Problem Set 9 Solutions

1. *A monopoly insurance company provides accident insurance to two types of customers: low risk customers, for whom the probability of an accident is 0.25, and high risk customers, for whom the probability of an accident is 0.5. There is an equal number of both types of customers. Without insurance, each customer's wealth is 16 if there is no accident, but 0 if there is an accident. Customers' von Neumann Morgenstern utility of wealth is:* $u(w) = \sqrt{w}$.*The insurance company cannot identify the type of a customer when the customer applies for an insurance contract. Suppose the insurance company offers the following two contracts. The first contract offers a payout of 8 in case there is an accident, and requires customers to pay a premium of 7. The second contract offers a payout of 16 in case there is an accident, but requires customers to pay a premium of 10.*

   (a) *Determine for low risk customers and then also for high risk customers which, if any, of these contracts they will buy.* Consider the low risk (LR) customer first.

   Solution

$$
\begin{aligned}
Eu^{LR} &= \frac{3}{4}\sqrt{16} + \frac{1}{4}\sqrt{0} = 3 \\
Eu^{LR}(I) &= \frac{3}{4}\sqrt{16-7} + \frac{1}{4}\sqrt{0+8-7} = \frac{9}{4} + \frac{1}{4} = \frac{5}{2} = 2.5 \\
Eu^{LR}(II) &= \frac{3}{4}\sqrt{16-10} + \frac{1}{4}\sqrt{0+16-10} = \sqrt{6} \approx 2.4
\end{aligned}
$$

   Since $Eu^{LR} > Eu^{LR}(I) > Eu^{LR}(II)$ the low risk consumer will not participate in the insurance market and, therefore, will not buy any of the offered contracts.

   Now consider the high risk (HR) consumer.

$$
\begin{aligned}
Eu^{HR} &= \frac{1}{2}\sqrt{16} + \frac{1}{2}\sqrt{0} = 2 \\
Eu^{HR}(I) &= \frac{1}{2}\sqrt{16-7} + \frac{1}{2}\sqrt{0+8-7} = 2 \\
Eu^{HR}(II) &= \frac{1}{2}\sqrt{16-10} + \frac{1}{2}\sqrt{0+16-10} = \sqrt{6} \approx 2.4
\end{aligned}
$$

   Since $Eu^{HR}(II) > Eu^{HR}(I) = Eu^{HR}$ the high risk consumer will buy the second contract.

   (b) *Does the insurance company manage to screen its customers with these contracts?*

   Solution

The insurance company does mange to screen consumers with these two contracts: The company can infer that all of the people who show up to buy the second contract are the high risk consumers.

(c) *Calculate the insurance company's expected profit if it offers these contracts.*

Solution

The expected profit of the insurance company is given by

$$E\pi = E\pi^{HR}(II) = 10 - \frac{1}{2}16 = 2$$

2. *The driver of a car can exert effort to avoid an accident ($e = 1$) or not exert any effort ($e = 0$). If $e = 1$, the probability of an accident is $1/2$. If $e = 0$, the probability of an accident is 1. The driver's wealth without accident is: $w = 100$. In case of an accident, the repair of the car costs 64. So, if there is an accident, the driver has $w = 100 - 64 = 36$ left. The driver's utility of wealth is $\sqrt{w}$, that is, the driver is risk averse. The cost of effort, $C(e)$, are 0 if effort is $e = 0$, and 1 if effort is $e = 1$. The driver's von Neumann Morgenstern utility function is: $u(w, e) = \sqrt{w} - C(e)$.*

(a) *Will the driver choose to exert effort? Compare the expected utility of the driver when exerting effort ($e = 1$) with the expected utility when exerting no effort ($e = 0$).*

Solution

The driver compares the expected utility from exerting effort with that of not exerting effort and picks the one that gives higher expected utility.

be used even if she is substituting in her constraints instead of setting up lagrange. Just wanted to make sure that was true?

$$\begin{aligned} E(U|e &= 0) = 0U(100) + 1U(36) - 0 && (1)\\ &= \sqrt{36} && (2)\\ &= 6 && (3) \end{aligned}$$

$$\begin{aligned} E(U|e &= 1) = 0.5U(100) + 0.5U(36) - 1 && (4)\\ &= 0.5\sqrt{100} + 0.5\sqrt{36} - 1 && (5)\\ &= 7 && (6) \end{aligned}$$

The driver will choose to exert effort since the expected utility from exerting effort is higher than that from not exerting effort.

(b) *Now suppose there is a risk neutral insurance company. This insurance company acts like a principal with the driver being the agent. Suppose the insurance company cannot monitor the driver's behavior. The insurance company considers three contracts, labeled A, B and C. Each contract specifies the price p and the amount of money the driver gets in case of an accident, d. Given p and d, the final wealth of the driver in case of no accident is $w_0 = 100 - p$ and the final wealth in case of an accident is $w_A = 36 - p + d$. The contracts are:*

Contract	Price p	Payment d
A	36	64
B	19	47
C	19	32

For each of the contracts, calculate the final wealths, w_0 and w_A, in the two outcomes, and list them in a table of the following type:

Contract	w_0	w_A
A		
B		
C		

Which of these contracts offers full insurance to the driver?

Solution

Contract	w_O	w_A
A	$100 - 36 = 64$	$36 + 64 - 36 = 64$
B	$100 - 19 = 81$	$36 + 47 - 19 = 64$
C	$100 - 19 = 81$	$36 + 32 - 19 = 49$

Contract A offers full insurance.

(c) *For each of these contracts, determine which of the two effort levels, $e = 0$ or $e = 1$, would be expected utility maximizing for the driver if he accepted that contract. Assume that the driver, if both effort levels yield the same expected utility, chooses $e = 1$.*

Solution

For contract A:

$$\begin{aligned} E(U|e &= 0) = 0U(64) + 1U(64) - 0 && (7) \\ &= \sqrt{64} && (8) \\ &= 8 && (9) \end{aligned}$$

$$\begin{aligned} E(U|e &= 1) = 0.5U(64) + 0.5U(64) - 1 && (10) \\ &= 0.5\sqrt{64} + 0.5\sqrt{64} - 1 && (11) \\ &= 7 && (12) \end{aligned}$$

If the driver chooses contract A it is optimal for him/her to chose $e = 0$.

For contract B:

$$
\begin{aligned}
E(U|e &= 0) = 0U(81) + 1U(64) - 0 && (13)\\
&= \sqrt{64} && (14)\\
&= 8 && (15)
\end{aligned}
$$

$$
\begin{aligned}
E(U|e &= 1) = 0.5U(81) + 0.5U(64) - 1 && (16)\\
&= 0.5\sqrt{81} + 0.5\sqrt{64} - 1 && (17)\\
&= 7.5 && (18)
\end{aligned}
$$

If the driver chooses contract B it is optimal for him/her to chose $e = 0$.

For contract C:

$$
\begin{aligned}
E(U|e &= 0) = 0U(81) + 1U(49) - 0 && (19)\\
&= \sqrt{49} && (20)\\
&= 7 && (21)
\end{aligned}
$$

$$
\begin{aligned}
E(U|e &= 1) = 0.5U(81) + 0.5U(49) - 1 && (22)\\
&= 0.5\sqrt{81} + 0.5\sqrt{49} - 1 && (23)\\
&= 7 && (24)
\end{aligned}
$$

If the driver chooses contract C it is optimal for him/her to chose $e = 1$.

(d) *Which of these contracts are such that the driver would accept the contract rather than staying uninsured? (For each contract, compare the expected utility from being uninsured with the expected utility when having the contract and choosing the optimal effort level. Assume that the driver accepts a contract if indifferent between insuring and not insuring.)*

<u>Solution</u>

Note that the optimal payoff from chosing any of the three contracts is atleast as high as that from being uninsured. Hence the driver would accept any of the three contracts rather than remaining uninsured.

(e) *Which of the three contracts gives the insurance company the highest expected profits? What are the expected profits of the insurance company if it offers this contract?*

Solution

Notice that if the insurance company offers contract A then the agent chooses to exert no effort and the profit of the firm is $36 - 64 = -28$. Similarly, if the insurance company offers contract B then the agent chooses to exert no effort and the profit of the firm is $19 - 47 = -28$. Lastly, if the insurance company offers contract C then the agent chooses to exert effort ($e = 1$) and the profit of the firm is $0.5(19) + 0.5(19 - 32) = 3$. Contract C maximizes the firms expected profit.

(f) Consider the model of second degree price discrimination that we introduced in class.

i. Complete the graphical representation of the optimal pricing problem of the monopolist, and describe visually the payments that he will receive from the buyers.

Solution

In the diagram below, the red area represents the amount that the monopolist receives from each low type consumer, and the red plus the green area represents the amount that the monopolist receives from each high type consumer.

ii. Complete the analysis of the optimal quality provision by analyzing the associated transfer payments in equilibrium. Verify that the remaining participation and incentive constraints that we assumed to be slack, i.e. nonbinding, are indeed nonbinding in the optimal solution.

Solution

Starting from the beginning, the problem is to solve

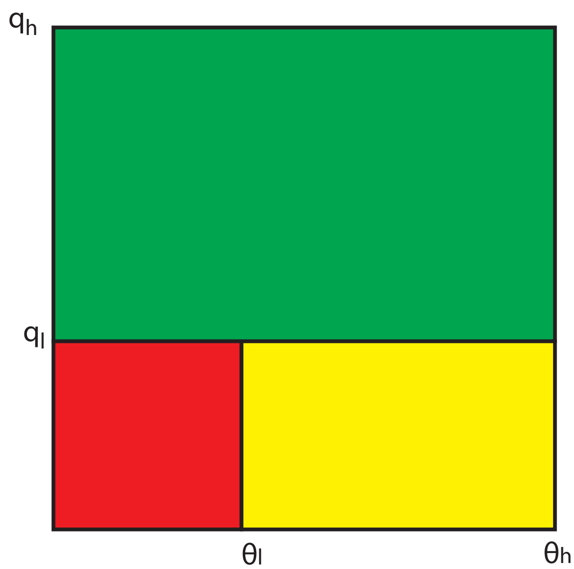
$$\begin{aligned} \max_{t_l, t_h, q_l, q_h} \alpha(t_l - \frac{1}{2}q_l^2) + (1-\alpha)(t_h - \frac{1}{2}q_h^2) \text{ s.t.}& \\ t_l &\leq \theta_l q_l \\ t_h &\leq \theta_h q_h \\ \theta_l q_h - t_h &\leq \theta_l q_l - t_l \\ \theta_h q_l - t_l &\leq \theta_h q_h - t_h \end{aligned}$$

Assuming that the first and fourth constraints are binding in equilibrium, and substituting these constraints into the maximization problem and simplifying, this simplifies to the unconstrained maximization problem

$$\max_{q_l, q_h} \alpha(\theta_l - q_l) + (1-\alpha)(\theta_h q_h - q_l(\theta_h - \theta_l)$$

The first order conditions are:

9 Figure 1.pdf



Key: Red=transfer from low type
Red+Green=transfer from high type
Yellow=High type information rent

$$\alpha(\theta_l - q_l) - (1-\alpha)(\theta_h - \theta_l) = 0$$
$$(1-\alpha)\theta_h - q_h = 0$$

Solving for q_l and q_h yields

$$q_l = \theta_l - \frac{1-\alpha}{\alpha}(\theta_h - \theta_l)$$
$$q_h = \theta_h$$

Plugging these values into the binding first and fourth constraints from the first statement of the problem yields the transfer payments

$$t_l = \theta_l^2 - \theta_l \frac{1-\alpha}{\alpha}(\theta_h - \theta_l)$$
$$t_h = \theta_h^2 - \theta_l(\theta_h - \theta_l) + \frac{1-\alpha}{\alpha}(\theta_h - \theta_l)^2$$

To verify that the second constraint is not binding, rewrite the binding fourth constraint as

$$t_h = \theta_h q_h - q_l(\theta_h - \theta_l)$$

Plugging this into the second contraint shows that the second constraint is not binding as long as $q_l > 0$.
To verify that the third constraint is not binding, plug the binding first constraint into the right hand side of the third constraint and plug the rewritten fourth constraint into the left hand side of the third constraint to get

$$(\theta_l - \theta_h)q_h - q_l(\theta_h - \theta_l) \leq 0$$

which holds strictly, showing that the third constraint is also not binding.

iii. Describe in detail the nature of the solution as either α becomes small or $\theta_h - \theta_l$ becomes large. Describe the economic intuition behind the solution.
Solution
For α small or $\theta_h - \theta_l$ large, the quality sold to the low valuation consumer goes to zero, while the transfer from the high valuation consumer goes to the complete information transfer $t_h = \theta_h^2$. The intuition is that as the low valuation consumers become less important, the firm becomes less willing to pay the information rent to the high valuation consumers in order to sell to the low valuation consumers, until the firm stops selling to the low valuation consumers entirely.

iv. Finally suppose that there are three different types of customers

$$0 < \theta_l < \theta_m < \theta_h,$$

with prior probabilities $0 < \alpha_l, \alpha_m, \alpha_h < 1$. Extend the analysis of the optimal second degree price discrimination from two to three types.

A. Start with the guess that the only binding constraints are the individual participation constraint of θ_l and that the binding incentive constraints are θ_m to θ_l and θ_h to θ_m. Give an interpretation of the binding constraint and give an argument as to why these might be the only binding constraints.

Solution

These binding constraints imply that the monopolist appropriates all of the surplus generated by the lowest valuation type, while the middle and high valuation types must be given information rents in order for them to be willing to separate themselves from the lower valuation types. The middle type must be given a rent to separate from the lowest type, and the high type must be given a (larger) rent to separate from the middle type.

B. Now compute the optimal solution under the above hypothesis. What can you say about the relative size of q_l, q_m, q_h in the first best case (perfect price discrimination or social welfare maximizing) and the second best (revenue maximizing solution under incomplete information.)

Solution

The monopolist solves

$$\max_{q_l,q_m,q_h,t_l,t_m,t_h} \alpha_l(t_l - \frac{1}{2}q_l^2) + \alpha_m(t_m - \frac{1}{2}q_m^2) + \alpha_h(t_h - \frac{1}{2}q_h^2) \text{ s.t.}$$

$$\begin{aligned} t_l &\leq \theta_l q_l \\ t_m &\leq \theta_m q_m \\ t_h &\leq \theta_h q_h \\ \theta_i q_j - t_j &\leq \theta_i q_i - t_i \text{ for all } i \neq j \end{aligned}$$

The binding constraints are:

$$\begin{aligned} t_l &= \theta_l q_l \\ \theta_m q_l - t_l &= \theta_m q_m - t_m \\ \theta_h q_m - t_m &= \theta_h q_h - t_h \end{aligned}$$

Plugging the binding constraints into the maximization problem yields the unconstrained maximization problem

$$\begin{aligned} &\max_{q_l,q_m,q_h} \alpha_l(\theta_l q_l - \frac{1}{2}q_l^2) + \alpha_m(\theta_m q_m - q_l(\theta_m - \theta_l) - \frac{1}{2}q_m^2) \\ &+ \alpha_h(\theta_h q_h - q_m(\theta_h - \theta_m) - q_l(\theta_m - \theta_l) - \frac{1}{2}q_h^2) \end{aligned}$$

The FOCs are:

$$\begin{aligned}
\alpha_l(\theta_l - q_l) - \alpha_m(\theta_m - \theta_l) - \alpha_h(\theta_m - \theta_l) = 0 \\
\alpha_m(\theta_m - q_m) - \alpha_h(\theta_h - \theta_m) = \\
\alpha_h(\theta_h - q_h) = 0
\end{aligned}$$

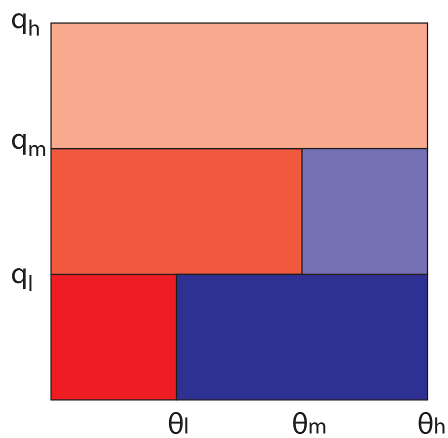
Solving yields:

$$\begin{aligned}
q_l &= \theta_l - \frac{\alpha_m + \alpha_h}{\alpha_l}(\theta_m - \theta_l) \\
q_m &= \theta_m - \frac{\alpha_h}{\alpha_m}(\theta_h - \theta_m) \\
q_h &= \theta_h
\end{aligned}$$

The optimal transfers can be found by plugging these quantities into the binding constraints.
In the first best, $q_i = \theta_i$ for all i. In the second best, q_l and q_m are less than the first best qualities.

C. Finally, illustrate the revenue that the firm and the net utility that the agents get in the (x, y) diagram used above, where the x-axis describes the type and the y-axis the quantity.
<u>Solution</u>

9 Figure 2.pdf



The darkest shade of red is the transfer paid by the lowest type. The sum of the two darker shades of red is the transfer paid by the medium type, and the sum of all three shades of red is the transfer paid by the highest type. The darkest shade of blue is the information rent to the medium type and the sum of the two shades of blue is the the information rent to the highest type.