazard rate, in the sense of Definition 4. Then there exists a solution of OPT3 with an allocation scheme that satisfies monotonicity (i.e. valuation, entry and exit monotonicity). In particular, the allocation rule computed by the algorithm coupled with pricing rule (18) solves OPT3.*

Example 1 provides an example of an environment which satisfies our conditions. It calculates the thresholds for the algorithm we propose, and demonstrates their optimality.

Example 1 *There is* 1 *unit for sale, over* 2 *time periods,* 1 *and* 2*. In period* 1*,* 1 *or* 2 *agents arrive with equal probability. In period* 2*, only* 1 *agent arrives. Conditional on arriving in period* 1*, agents have an exit time of* 1 *or* 2 *with equal probability.*

Agents with entry-exit time combinations $(1, 2)$ *and* $(2, 2)$ *have valuations drawn from a uniform distribution on* $[1, 2]$*. This implies that the virtual valuation of an agent of type* $(v, ., 2) = 2v - 2$*; and virtual valuations are uniform on* $[0, 2]$

has expected value 0 according to the true distribution $N(\cdot)$, but a strongly negative value for a misreporting buyer.

Agents with entry and exit time 1 *have valuations drawn from a uniform distribution* $[1, 2.5]$. *Therefore, the virtual valuation of an agent of type* $(v, ., 2) = 2v-2$*; and virtual valuations are uniform on* $[-0.5, 2.5]$.

It is easy to see that

$$\begin{aligned} R_2(\phi) &= 1, \\ R_2((v, 1, 2)) &= v^2 - 2v + 2. \end{aligned}$$

If a type $(v, 1, 1)$ *arrives at time* 1-

- *(With probability* $\frac{1}{2}$*) No other agent shows up in that period, and it is allotted as long as* $2v - 2.5 > 1 \Rightarrow v > 1.75$.
- *(With probability* $\frac{1}{4}$*) Another agent of type* $(v', 1, 1)$ *shows up, it is allotted as long as* $v > v'$ *and* $v > 1.75$*; i.e. wp* $\frac{v-1}{2.5-1} = \frac{2(v-1)}{3}$.
- *(With probability* $\frac{1}{4}$*) Another agent of type* $(v', 1, 2)$ *shows up, and our agent is allotted as long as* $2v - 2.5 > v'^2 - 2v' + 2$*, i.e.* $v' < 1 + \sqrt{2v - 3.5}$*; therefore w.p.* $\sqrt{2v - 3.5}$.

Therefore

$$A(v, 1, 1) = \begin{cases} \frac{1}{2} + \frac{(v-1)}{6} + \frac{\sqrt{2v-3.5}}{4} & \text{if } v > 1.75 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly if a type $(v, 1, 2)$ *arrives at time* 1-

- *(With probability* $\frac{1}{2}$*) No other agent shows up in that period, and it is allotted as long as the agent who arrives in period* 2 *has a lower valuation, i.e.* $v - 1$.
- *(With probability* $\frac{1}{4}$*) Another agent of type* $(v', 1, 2)$ *shows up, and our agent is allotted as long as* $v > v'$*, and the valuation of the agent who arrives in period* 2 *is lower i.e. w.p.* $(v - 1)^2$.
- *(With probability* $\frac{1}{4}$*) Another agent of type* $(v'1, 1)$ *shows up; and our agent is allotted as long as* $2v' - 2.5 < v^2 - 2v + 2 \Rightarrow v' < \frac{v^2-2v+4.5}{2}$ *and the agent who arrives in period* 2 *has a lower valuation; i.e. wp* $(v - 1)\frac{(v^2-2v+2.5)}{3}$.

Therefore

$$A(v, 1, 2) = \frac{v - 1}{2} + \frac{(v - 1)^2}{4} + \frac{(v - 1)(v^2 - 2v + 2.5)}{3}.$$

We only need ensure that $A(v, 1, 2) \geq A(v, 1, 1)$ *in the range* $[1.75, 2]$. *To see this note that* $A(2, 1, 1) \approx 0.843$*; while* $A(1.75, 1, 2) \approx 0.906$.

Next, we give an example where the optimal incentive compatible allocation rule is non-monotone, but the payment rules turn out to be the same as those computed by Theorem 1. This shows that monotonicity of the allocation rule is sufficient but not necessary for our pricing rule to be incentive compatible. In particular, the allocation rule violates part 3 of Definition 3.

Example 2 *Let* $T = 2$, $C = 1$. *There are* 2 *possible valuations, i.e.* $I = \{1, 2\}$. *Therefore there will be* 6 *possible types, corresponding to the* 3 *possible entry-exit times* $(1, 1)$, $(1, 2)$ *and* $(2, 2)$ *and* 2 *possible valuations. Let the arrival rate be such that*

in each period 1 *agent arrives with probability* 0.49 *(and no-agent arrives otherwise). Suppose that conditional on arrival at time* 2*, type* $(2, 2, 2)$ *has probability* 1*. Further suppose that conditional on arrival at time* 1*, types* $(2, 1, 2)$ *and* $(2, 1, 1)$ *have probability* 0*. It is easy to show that this distribution meets the monotone hazard rate condition. However it does not meet Definition 4, since the types* $(v, 1, 2)$ *and* $(v, 1, 1)$ *have the same virtual valuations. Finally consider the following allocation rule:* $A(1, 1, 1) = 1$*,* $A(2, 2, 2) = 1$*,* $A(1, 1, 2) = .51$*. Note that this rule is clearly not monotonic in the sense of Definition 3. However, coupled with the payment rule* $P(1, 1, 1) = 1$*,* $P(1, 1, 2) = .51$ *and* $P(2, 2, 2) = 2$*, this is incentive compatible.*

To see that this is the optimal allocation rule: if type $(1, 1, 1)$ *arrives at time* 1*, allot him for a payment of* 1 *(in expectation this is better than not allotting him). On the other hand if* $(1, 1, 2)$ *arrives, then it is optimal to wait, since, potentially in period* 2 *an agent of higher valuation arrives. Finally note that the payments are the same as would have been calculated from equation (18).*

5 Conclusion

In this paper, we formulate and solve a multi-period dynamic auction design problem for the sale of multiple identical items. A novel feature is that we allow agents to be strategic with respect to the revelation of their arrival and departure times. The solution turns out to be intuitive in that it is the natural generalization of the optimal auction for the static case to this dynamic framework. We are unable to characterize the solution to this problem unless the distribution over buyer types is 'sufficiently' regular– recent work suggests that no interesting characterization exists in this case.

We see several important extensions to this model. Intuitively, the assumption driving our results is the preferences of the buyer. We assume throughout that the buyer desires exactly 1 unit of a homogenous good, and enjoys no utility from being allotted outside his time in the system. This cuts down on the number of ways a buyer may misreport his private information. Recall that we motivated this problem (rather, this entire model has been motivated) as particularly relevant to the airline and hotel industries. Given that several companies and online retailers in this sector are trying to sell package deals, where they sell, for example, airline tickets and hotel reservations together, we believe it may be of interest to study how this can be done optimally, while relaxing our rather strong assumption on preferences. Models of selling heterogenous goods dynamically have been studied, but there has been no work we are aware of where buyers are allowed to be strategic with respect to their time preferences.

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A Monotonicity

In this appendix we give a proof for Theorem 3. Since the proof is reasonably elaborate, we build intuition by examining the 1-D case ($C = 1$, $T = 1$) where agents cannot over-report their valuation. We believe this case captures the crux of the argument. In the one-dimensional types case, a variant of the classical Spence-Mirrlees condition tells us that if an agent can misreport his type both up and down, only monotonic allocation rules can be Incentive Compatible. When misreports can only take place in one direction, as is the case in our dynamic setting for entry and exit times, monotonicity is not implied by Incentive Compatibility.

We first consider the 1-D case as described above (i.e. agents cannot over-report their valuation). Then we prove prove Proposition 3 for our model. There are other more distinguished papers that have considered similar problems, i.e. only one direction of misreport in static settings, for example, Celik [6], Blackorby and Szalay [3], and Beaudry, Blackorby and Szalay [1].

A.1 Single Dimensional Types

There are n risk-neutral buyers bidding for a single object to be sold by a risk-neutral seller. Bidders have private valuations $v \in V = \{1, 2, \ldots, V\}$. In terms of our model, $C = 1$, $T = 1$, $Q = \{1\}$. Types are i.i.d. by some distribution f, which satisfies the monotone hazard rate condition, i.e. $\frac{f(v)}{1-F(v)}$ is increasing in v. There are n agents/buyers. Possible realizations of types are $\pi \in \Pi = V^n$. Suppose no agent can lie 'upward', i.e. an agent of type v cannot report that he is of type $v' > v$. Hence only the downward IC constraints need be imposed. Border [4] shows how to write the space of feasible (interim) allocation rules as a set of linear constraints.

The revenue maximizing program can be rewritten:

$$\max_{a,p} \sum_{v=1}^{V} f(v)p_v \tag{P}$$

$$\begin{aligned} \text{s.t. } & va_v - p_v \geq va_{v'} - p_{v'} \quad \forall v' < v \\ & va_v - p_v \geq 0 \forall v \\ & a \text{ feasible} \end{aligned}$$

To solve this consider a relaxed program where agent of type v can only report his type as v or $v-1$:

$$\begin{aligned} &\max_{a,p} \sum_{v=1}^{V} f(v)p_v \\ \text{s.t. } & va_v - p_v \geq va_{v-1} - p_{v-1} \quad \forall v \\ & va_v - p_v \geq 0 \quad \forall v \\ & a \text{ feasible} \end{aligned} \tag{P'}$$

Claim 1 *Given any (feasible) allocation rule a, the prices that maximize revenue for P' will be $p_v = va_v - \sum_{j=1}^{v-1} a_v$.*

Proof: We prove this by induction. Consider the lowest type v that gets allotted. Clearly in any revenue maximizing auction it will pay va_v. Induction Hypothesis: $(v+k)$ pays : $(v+k)a_{v+k} - \sum_{j=1}^{v+k-1} a_j$.

Induction Step: In the optimal solution, $v+k+1$ is indifferent to being $v+k$ (else increase his price till he is: recall that in this relaxed program, an agent with valuation v can only report his type as v or $v-1$).

$$(v+k+1)a_{v+k+1} - p_{v+k+1} = (v+k+1)a_{v+k} - p_{v+k} = \sum_{j=1}^{v+k} a_j.$$

■

Claim 2 *Therefore, given a, the net revenue is $\sum_{v=1}^{V}(v - \frac{1-F(v)}{f(v)})f(v)a_v$.*

Pick the following solution to program P'. In each profile allot the highest type as long as it has a positive virtual value. Call this allocation rule a^*. Note that a^* is clearly monotonic. Given the monotone hazard rate assumption, a^* is optimal for P'. Monotonicity of a^* implies that it is feasible in P. Therefore it must also be optimal in P.

Claim 3 *The allocation rule a^* and the prices $p_v = va_v^* - \sum_{j=1}^{v-1} a_j^*$ are optimal for P given our monotone hazard rate assumption.*

A.2 A Dynamic Allocation Rule

The idea of the proof is the same as above. We consider the solution to ROPT:MONO. We show that if the monotone hazard rate condition (4) holds, the resulting allocation rule is valuation and entry monotonic. We then show that if the virtual valuations are sufficiently monotonic in the sense of Definition 4, exit monotonicity will be satisfied as well. The proof is involved, and to aid understanding we begin by considering the case $C = 1$, i.e. there is only 1 unit for sale.

A.2.1 The 1 Unit Case

ROPT:MONO is a standard dynamic allocation problem: At any time t, n_t buyers enter, each having valuation and exit time $(\varphi_{v,t,\bar{t}}, \bar{t})$ with probability $\frac{f_{t,\bar{t}}(v)}{\sum_{v',\bar{t}'} f_{t,\bar{t}'}(v')}$. Further

any unallocated buyers with exit time t exit. If a unit has not already been allotted, the program can choose to allot it to any buyer currently in the system (including the ones exiting the system in that period). Standard arguments show that in any period, the expected value maximizing program will allot if at all to agents exiting the system in that period. Further, at each time t, there exists a function $R_{t+1}(\cdot)$, whose argument is the types of buyers in the system arriving at or before period t, and departing in period $t+1$ or later (denoted by I^t_{t+1}). [21] The optimal policy at time t allots to the highest virtual valuation exiting at time t conditional on it being higher than the cutoff $R_{t+1}(\cdot)$. This cutoff, R_{t+1} represents the expected value of following the optimal policy from $t+1$ onward, given the buyers currently in the system. One can easily specify the family of functions $\{R_t\}_{t=1}^T$ by backward induction. The notation $()^+$ in the sequel refers to the maximum element in a set.

$$R_{T+1} = 0; \tag{24}$$

$$R_T(I) = \mathbb{E}_{A^T}[(I \bigcup A^T)^+]; \tag{25}$$

$$\begin{aligned} R_t(I) = \mathbb{E}_{A^t}\Big[&\mathbb{I}_{\{(L_t(A^t)\bigcup L_t(I))^+ > R_{t+1}(A^t_{t+1}\bigcup I_{t+1})\}}\big(L_t(A^t)\bigcup L_t(I)\big)^+ \\ &+\mathbb{I}_{\{(L_t(A^t)\bigcup L_t(I))^+ \leq R_{t+1}(A^t_{t+1}\bigcup I_{t+1})\}} R_{t+1}(A^t_{t+1}\bigcup I_{t+1})\Big]. \end{aligned} \tag{26}$$

Equation (24) says that the value of the unit after period T is 0. Equation (25) says that the value of the unit in the last period is the expected maximum virtual valuation among the remaining inventory and the buyers that arrive in that period. Equation (26) inductively defines the value of the unit in period t as a function of the inventory. The allocation rule can allot either to the agent leaving in period t with the highest virtual valuation, which it will do if this is larger than $R_{t+1}(A^t_{t+1}\bigcup I_{t+1})$. Alternately, it can let those bids expire and pick up $R_{t+1}(A^t_{t+1}\bigcup I_{t+1})$ (in expectation).

Before we can study properties of this allocation rule, we prove some (intuitive) properties of the cutoff rule $R_t(\cdot)$. Our first observation confirms our intuition that a better inventory leads to a higher future expected payoff $R_t(\cdot)$.

Observation 4 *If $I_t \succeq I'_t$, $R_t(I_t) \geq R_t(I'_t)$.*

Proof: The proof is by induction on t. Clearly if $I_T \succeq I'_T$, we have that $R_T(I_T) \geq R_T(I'_T)$. Assume that for all $t' \geq t+1$,

$$I_{t'} \succeq I'_{t'} \Rightarrow R_{t'}(I_{t'}) \geq R_{t'}(I'_{t'}).$$

We show this is true for t.

We know that $I_t \succeq I'_t$, let $S_t \subseteq I_t$ be the set of bids that dominate $L_t(I'_t)$. Then for any set of people arriving in period t, A^t; $I_{t+1}\bigcup A^t_{t+1}\backslash S_t \succeq I'_{t+1}\bigcup A^t_{t+1}$. By our inductive hypothesis,

$$R_{t+1}(I_{t+1}\bigcup A^t_{t+1}\backslash S_t) \geq R_{t+1}(I'_{t+1}\bigcup A^t_{t+1}). \tag{27}$$

[21] We do not formally specify the domain of the function R_t for ease of notation.

It remains to show that $R_t(I_t) \geq R_t(I'_t)$. However,

$$\begin{aligned} R_t(I_t) \quad \geq \quad & \mathbb{E}_{A^t}\Big[\mathbb{I}_{\{(L_t(A^t)\bigcup L_t(I_t)\bigcup S_t)^+ > R_{t+1}(A^t_{t+1}\bigcup I_{t+1}\setminus S_t)\}}\big(L_t(A^t)\bigcup L_t(I_t)\bigcup S_t\big)^+ \\ & +\mathbb{I}_{\{(L_t(A^t)\bigcup L_t(I_t)\bigcup S_t)^+ > R_{t+1}(A^t_{t+1}\bigcup I_{t+1}\setminus S_t)\}} R_{t+1}(A^t_{t+1}\bigcup I_{t+1}\setminus S_t)\Big], \end{aligned}$$

where the right hand side of the inequality above corresponds to the suboptimal policy of allotting either the leaving agents at time t or agents in S_t. By definition,

$$\begin{aligned} R_t(I'_t) \quad = \quad & \mathbb{E}_{A^t}\Big[\mathbb{I}_{\{(L_t(A^t)\bigcup L_t(I'_t))^+ > R_{t+1}(A^t_{t+1}\bigcup I'_{t+1})\}}\big(L_t(A^t)\bigcup L_t(I'_t)\big)^+ \\ & +\mathbb{I}_{\{(L_t(A^t)\bigcup L_t(I'_t))^+ \leq R_{t+1}(A^t_{t+1}\bigcup I'_{t+1})\}} R_{t+1}(A^t_{t+1}\bigcup I'_{t+1})\Big] \end{aligned}$$

Consider the terms inside the expectation operator in the two inequalities above. It is easy to see that

$$\big(L_t(A^t)\bigcup L_t(I_t)\bigcup S_t\big)^+ \geq \big(L_t(I^t_t)\bigcup L_t(I'_t)\big)^+,$$

combining this with equation (27), we get our desired result. ■

Armed with this result, we can pursue our original goal of showing that the interim allocation probabilities are monotonic as per Definition 3. We will prove each of the 3 parts of Definition 3 separately, hence proving Theorem 3

Valuation Monotonicity Since the allocation rule is a cutoff rule, and virtual valuations are monotone, any inventory along which a type $(v, t, \bar{t})$ gets allotted will also result in the allocation of $(v', t, \bar{t})$ for $v' > v$ in the corresponding inventory which has the report of the latter type in place of the former. ■

Entry Monotonicity This requires that the interim probability of getting allotted is decreasing in entry time, holding exit time and valuation constant. To see this, fix two types, $(v, t, \bar{t})$ and $(v, t+1, \bar{t})$ (by Lemma 2, monotonicity in adjacent entry times is necessary and sufficient). Consider an I such that $(v, t, \bar{t})$ is in I^t_t. If $(v, t+1, \bar{t})$ is in I^{t+1}_{t+1}, it will never be allotted, since at $\bar{t}$, $(v, t, \bar{t})$ will also be present, but it has a higher virtual valuation. So suppose $(v, t+1, \bar{t})$ is not in I^{t+1}_{t+1}.

Consider the inventory I' constructed as:

$$\begin{aligned} I'^t_t &= I^t_t \setminus \{(v, t, \bar{t})\}, \\ I'^{t+1}_{t+1} &= I^{t+1}_{t+1} \bigcup \{(v, t+1, \bar{t})\}, \end{aligned}$$

with I'^{t+2}_{t+2} through $I'^{\bar{t}}_{\bar{t}}$ defined appropriately. The set of all I''s as constructed above is the set of all inventories that contain the report $(v, t+1, \bar{t})$, but not $(v, t, \bar{t})$. [22] We

[22] It is at this precise step that we need the property that n_t has full support on $\mathbb{Z}_+$. Without it, on some inventories, an agent misreporting his entry time could produce an inventory that was impossible under the problem parameters. For example if in a given period, n_t is such that at most n people can arrive at time t, then a misreport of entry time by an agent who arrived earlier could lead to $n+1$ agents claiming a t

show that whenever $(v, t+1, \bar{t})$ is allotted in I',$(v, t, \bar{t})$ is allotted in I. To see that, $(v, t+1, \bar{t})$ will be allotted along this sequence if:

1. The good was not allotted before $t+1$.
2. The good was not allotted from time $t+1$ through $\bar{t}$.
3. $(v, t+1, \bar{t})$ has the highest virtual valuation among $I'^{\bar{t}}_{\bar{t}}$; and this is greater than $R_{\bar{t}+1}(I'^{\bar{t}}_{\bar{t}+1})$.

Further note that since $I^{t'}_{t'} \succeq I'^{t'}_{t'}$ for any t', it must be that $R_{t'}(I^{t'}_{t'}) \geq R_{t'}(I^{t'}_{t'})$. Therefore:

1. If the good was not allotted before $t+1$ along I', it would not have been allotted up to $t+1$ along I (all bids stay the same except that I has the report $(v, t, \bar{t})$ which I' does not, which weakly increases $R_{t+1}(.)$).
2. If the good was not allotted from time $t+1$ through $\bar{t}$ along I', it would not have been allotted along I either, by the same token.
3. Finally, by assumption the good was allotted to $(v, t+1, \bar{t})$ at $\bar{t}$. Therefore this was the highest virtual valuation among $I'^{\bar{t}}_{\bar{t}}$; and this is greater than $R_{\bar{t}+1}(I'^{\bar{t}}_{\bar{t}+1})$. But then $(v, t, \bar{t})$ has the highest virtual valuation among $I^{\bar{t}}_{\bar{t}}$ (by construction); and this is greater than $R_{\bar{t}+1}(I^{\bar{t}}_{\bar{t}})$ since $I'^{\bar{t}}_{\bar{t}} = I^{\bar{t}}_{\bar{t}}$.

Therefore $A(v, t, \bar{t}) \geq A(v, t+1, \bar{t})$. ■

Exit Monotonicity Fix two types $(v, t, \bar{t})$ and $(v, t, \bar{t}+1)$, we refer to them in the rest of this proof as τ and τ' respectively. Denote their virtual valuations by φ_τ and $\varphi_{\tau'}$. We need to show that $A(v, t, \bar{t}) \leq A(v, t, \bar{t}+1)$. Let $I = (I^1_1, \ldots, I^T_T)$ be a sequence of inventories as before. The probability that the good is allotted to type $(v, t, \bar{t})$, $A(v, t, \bar{t}) =$

$$\begin{aligned}
& \mathbb{P}[\text{Good not allotted before } t] && (28)\\
\times\ & \mathbb{P}[\text{Good not allotted up to } \bar{t}-1 \mid \text{ Good not allotted before } t, \tau \in A^t] && (29)\\
\times\ & \mathbb{P}[\varphi_\tau \text{ is largest virtual valuation in } L_{\bar{t}}(I^{\bar{t}}_{\bar{t}}), \varphi_\tau \geq R_{\bar{t}+1}(I^{\bar{t}}_{\bar{t}+1}) \\
& \quad \mid \text{ Good not allotted up to } \bar{t}-1, \tau \in A^t]. && (30)
\end{aligned}$$

Define $\varphi'_{\tau'} = \varphi_\tau + c$ for $c > 0$. Compute R' as in Equations (24-26) with the virtual valuation of τ' taken as $\varphi'_{\tau'}$. Recompute the allocation rule with this R'. Note that for any time $t' \geq \bar{t}+2$, $R'_{t'} = R_{t'}$. The probability that this good is allotted to type $(v, t, \bar{t}+1)$, $A'(v, t, \bar{t}+1)$, with its virtual valuation defined as $\varphi'_{\tau'}$ is:

$$\begin{aligned}
& \mathbb{P}[\text{Good not allotted before } t] && (31)\\
\times\ & \mathbb{P}[\text{Good not allotted up to } \bar{t}-1 \mid \text{ Good not allotted before } t, \tau' \in A^t] && (32)\\
\times\ & \mathbb{P}[\text{Good not allotted at } \bar{t} \mid \text{ Good not allotted before } \bar{t}, \tau' \in A^t] && (33)\\
\times\ & \mathbb{P}[\varphi'_{\tau'} \text{ is largest virtual valuation in } L_{\bar{t}+1}(I^{\bar{t}+1}_{\bar{t}+1}), \varphi_{\tau'} \geq R_{\bar{t}+2}(I^{\bar{t}+1}_{\bar{t}+2}) \\
& \quad \mid \text{ Good not allotted up to } \bar{t}+1,\ \tau' \in A^t.] && (34)
\end{aligned}$$

We note that $A'(v, t, t \bar{+} 1)$ is increasing in c:

arrival; which the auctioneer knows is impossible. This would make misreports easier to detect and punish-something a full support assumption would rule out.

1. (31) is weakly increasing in c- as the value of a possible future arrival increases; the allocation rule is more conservative about allotting at earlier times.
2. (32) and (33) are weakly increasing in c- as the value of a current type already present in the system increases, the allocation rule is more conservative about allotting to another type.
3. (34) is weakly increasing in c- as the value of one of the exiting types increases, it is more likely to be the highest valued exiting type. Further it is more likely that it is larger than the expected future payoff from not allotting in that period, $R_{\bar{t}+2}(I_{\bar{t}+2}^{\bar{t}+1})$.

Further for some c large enough; clearly $A_{(v,t,\bar{t}+1)} \geq A(v,t,\bar{t})$ (for very large values of c, $A_{(v,t,\bar{t}+1)} \to 1$). Let c^* be the lowest such c. If

$$\varphi_{\tau'} \geq \varphi_\tau + c^*, \tag{35}$$

it follows that $A(v,t,\bar{t}+1) \geq A(v,t,\bar{t})$. By Definition 4, the distribution is such that (35) is satisfied. As we pointed out earlier, we are unable to analytically characterize c^*. Example 1 shows that there are meaningful distributions that meet this condition. ■

A.3 The C-unit case

In this case, the optimal allocation policy will depend also on the number of units k left in the system. Therefore the cutoffs will be of the form $\{R_t^k(\cdot)\}_{k=1}^C$, which are defined inductively as:

$$\forall t \qquad R_t^0(I) = 0; \tag{36}$$

$$\forall k \qquad R_{T+1}^k = 0; \tag{37}$$

$$R_T^1(I) = \mathbb{E}_{A^T}[(I \bigcup A^T)^+]; \tag{38}$$

$$\forall k > 1 \qquad R_T^k(I) = \mathbb{E}_{A^T}[(I \bigcup A^T)^+ + R_T^{k-1}(I \backslash (I \bigcup A^T)^+)]; \tag{39}$$

$$\begin{aligned} \forall k \qquad R_t^k(I) = \mathbb{E}_{A^t}\Big[& \mathbb{I}_{\{(L_t(A^t) \bigcup L_t(I))^+ > R_{t+1}^k(A_{t+1}^t \bigcup I_{t+1})\}} \\ & \times \big(\big(L_t(A^t) \bigcup L_t(I)\big)^+ + R_t^{k-1}(I \backslash \big(L_t(A^t) \bigcup L_t(I)\big)^+) \big) \\ & + \mathbb{I}_{\{(L_t(A^t) \bigcup L_t(I))^+ \leq R_{t+1}(A_{t+1}^t \bigcup I_{t+1})\}} \\ & \times R_{t+1}^k(A_{t+1}^t \bigcup I_{t+1}) \Big] \end{aligned} \tag{40}$$

It is easily noted the appropriate version of Observation 4 remains true in this case- if $I_t \succeq I_t'$ then for any k, t, we have that $R_t^k(I_t) \geq R_t^k(I_t')$. Further, we have that $R_t^k(I_t) \geq R_t^{k-1}(I_t)$. Therefore the proofs of Parts 1 and Part 2 carry over as before.

To obtain the correct version of our proof of exit monotonicity requires one to notice that the number of units allotted up to time $\bar{t}$ when the inventory contains $(v,t,\bar{t})$ is weakly more than the number of units allotted when the inventory contains $(v,t,\bar{t}+1)$. Conditional on any number of units allotted up to $\bar{t}$, we once again need the virtual

value to drop sufficiently to ensure that the probabilities of being allotted for $(v, t, \bar{t})$ are less than $(v, t, \bar{t}+1)$.

B The Revelation Principle

This section outlines a revelation principle for this environment. We first formally state and prove the revelation principle as it applies here. We then discuss why this, in our opinion, is not restrictive in the sense that any Bayesian-Nash Equilibrium where our revelation principle does not apply can be 'transformed' into one where it does (i.e. all the buyers are indifferent between the two equilibria, as is the seller). We conclude with an example which illustrates why we cannot prove a revelation principle in full generality.

Fix the environment as described in Section 2. Recall that the game takes place over periods 1 through T. The game form we propose is as follows: in each period t, there is a message space M_t. An agent with entry time t and exit time $\bar{t}$ can send messages over any contiguous subset of periods t through $\bar{t}$. We do not allow a buyer of type $(v, t, \bar{t})$ to communicate *before* period t or after $\bar{t}$. Since our assumption is that the buyer will have no information other than his own type and the (common knowledge) model, we do not need to consider any communication from the seller.

We denote the history of all messages received by the seller from each player up to (and including) time t by h_t, and the set of all possible histories by H_t.

The allocation rule is a sequence of functions $\{a_t\}_{t=1}^{T}$, where a_t is a function from an element $h_t \in H_t$ to allocations up to time t (the formal definitions can be seen in Section 2, we do no repeat notation here).

Further suppose that this allocation rule is feasible, in that over any possible history, the allocation rule allots no more than C units, and does not allot players in a period in which they do not send a message. A payment rule P associates with the string of messages sent by a player, a non-negative payment to be made by him.

A Bayesian-Nash Equilibrium has its usual meaning: suppose that in equilibrium each type $(v, t, \bar{t})$ sends messages $\mathbf{m} = (m_1, m_2, \ldots, m_k)$, starting some time $t' \geq t$ and finishing some time $t' + k \leq \bar{t}$. Then it must be the case that

$$\mathbb{E}[V(A_{t'+k}(h_{t'+k})|(v, t, \bar{t}))] - P(m) \geq \mathbb{E}[V(A_{\bar{t}}(h'_{\bar{t}})|(v, t, \bar{t}))] - P(m'),$$

for any other sequence of messages that a player can send, $\mathbf{m}'$. (The expectation here is over messages sent by other agents in equilibrium.)

We show the revelation principle for equilibria in which each type $(v, t, \bar{t})$ is allotted exactly at time $\bar{t}$ if at all.

Lemma 3 *Suppose we have an equilibrium such that each type $(v, t, \bar{t})$ is allotted a unit at time $\bar{t}$ if at all. Then there exists a direct revelation mechanism (DRM) which gives the same expected utility to all buyers, and the same expected revenue to the seller.*

Proof: Let the allocation and payment rules for the direct revelation mechanism be the same as for the original game. We are left to show that this mechanism is incentive compatible. To this end suppose that the DRM is not incentive compatible, and suppose instead type $(v, t, \bar{t})$ can profitably misreport his type as $(v', t', \bar{t}')$. Then:

- $\bar{t}' \leq \bar{t}$, since by assumption, a buyer reporting exit time t' gets allotted at time t', and a buyer of true type $(v, t, \bar{t})$ gets no utility from objects allotted after $\bar{t}$.
- $t' \geq t$, since an agent arriving at time t cannot report his type as one with entry time strictly less than t.

But this implies that type $(v, t, \bar{t})$ could have sent the messages corresponding to $(v', t', \bar{t}')$, and deviated profitably in the original game, contradicting our hypothesis that the messages in the original game constituted a Bayes-Nash equilibrium. ■

Next we show that this result is in some sense generic. We show that any other equilibrium (i.e. where some types get allotted in quantities other than their requirement and/or before their exit time) can be converted into one where all types get allotted with the same (interim) probability. However in this new equilibrium, each type gets allotted exactly at their exit time. Further in this new equilibrium, each type has the same expected utility and the seller has the same expected revenue. However we may need to expand the message space in order for our construction to work.

PROPOSITION 2 *Suppose there is an equilibrium in the original game where an agent of type $(v, t, \bar{t})$ sends messages $m_{(v,t,\bar{t})}$ in equilibrium. Then there is an equilibrium (in a potentially modified game with an expanded message space) such that in this equilibrium:*

1. *Each type $(v, t, \bar{t})$ sends messages $m'_{(v,t,\bar{t})}$ starting period t through $\bar{t}$.*
2. *Messages are prefix-free: for any two types $(v, t, \bar{t})$, $(v', t, \bar{t}')$ s.t. $\bar{t} \geq \bar{t}'$, the messages sent by the two types from period t through $\bar{t}'$ are not the same.*
3. *Each type $(v, t, \bar{t})$ is allotted at $\bar{t}$. Further all types have the same expected utility and the seller has the same expected revenue as in the original equilibrium.*

PROOF: We shall construct a potential equilibrium with the properties described above, and proceed to show that it is in fact an equilibrium. So (expanding the message space if necessary) construct a sequence of prefix-free messages, one for each possible type. Note that this set of equilibrium messages is 'fully separating', in the sense that at no stage can a type be confused for another.

For each type $(v, t, \bar{t})$, and any possible profile of types of other agents, let the allocation rule in our new equilibrium reallocate all probabilities of getting a unit during any period t through $\bar{t}$ to getting it at exactly $\bar{t}$, and discard the rest of the allocations. Doing this for each type gives us an allocation rule, which is feasible since it allots weakly fewer units than the old rule (which was feasible by assumption). Further let the payment of each type be the same as in the original mechanism. It is clear that each type will be indifferent between the old allotment and the new one.

Therefore it is left to show that this new allocation rule coupled with the old message strategies is in fact an equilibrium. Once again suppose not, i.e. suppose that some type $(v, t, \bar{t})$ can profitably deviate by sending messages corresponding to $(v', t', \bar{t}')$. By the construction of the allocation rule, it must be the case that $t \leq t'$ and $\bar{t} \geq \bar{t}'$. But then clearly agent $(v, t, \bar{t})$ could also have profitably deviated by sending messages corresponding to $(v', t', \bar{t}')$ in the original game as well. ■

Finally we show why there cannot be a revelation principle for equilibria where types get allotted at times other than their exit time.

An Example Suppose $T = 2$, $C = 1$. Suppose there is exactly 1 potential buyer, who has two possible types: $(1, 1, 1)$ and $(1, 1, 2)$, i.e. the first type has valuation 1 and enters and exits at time 1; the second type also has valuation 1 and enters at time 1 but exits at time 2. Suppose $M_1 = \{1, 2\}$, and $M_2 = \{1\}$. Suppose further that

$$a_1(1) = 0;\ a_1(2) = 1;\ a_2(1, 1) = a_2(2, 1) = 0;$$

i.e. the allocation rule allots the unit to an agent announcing 2 in the first period, and nothing otherwise. Further suppose that

$$P(1) = P(1, 1) = 0;\ P(2) = 3;\ P(2, 1) = 0.99.$$

Consider the following equilibrium: type $(1, 1, 1)$ announces 1 in the first period, type $(1, 1, 2)$ announces 2 in the first period and 1 in the second. The only deviation $(1, 1, 1)$ has is to announce 2 in the first period. But this gets him the unit at the price of 3, which is not profitable. Similarly one can check that $(1, 1, 2)$ cannot profitably deviate.

However, this allocation rule cannot be implemented by a Direct revelation mechanism- type $(1, 1, 1)$ can report $(1, 1, 2)$, get the unit in period 1 and pay 0.99, which gets him a surplus of 0.01, whereas announcing $(1, 1, 1)$ gets him a surplus of 0. In short, if you allocate buyers before their stated exit times, misreporting a later exit time in a direct revelation mechanism is possible.

Note however that is easy to construct an 'equivalent' equilibrium (all types and the seller are indifferent)- simply allot type $(1, 1, 2)$ in period 2 rather than period 1. ■

Even if buyers can communicate past their exit times, considering allocation rules that allot only at exit is without loss of generality— see Observations 1, 5.

C Miscellany

C.1 Proof of Theorem 1

Proof: To begin, suppose all agents have exit time T. We therefore drop exit times from the agents' type wlog. Consider the 4 adjacent types $(v, t), (v + 1, t), (v, t - 1), (v + 1, t - 1)$, as shown in Figure 1. There are 2 possible paths from (v, t) to $(v + 1, t - 1)$- edge 1 followed by 2 and edge 4 followed by 3. The length of the former path is $(v + 1)(A(v + 1, t - 1) - A(v, t))$. The length of the latter path is $(v + 1)A(v + 1, t - 1) - vA(v, t) - A(v, t - 1)$. By monotonicity $A(v, t - 1) \geq A(v, t)$ and therefore the latter path is shorter than the former. By the Principle of Optimality, in a setting where all agents had exit time T, the shortest path from $(1, T)$ to a generic type (v, t) would be $(1, T) \to (1, T - 1) \ldots \to (1, t) \to (2, t) \ldots \to (v, t)$.

Now, we drop the assumption that all agents have the same exit time T. Suppose we wish to find the shortest path from $(v, t, \bar{t})$ to $(v, t - 1, \bar{t} + 1)$. There are 4 possible paths: [23]

1. $(v, t, \bar{t}) \to (v, t, \bar{t} + 1) \to (v, t - 1, \bar{t} + 1) \to (v + 1, t - 1, \bar{t} + 1)$.
2. $(v, t, \bar{t}) \to (v, t - 1, \bar{t}) \to (v + 1, t - 1, \bar{t}) \to (v + 1, t - 1, \bar{t} + 1)$.

[23] This list is not exhaustive, but other paths can be ruled out by the previous argument.

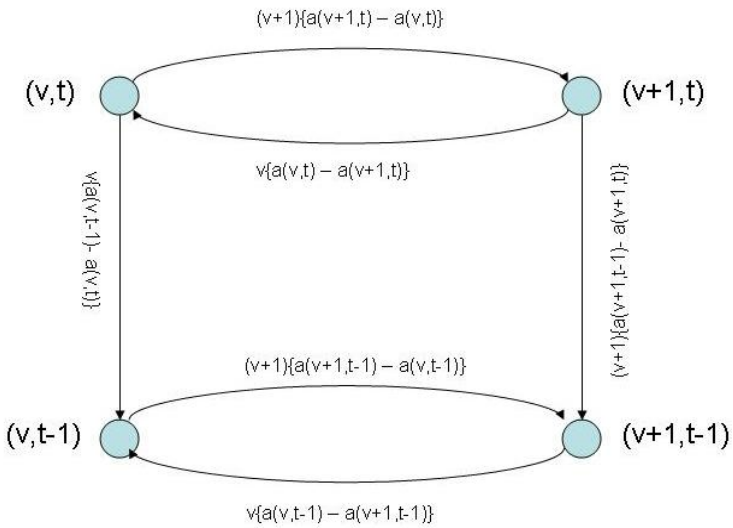

FIGURE 1: The Shortest Path Graph

3. $(v, t, \bar{t}) \to (v, t-1, \bar{t}) \to (v, t-1, \bar{t}+1) \to (v+1, t-1, \bar{t}+1)$.
4. $(v, t, \bar{t}) \to (v+1, t, \bar{t}) \to (v+1, t, \bar{t}+1) \to (v+1, t-1, \bar{t}+1)$.

The lengths of these paths are:

1. Paths 1 and 3: $(v+1)A(v+1, t-1, \bar{t}+1) - vA(v, t, \bar{t}) - A(v, t-1, \bar{t}+1)$.
2. Path 2: $(v+1)A(v+1, t-1, \bar{t}+1) - vA(v, t, \bar{t}) - A(v, t-1, \bar{t})$.
3. Path 4: $(v+1)(A(v+1, t-1, \bar{t}+1) - A(v, t, \bar{t}))$.

By monotonicity $A(v+1, t-1, \bar{t}+1) \geq A(v, t-1, \bar{t}) \geq A(v, t, \bar{t})$, and therefore path 1 is the shortest path. Once again, applying the principle of optimality, the shortest path is of the type described, and Equation (18) follows. ■

C.2 Allot-at-Exit Allocation Rules

This section rounds out some other results alluded to in the paper.

OBSERVATION 5 *Suppose the pricing rule is a non-negative function of types, and the allotment rule is of the type identified in Observation 1. Then the IC constraint corresponding to an agent reporting a late-departure is redundant.*

PROOF: Consider an agent of type $(v, t, \bar{t})$ misreporting type as $(v, t, \bar{t}')$, where $\bar{t} < \bar{t}'$. The relevant IC constraint is:

$$V(A(v, t, \bar{t})|(v, t, \bar{t})) - P(v, t, \bar{t}) \geq V(A(v, t, \bar{t}')|(v, t, \bar{t})) - P(v, t, \bar{t}').$$

By Observation 1, an agent reporting an exit time of $\bar{t}'$ gets allotted at $\bar{t}'$. By our assumptions on the functional form of $V(.)$, $V(A(v, t, \bar{t}')|(v, t, \bar{t})) = 0$. Therefore,

$$V(A(v, t, \bar{t})|(v, t, \bar{t})) - P(v, t, \bar{t}) \geq -P(v, t, \bar{t}).$$

This is redundant given Individual Rationality (6) and the fact that $P \geq 0$. ■