THE DYNAMIC VICKREY AUCTION

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Abstract. We construct a simple payment scheme that implements the efficient allocation rule for a single indivisible object over T time periods. Buyers arrive randomly over time. Private information is multidimensional because valuations depend on the time at which the object is sold. It is shown that each type has a unique potential winning period and only the valuation for this period is important for the allocation decision. Therefore, types can be reduced to essentially one dimension and there is a natural order on the type space by which buyers can be compared.
These properties allow to define a simple payment rule in which (A) only the winner has to make payments, (B)transfers are ex-post individually rational, (C) there are not subsidies, and (D) payments can be made online. The payment rule is a generalization of the static Vickrey auction in which the winner pays the lowest valuation for the winning period that would suffice to win. Losers pay nothing.
Furthermore, in each period, there is only one buyer who has a chance to win the object in the future, all other buyers can be dismissed and will never be recalled. This allows to define a generalized ascending auction that implements the efficient allocation rule and the same payment rule as the dynamic Vickrey auction. Both the dynamic Vickrey auction and the generalized ascending auction are periodic ex-post incentive compatible.
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1. Introduction

Standard auction models usually assume that all potential buyers are available at the same time, and that the valuations of buyers do not depend on the time of the allocation. In many allocation problems, however, time is an important factor. In online auctions, buyers typically arrive over time and since auctions usually last several days, some buyers may not be willing to wait until the end of the auction (for example think of buying a last-minute birthday present). Internet platforms like eBay offer a feature that allows to end the auction immediately for a predetermined price. One explanation why this buy-price is used, is that buyers are impatient and willing to pay a high price for closing a deal immediately (Mathews, 2004; Gallien and Gupta, 2007).[1] Time preferences of buyers, as well as dynamic arrival are important in many other markets. For example, in the housing market, and in the markets for airline tickets or hotel reservations, a long time elapses between the start and the end of the selling mechanism.

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[1]Another explanation is risk-aversion (Budish and Takeyama, 2001; Hidvégi, Wang, and Whinston, 2006; Reynolds and Wooders, 2006).

In this paper, we study the dynamic allocation of a single object over a finite time horizon. The model generalizes the standard independent private values framework. Potential buyers arrive randomly over time, they are long-lived, and the valuation they derive from getting the object may depend on the time of the allocation in an arbitrary way. We show that the efficient allocation rule can be implemented by a mechanism with a *simple payment rule* that generalizes the static Vickrey auction (Vickrey, 1961).

The implementability of the efficient allocation rule has been demonstrated in various settings by Parkes and Singh (2003), Bergemann and Välimäki (2010), Athey and Segal (2007) and Cavallo, Parkes, and Singh (2010). To ensure incentive compatibility, expected payments of each buyer have to be equal to the expected change in the welfare enjoyed by the other agents due to the report of the buyer. This is an application of the logic of Vickrey-Clarke-Groves (VCG) mechanisms to the dynamic framework (Vickrey, 1961; Clarke, 1971; Groves, 1973). Incentive compatibility thus pins down the *expected* payments conditional on all information available at the time when agents observe their private information (i.e. their arrival time in this paper). There are many ways, however, in which *ex-post* payments can be distributed over different states of the world, while maintaining the VCG-property of the expected payments.

The central question of this paper is whether the flexibility in the choice of ex-post payments allows to define a *simple payment rule* that implements the efficient allocation rule. A simple payment rule is one where

(A) *only the winner makes a payment,*
(B) payments are *ex-post individually rational*,
(C) the mechanism *never transfers money* to any buyer, and
(D) payments are made *online*, i.e., all information that is needed to determine the payment must be available at the time of allocation.

Properties (A)–(C) are fulfilled by standard static auction formats. Moreover, if we leave the ideal world of the abstract mechanism design model, properties (A)–(D) are obviously desirable. Property (A) minimizes the number of transactions. This is important if the buyers or the seller incur transaction costs for financial transactions. Property (B) is important because ex-post individually rational payments are easier to enforce. A winner who has to pay more than his value may feel a strong desire to renege on his bid. Property (C) is convenient because payments to buyers may encourage persons who are not interested in the object, to speculate on getting such subsidies and trying to renege on their bids in case they are selected as the winner. While it may be possible to prevent such abuse by strict enforcement of the mechanism's rules, this will certainly involve additional costs. The last property (D) seems indispensable. If payments cannot be determined online, this means that additional information has to be collected from future buyers after the winner has already been determined. Incentives for reporting such information are weak. Furthermore, online payments allow to match payments with delivery, which makes it easier to enforce payments.

The Dynamic Vickrey Auction proposed in this paper yields expected payments that are the same as in the mechanisms of the papers cited above.[2] Ex-post, however, the payment

[2]This is a necessary condition for incentive compatibility.

of an agent corresponds neither to the expected change in the other agents' welfare as in the *online VCG mechanisms* proposed by Parkes and Singh (2003), nor to the sum of *flow marginal contributions* as in the *dynamic pivot mechanism* proposed by Bergemann and Välimäki (2010).[3] Instead, following the logic of the static Vickrey (or second-price) auction, the payment of the winner is equal to the lowest valuation for the winning period, with which she could have won the object. This valuation is called the *critical type* of the winning buyer. In a static model, the critical type is equal to the second highest valuation. In the dynamic model, this is not the case because the winner does not only compete with buyers in the same period. The critical type of the winner also depends on the value of past allocations that the winner has prevented by reporting a high type, and the expected value of future allocations. Therefore, the payment can differ from the second highest valuation for the winning period. Since only the winner has to make a payment, and since the critical type never exceeds the winner's valuation, properties (A)–(C) are satisfied. Moreover, since the critical type determines the allocation decision, it can only depend on information that is available in the winning period. Therefore, the payment does not have to be delayed beyond the time of the allocation of the object. It can be determined *online* and property (D) is satisfied.

The design of the Dynamic Vickrey Auction is non-trivial in two ways. First, the definition of a critical type is not straight-forward because types are multidimensional. The central result of this paper is that the information about a buyer's type, that is relevant for determining the efficient allocation, is essentially one-dimensional. For each type, there is a unique period in which she can possibly win the object. Therefore, only the valuation for this period matters for the efficient allocation rule. This reduction to one dimension allows to define an appropriate order on the type-space that makes the notion of a critical type meaningful. As a corollary of this result, it is shown that at each point in time, there is only one bidder who has a positive chance of winning the object. All other bidders can be dismissed immediately and will never be recalled. In other words, the efficient allocation rule only needs a queue of bidders of length one. These properties also allow to design a generalized ascending auction that implements the efficient allocation rule in periodic ex-post equilibrium.

Second, impossibility results by Gershkov and Moldovanu (2009, 2010) show that in general, the efficient allocation rule cannot always be implemented by a simple payment rule, that satisfies (A)–(D). In these papers, the designer learns about future buyers' type distributions or the arrival distributions, respectively, from current arrivals and current buyers' types. These models violate an independence assumption that is made in the analyses of Parkes and Singh (2003), Bergemann and Välimäki (2010), and Cavallo, Parkes, and Singh (2010), in order to show that the efficient allocation rule is implementable by online payments (D). Correlated types or arrivals lead to an informational externality of a buyers report on the value that the designer can achieve by withholding the object and allocating to future buyers. In general, this informational externality can only be reflected in the payments to the buyer if the designer can condition payments on the realized types

[3]In should be noted, however, that the *dynamic pivot mechanism* proposed by Bergemann and Välimäki (2010) does yield a simple payment rule if it is applied to a scheduling problem rather than the allocation of a single object.

and arrivals of future buyers, which are drawn from the correct distribution (see also Mezzetti, 2004; Athey and Segal, 2007). In the case of correlated valuations, Gershkov and Moldovanu (2009) demonstrate that online payments are insufficient to implement the efficient allocation rule if the informational externalities are too strong. In the case of correlated arrivals, there is another possibility to provide incentives for the buyers to report their arrivals truthfully (Gershkov and Moldovanu, 2010). Since buyer's can only delay but not bring forward their arrivals, the mechanism could pay a subsidy for early arrivals that reflects a positive informational externality. While this allows to implement the efficient allocation rule with online payments, the subsidy has to be paid to all buyer's, not only the winner. Therefore condition (C) is violated by this payment scheme. In this paper, we restrict attention to a model that satisfies the independence assumption mentioned above. This implies that there is no fundamental reason that precludes online payments. Little is known, however, under which conditions also the other properties (A)–(C) can be achieved simultaneously with (D) in the implementation of the efficient allocation rule.

The paper is structured as follows: In Section 2, the formal model is introduced, the efficient allocation rule is defined, and the mechanisms proposed by Parkes and Singh (2003) and Bergemann and Välimäki (2010) are discussed. In Section 3, it is shown that for each type, there is a unique potential winning period. In Section 4, the payment rule is constructed. Section 5 describes the generalized ascending auction. Section 6 concludes with a discussion of possible generalizations and relationships of the results to revenue-maximizing auctions.

2. The Model

2.1. **Setup and Notation.** A seller wants to sell a single indivisible object within T time periods. In each period t, a random number of buyers n_t arrives. The numbers n_t are independent random variables and the probability that $n_t = k$ is denoted by $\rho_k^t \geq 0$, with $\sum_{k=0}^{\infty} \rho_k^t = 1$. $N_t := \sum_{\tau=1}^{t} n_\tau$ denotes the number of buyers that have arrived in or before period t. Buyers are indexed in the order of arrival. Within periods, indexing is random. $I_t = \{1, \ldots, N_t\}$ denotes the set of buyers that arrive in or before period t.

A typical buyer j who arrives in period t, attaches a monetary value of $v_j^\tau \in [0, \overline{v}]$ to the object if she gets it in period $\tau \geq t$, where $\overline{v} > 0$. Each buyer is completely characterized by her vector of valuations starting with her arrival period. Her type is thus $v_j = (v_j^t, \ldots, v_j^T)$. Buyers' types are independent random variables with distribution functions $\Phi_t : [0, \overline{v}]^{T-t+1} \to [0, 1]$ and strictly positive densities on $[0, \overline{v}]^{T-t+1}$, where t is the arrival period of the respective buyer. We allow for dependencies between the components of a buyer's type. $(\rho_k^t)_{t=1,\ldots T, k\in\mathbb{N}}$ and $(\Phi_t)_{t=1,\ldots,T}$ are commonly known by the buyers and the seller. Realizations of type and arrival period are private information of each buyer, and we assume that she knows her complete type in the arrival period. The valuation for the object can depend on the time of allocation as described by the type-vector, but this relationship is completely determined when the buyer arrives.

Prior to the arrival period, the type of a buyer is not known to anybody. To emphasize this informational constraint, we distinguish the type of a buyer who arrives in period t: $(v_j^t, \ldots, v_j^T) \in [0, \overline{v}]^{T-t+1}$ from the type of a buyer who arrives in period $\tau < t$ and

has the same valuations for periods $t, \ldots, T$ but valuations of zero for periods before t: $(0, \ldots, 0, v_j^t, \ldots, v_j^T) \in [0, \bar{v}]^{T-\tau+1}$.

Buyers are risk-neutral. If a buyer j has to make a total expected payment of p_j, and q_τ is the probability of getting the object in period τ, then her expected utility is given by $\sum_{\tau=t}^{T} v_j^\tau q_\tau - p_j$. The seller's valuation for the object is normalized to zero.

For $\tau \geq t$, let $\eta_\tau(I_t) := \max_{i \in I_t}\{v_i^\tau\}$ be the highest valuation for getting the object in period τ among all buyers in I_t and define $\eta(I_t) := (\eta_t(I_t), \ldots, \eta_T(I_t))$ and $\eta := (\eta(I_1), \ldots, \eta(I_T))$.

2.2. **The Efficient Allocation Rule under Complete Information.** In period t, allocation decisions can only depend on the types of buyers that have already arrived. The value of the object, on the other hand, depends on the time when a buyer gets it. Consequently, the ex-post efficient allocation rule is not feasible. It is not always possible to identify the buyer with the ex-post highest valuation in the period where this valuation can be realized. Instead, we consider the ex-ante efficient allocation rule, i.e., the allocation rule that maximizes the expected value of the utility enjoyed by the buyers, subject to the informational constraint that information about future types cannot be used. This allocation rule is the optimal policy for the following dynamic program.

The *state* (h_t, a_t) at time t, consists of the history of types of all buyers in I_t, denoted h_t, and the availability of the object, denoted $a_t \in \{0, 1\}$. a_t equals zero if the object has already been allocated, and one if the object is still available in period t.

A *policy* is a family of decision rules $x = (x_t(.,.))_{t=1}^T$ where $x_t(h_t, a_t) \in \{0, 1, N_t(h_t)\}$ if $a_t = 1$ and $x_t(h_t, 0) = 0$.[4] The total value (or welfare) in period T at history h_T after decisions $x = (x_1, \ldots, x_T)$ is given by $\sum_{t=1}^{T} v_{x_t}^t$ (define $v_0^t := 0, \forall t$). The *efficient allocation rule* is the optimal policy x^*, for the dynamic program $\mathcal{P}$:

$$(\mathcal{P}) \qquad \max_{(x_\tau(.,.))_{\tau=1}^T} E\left[\sum_{\tau=1}^{T} v_{x_t(h_t,a_t)}^\tau\right].$$

The value function is given by $V_{T+1}^*(h_{T+1}, a_{T+1}) := 0$ and

$$\begin{aligned} V_t^*(h_t, a_t) &:= \max_{(x_\tau(.,.))_{\tau=t}^T} E\left[\sum_{\tau=t}^{T} v_{x_\tau(h_\tau,a_\tau)}^\tau\right] \\ &= v_{x_t^*(h_t,a_t)}^t + E\left[V_{t+1}^*(h_{t+1}, a_{t+1}) \,\middle|\, h_t, a_t, x_t^*(h_t, a_t)\right]. \end{aligned}$$

The efficient allocation rule always allocates to a buyer who has the highest valuation for the selling period. Hence, the value function can be considered as a function of $\eta(I_t)$ instead of h_t.

$$V_t^*(h_t, a_t) = V_t^*(\eta(I_t), a_t).$$

The option value of retaining the object in period t is a function of the highest valuations for all future periods $\hat{v}_t : [0, \bar{v}]^{T-t+1} \to \mathbb{R}$, defined by

$$\hat{v}_t(\eta(I_t)) := E\left[V_{t+1}^*(\eta(I_{t+1}), 1) \,\middle|\, \eta(I_t)\right]. \tag{2.1}$$

[4]Without loss of efficiency we can restrict the decision space to deterministic decisions.

It is efficient to allocate the object in period t, if the option value is below the highest valuation in period t. Hence, the object is allocated in the first period for which

$$\eta_t(I_t) > \hat{v}_t(\eta(I_t)). \tag{2.2}$$

In this period, the object is awarded to a buyer with valuation $\eta_t(I_t)$. If there is a tie, the buyer with the lowest index is chosen.[5]

2.3. **Incentive Compatibility.** In order to study possible implementations of the efficient allocation rule, we consider incentive compatible *direct mechanisms* (S, x^*, π). $S = (S_t)_{t\in\{1,\dots,T\}}$ is the *sequence of signal spaces*, where $S_t = [0, \overline{v}]^{T-t+1}$ is the signal space for period t. In each period, buyers can only report a complete type of a buyer that arrived in the same period. A buyer can make a report in any period after her arrival but we assume for simplicity that each buyer makes at most one report. x^* is the efficient policy described in the previous section. $\pi : h_T \mapsto (\pi_i)_{i=1,\dots,N_T} \in \mathbb{R}^{N_T}$ is the payment rule. $\pi_j(h_T)$ specifies the payment of buyer j at the terminal history h_T. A mechanism does not explicitly specify the time at which payments have to be made. Payments may depend on all reports until the last period. If, however, in period t, and for some h_t, $\pi_j(h_T)$ is independent of all reports after period t, then the payments of buyer j can already be made in period t.

Now consider a buyer j, who arrives in period t and plans to make a report $v' \in S_r$ in period $r \geq t$. Denote the history of reports of all buyers except j, by h_t^{-j}. For given h_t^{-j} and assuming that all other buyers report truthfully, the *winning probability* of buyer j for period τ conditional on all information available in the arrival period is given by

$$q_\tau(v', h_t^{-j}) := \text{Prob}\left[x^*_\tau(h_\tau) = j \;\middle|\; h_t^{-j}, v_j = v'\right],$$

where the time of the report is implicitly given by the length of v'. We omit a_t as an argument of the winning probability because $q^r_\tau(h_t^{-j}, v')$ will only be used when $a_t = 1$. The *expected payment* is given by

$$p(v', h_t^{-j}) := E\left[\pi_j(h_T) \;\middle|\; h_t^{-j}, v_j = v'\right].$$

With these definitions, the *expected utility* from participating in the mechanism with a reported type $v' \in S_r$ and true type $v \in S_t$ is

$$U(v, v', h_t^{-j}) := \sum_{\tau=r}^{T} v^\tau q^r_\tau(v', h_t^{-j}) - p^r(v', h_t^{-j}).$$

The expected utility from truth-telling is abbreviated $U(v, h_t^{-j}) := U(v, v, h_t^{-j})$.

A mechanism is *periodic ex-post incentive compatible* if for all $t, r \in \{1, \dots, T\}$ with $r \geq t$, all $v \in S_t$, $v' \in S_r$, and all possible histories of reports h_t^{-j},

$$U(v, h_t^{-j}) \geq U(v, v', h_t^{-j}). \tag{2.3}$$

[5]This implies a random selection among buyers with the same arrival period because indices are assigned randomly. Furthermore, there is preference for buyers that arrived earlier. In this paper, the *efficient allocation rule* always refers to the allocation rule that has just been described. There are other allocation rules which achieve the same total expected welfare but employ different tie-breaking rules. For example, the allocation rule that allocates in the first period for which $\eta_t(I_t) \geq \hat{v}_t(\eta(I_t))$, is also efficient. Also, other tie-breaking rules could be used.

Periodic ex-post incentive compatibility is a hybrid concept that reflects the informational constraint of the dynamic model. Expectations are taken with respect to the types of future buyers. In this sense, it resembles Bayes-Nash incentive compatibility. With respect to past and current buyers, incentive compatibility constraints must hold for every profile of types. Therefore, ex-post incentive compatibility is required only for information that is already realized at the time when a buyer makes a report.[6]

Incentive compatibility of the efficient allocation rule has been shown by Parkes and Singh (2003) for discrete type-spaces[7] and by Bergemann and Välimäki (2010) for continuous type-spaces.[8]

Adapted to the model of this paper, the *online VCG mechanism* of Parkes and Singh (2003) uses the following payment rule. The payment of j, when she makes a report $v'_j \in S_r$, is defined as

$$\pi_j^{\text{oVCG}}(h_T^{-j}, v') = \sum_{\tau=r}^{T} v'^{\tau} \mathbf{1}_{\{x_\tau^*((h_\tau^{-j}, v'))=j\}} - \left[V_r^*((h_r^{-j}, v'), a_r) - V_r^*(h_r^{-j}, a_r)\right]. \tag{2.4}$$

The first term is equal to the private utility j enjoys according to her reported type. If j wins the object in period τ, this is equal to v'^{τ}. If she does not win the object in any period, it is zero. The term in parentheses is the change in expected total welfare due to the report of buyer j given the information available in period r. As in the standard static VCG mechanism, the payment replaces the private surplus of each buyer by the (expected) change in total welfare due to her report. The allocation rule maximizes welfare subject to the informational constraint that future types are not known. Therefore, it is optimal for j to report her true type, because she faces the same informational constraint.

Theorem 1 (Parkes and Singh (2003))**.** *The mechanism* (S, x^*, π^{oVCG}) *is periodic ex-post incentive compatible.*

π^{oVCG} is not the only payment rule that implements the efficient allocation. The *dynamic pivot mechanism* of Bergemann and Välimäki (2010) does not aggregate payments over periods. The payment of a buyer j in period $s \geq r$ is defined as

$$\begin{aligned}\pi_j^{\text{DP}}(h_s^{-j}, v') = {} & v_{x_s^*(h_s^{-j})} + E\left[V_{s+1}^*(h_{s+1}^{-j}) \middle| h_s^{-j}, x_s^*(h_s^{-j})\right] \\ & - \left(v_{x_s^*(h_s^{-j}, v')} + E\left[V_{s+1}^*(h_{s+1}^{-j}) \middle| h_s^{-j}, x_s^*(h_s^{-j}, v')\right]\right)\end{aligned}$$

[6]Note that (2.3) also rules out profitable deviations in which a buyer delays her report and reports different types in later periods, conditional on the valuations of buyers who have arrived in the meantime. For period T, (2.3) ensures that it is optimal to report v_j^T truthfully, for every history $h_{T,-j}$. This applies to buyers who arrived in period T as well as to buyers who delayed their report because the mechanism cannot distinguish between them. In period $T-1$, (2.3) rules out that a delayed but truthful report of v_j^T is a profitable deviation. Therefore in period $T-1$, it is optimal to report (v_j^{T-1}, v_j^T) truthfully and without delay. Working backwards in time it follows inductively, that (2.3) rules out all feasible reporting strategies except a truthful report in the arrival period.

[7]These authors use a very similar equilibrium concept. In their concept, ex-post incentive compatibility is required with respect to information of buyers with lower index. This excludes the types of buyers who arrive simultaneously but were assigned a higher index.

[8]Athey and Segal (2007) also show implementability of the efficient allocation rule. Their model focuses on incentives in teams and budget balance. The proposed mechanism requires all agents to be available in all periods.

Neither of the mechanism fulfills all properties (A)-(C). In the truth-telling equilibrium of the online VCG mechanism, j always receives a payoff given by

$$V_t^*(h_t, a_t) - V_t^*(h_t^{-j}, a_t). \tag{2.5}$$

The payoff is independent of the event that she wins the object. In particular, this implies that the mechanism must transfer a positive amount of money to every buyer who has a positive chance of winning at the time of arrival, even if that buyer does not win the object. This violates the no-subsidy property (C).

The dynamic pivot mechanism requires payments from buyers who are pivotal for postponing the allocation even if they do not win the object. To see this consider the following example. Let $T = 2$, $I_1 = \{1, 2\}$ with $v_1 = (v_1^1, 0)$ and $v_2 = (0, v_2^2)$, and assume that $\rho_{2,1} = 1$ so that $I_2 = \{1, 2, 3\}$. Furthermore, assume that $v_1^1 = \frac{5}{8}$, $v_2^2 = \frac{3}{4}$ and $v_3^2 \sim U[0, 1]$. In this case, it is efficient not to allocate in the first period because $v_1^1 = \frac{5}{8} < E\left[\max\left\{\frac{3}{4}, v_3^2\right\}\right] = \frac{25}{32}$. Buyer two is pivotal for the allocation decision. Without her, the object would be allocated to buyer one in the first period. Her payment in period one is therefore given by $\pi_2^1 = v_1 - E[v_3] = \frac{1}{8}$. If the realized valuation v_3^2 exceeds v_2^2, then buyer two does not receive the object and his payment in the second period is $\pi_2^2 = 0$. Hence, her total payment is $\frac{1}{8}$ which violates property (A) and ex-post individual rationality.[9]

3. Properties of the Efficient Allocation Rule

The efficient allocation rule allocates the object in the first period where $\eta_t(I_t) > \hat{v}_t(\eta(I_t))$. We show below that if we apply this condition to the type v_j of buyer j, then we get the unique period in which this type can win.

Definition 2. For $t \in \{1, \ldots, T\}$, the *potential winning period* θ_j of buyer j with type $v_j \in S_t$, is the earliest period $\theta \geq t$ for which $v_j^\theta > \hat{v}_\theta(v_j)$, i.e.,

$$\begin{aligned} v_j^\tau &\leq \hat{v}_\tau(v_j), \qquad \text{for } \tau \in \{t, \ldots, \theta_j - 1\}, \\ \text{and} \qquad v_j^{\theta_j} &> \hat{v}_{\theta_j}(v_j). \end{aligned} \tag{3.1}$$

Definition 2 partitions the type-space. There is a unique potential winning period for each type. It is the period in which buyer j would win, if no other buyers arrive. Therefore, θ_j only depends on the type of buyer j and the structure of the allocation problem, i.e., the arrival process and the distributions from which valuations are drawn. It does not depend on the realized types of the other buyers or the realized numbers of buyers.

Examples:

(1) A buyer with constant valuation $v_j = (v, v, \ldots, v)$ has potential winning period T. Knowing that the value of an allocation to buyer j does not change over time, the seller will always wait to see if a buyer with a higher valuation arrives, rather than allocate to buyer j.

[9]This does not contradict the individual rationality result of Bergemann and Välimäki (2010) because these authors consider *periodic* ex-post individual rationality. Indeed, the expected payoff of buyer two in the first period is $\frac{25}{32} - \frac{5}{8} = \frac{5}{32} > 0$.

(2) A buyer with $v_j^\tau = v$ for some τ and $v_j^t = 0$ for $t \neq \tau$, has potential winning period T if $v_j^\tau \leq \hat{v}_\tau(0, \ldots, 0)$, otherwise she has potential winning period τ. It is clear that the seller would never allocate to a buyer with current valuation zero, except in the last period where the seller is indifferent. In period τ where the buyer has a positive valuation, the seller would allocate to him only if the expected value of waiting is below the current valuation.

The following theorem states that under the efficient allocation rule, a buyer can win the object only in her potential winning period.

Theorem 3. *Fix a buyer j with type $v_j \in S_t$. If $x_s(h_s, a_s) = j$ for some $s \geq t$, and some h_s, then $s = \theta_j$.*

Proof. See Appendix. □

To get an intuition for the result, consider the case $T = 2$. Suppose buyer j arrives in period one and has type $v_j = (v_j^1, v_j^2)$. The theorem states that she can either win in period one or in period two, but not in both periods. Of course, it depends on the types of the other buyers whether she wins at all.

First, suppose that for some profile of types, it is efficient that j gets the object in period two. In this case, the highest valuation for the first period $\eta_1(I_1)$, must not be greater than the option value of retaining the object:

$$\eta_1(I_1) \leq \hat{v}_1(\eta(I_1)).$$

If j wins in the second period, she must have the highest valuation for period two among the buyers from period one: $v_j^2 = \eta_2(I_1)$. Hence, the option value of retaining the object only depends on her valuation: $\hat{v}_1(\eta(I_1)) = \hat{v}_1(v_j^2)$. On the other hand, her valuation for the first period cannot be greater than $\eta_1(I_1)$. We conclude that

$$v_j^1 \leq \hat{v}_1(v_j^2).$$

Loosely speaking, if j had a higher valuation for period one, she would overbid the option value defined by her own valuation for period two. But this must not be the case if j wins in the second period.

Second, suppose that for some other profile of types it is efficient to allocate the object to j in period one. Then, $v_j^1 = \eta_1(I_1)$ and hence

$$v_j^1 > \hat{v}_1(\eta(I_1)) \geq \hat{v}_1(v_j^2).$$

Loosely speaking, j's valuation for period one must overbid the option value of retaining the object. Especially, j must overbid her own valuation for period two (transformed by the option value function). The conditions on v_j for winning in the first and in the second period, cannot be fulfilled simultaneously. Hence, it is not possible that j wins in different periods for different profiles of the other buyers' types.

Theorem 3 greatly reduces the dimension of the signal space that is necessary to implement the value-maximizing allocation rule. The type of each buyer j can be summarized by θ_j and $v_j^{\theta_j}$. In addition, once the winning period θ_j is fixed, the notion of the *lowest type* that can win the auction for a particular state of the world becomes well defined. This property will be used to define the generalized Vickrey auction.

Theorem 3 has two important implications that are useful to define auction rules. First, in each period t, there is a unique buyer j_t^* among those who have already arrived, who has a chance of winning. This buyer is called the *tentative winner in period t*. Furthermore, if we partition the set of buyers in period t into two subsets A and B, two tentative winners can be determined under the assumption that only the buyers in A or B, respectively, have arrived. The tentative winner for the set of all buyers $(A \cup B)$, must be one of the two tentative winners determined for the subsets A and B. Formally, we have:

Corollary 4. *(i) For each period t and every state (h_t, a_t) with $a_t = 1$, there exists a unique buyer $j_t^* \in I_t$ such that $x_\tau(h_\tau, a_\tau) \notin I_t \backslash \{j_t^*\}$ for all $\tau \geq t$ and all future states (h_τ, a_τ) that can occur after (h_t, a_t).*

(ii) Suppose $I_t = A \cup B$ with $A, B \neq \emptyset$ and $A \cap B = \emptyset$. Let $a \in A$ be the tentative winner if the set of buyers is $I_t' = A$, and let $b \in B$ be the tentative winner if $I_t' = B$. Then $j_t^* \in \{a, b\}$.

Proof. (i) For $t \in \{1, \ldots, T\}$ consider a hypothetical buyer k with valuations $v_k = \eta(I_t)$. By Theorem 3, this buyer has a unique potential winning period θ_k. The stopping rule of the efficient allocation only depends on the highest valuation for each period (cf. condition (2.2)). Therefore, the time at which the object is allocated with I_t, is the same as with $I_t' = \{k\}$ and $v_k = \eta(I_t)$. Hence, buyers in I_t can only win in period θ_k. If a buyer from I_t wins, it must be $j_t^* = \min\left(\arg\max_{i \in I_t}\{v_i^{\theta_k}\}\right)$. (If there are ties, the tie-breaking rule described in Section 2.2 eventually selects the buyer with the lowest index. As the indices of the buyers in I_t are already known in period t, the identity of the tentative winner is unique in period t.)

(ii) Without loss of generality we can assume that $j_t^* \in A$. But then j_t^* must also be the tentative winner for $I_t' = A$ because with any other tentative winner, the expected value of the allocation for $I_t' = A$ must be weakly smaller than the expected value with j_t^* as tentative winner.

□

An immediate implication of Corollary 4 is that a buyer who was not tentative winner in period t, cannot become tentative winner in period $t + 1$.

Corollary 5. *For $t \in \{1, \ldots, T-1\}$, let j_t^* and j_{t+1}^* be the tentative winners in t and $t+1$, respectively. Then $j_{t+1}^* \in (I_{t+1} \backslash I_t) \cup \{j_t^*\}$.*

These properties of the efficient allocation rule imply that it can be implemented as follows: In each period t, new buyers are asked to report complete types to the auctioneer.[10] If it is efficient to allocate immediately, the object is sold and the auction ends. If it is efficient to retain the object for the next period, the auctioneer declares a tentative winner which is either the tentative winner from the previous period or a new buyer who has made a report in period t. All other buyers are informed that they cannot win the auction and will never be recalled.

[10] Alternatively, they could be asked to report their potential winning period and the valuation for that period.

Remarks:

(1) The properties of the efficient allocation rule carry over to optimal policies of any dynamic program that has a similar structure as $\mathcal{P}$. For example, one could consider quasi-efficient allocation rules that maximize excepted welfare after valuations have been transformed by strictly increasing functions:

$$J_t(v_j) = (J_t^t(v_j^t), \dots, J_t^T(v_j^T)),$$

where each J_t^τ is strictly increasing.

(2) Note also that the assumption of full support of the type distribution has not been used in this section. Therefore, the results carry over to a model with constant valuations and deadlines. In this model, the types of all buyer have the form $(v_{i,} \dots, v_i, 0, \dots, 0)$ where the valuation v_i is repeated from the arrival time to the deadline d_i. Theorem 3 is trivial in this case because $\theta_i = d_i$ for all types, but the less obvious Corollary 4 also carries over to the model with deadlines.

4. Payments

In this section, a simple payment rule is constructed, that generalizes the static Vickrey auction. To highlight the similarity, we briefly review the payment rule of the static Vickrey auction.

4.1. **The Static Vickrey Auction.** Consider the standard independent private values model with N bidders. Valuations are drawn from $\Theta = [a, b]$ with distribution function F. Let q be the probability of winning the object in the second-price auction without reserve price.

By payoff-equivalence, the expected payoff of bidder 1 with valuation $v_1 \in \Theta$ in the second-price auction is given by

$$U(v_1) = \int_a^{v_1} q(v)dv. \tag{4.1}$$

Writing the winning probability explicitly, this becomes

$$U(v_1) = \int_a^{v_1} \int_{\Theta^{N-1}} \mathbf{1}_{\{v \geq \max\{v_2, \dots, v_N\}\}} dF(v_2) \dots dF(v_N) dv.$$

Writing $v_{(1)} = \max\{v_2, \dots, v_N\}$ and changing the order of integration yields:

$$\begin{aligned} U(v_1) &= \int_a^{v_1} \int_a^b \mathbf{1}_{\{v \geq v_{(1)}\}} dF^{N-1}(v_{(1)}) dv, \\ &= \int_a^b (v_1 - v_{(1)}) \mathbf{1}_{\{v_1 \geq v_{(1)}\}} dF^{N-1}(v_{(1)}), \\ &= \underbrace{\int_a^b v_1 \mathbf{1}_{\{v_1 \geq v_{(1)}\}} dF^{N-1}(v_{(1)})}_{v_1 \cdot q(v_1)} - \underbrace{\int_a^b v_{(1)} \mathbf{1}_{\{v_1 \geq v_{(1)}\}} dF^{N-1}(v_{(1)})}_{p(v_1)}. \end{aligned}$$

This shows that the payment of any bidder can be defined as zero if she does not win and as the highest valuation of the other bidders if she wins.

4.2. **Construction of Payments for the Dynamic Vickrey Auction.** The construction of payments in the dynamic setting follows the same logic. First, the dimension reduction of the type-space, incentive compatibility and payoff-equivalence for multidimensional mechanisms are used to derive a formula similar to (4.1). The result is (4.3) below. Second, as in the one-dimensional case, the winning probability is written explicitly and the order of integration is changed. Instead of the second-highest valuation, we will then obtain a *critical type* $\underline{v}^t(\theta_j, \eta_{-j})$ that depends on the arrival period t, the potential winning period θ_j, and the profile of the other bidders' highest valuations $\eta_{-j} = (\eta_s(I_\tau \backslash \{j\}))_{\tau=1,\dots,T, s=\tau,\dots,T}$. This is used to define payments for the winning bidder.

Consider a bidder j with type v_j who arrives in period $t \in \{1, \dots, T\}$. Proposition 1 in Jehiel, Moldovanu, and Stacchetti (1999) implies that the expected payoff for j, from participating in an incentive compatible mechanism which implements the efficient allocation rule, is given by

$$U^t(v_j) = U^t(0) + \int_0^1 \left\langle q^t(\gamma(s)), \gamma'(s) \right\rangle ds \tag{4.2}$$

where $q^t(v) := (q_t^t(v, h_t^{-j}), \dots, q_T^t(v, h_t^{-j}))$ is the vector of winning probabilities for the remaining periods, $\gamma : [0,1] \to [0, \overline{v}]^{T-t+1}$ parametrizes a piecewise smooth curve that connects the origin with v_j, and $\langle ., . \rangle$ denotes the standard scalar product on $\mathbb{R}^{T-t+1}$. The argument h_t^{-j} is suppressed in the functions U and q, in order to simplify notation. Incentive compatibility of the efficient allocation rule implies that q^t is a conservative vector field (Jehiel, Moldovanu, and Stacchetti, 1999). Therefore, γ can be chosen such that it is composed of three straight lines. Let the first line connect the origin with $w_j^1 := (0, \dots, \hat{v}_{\theta_j}(0), 0, \dots)$. (This implies that the first line reduces to a point for $\theta_j = T$.) Let the second line connect w_j^1 with $w_j^2 := (0, \dots, v_j^{\theta_j}, 0, \dots)$. Let the third line connect w_j^2 with v_j.

If $\theta_j \neq T$, for each $v = \gamma(s)$ on the first line, $q_\tau^t(v) = 0$ for $\tau \neq T$ by Theorem 3 (see the example after Definition 2), and $\gamma_T'(s) = 0$ by the choice of γ. Hence, the path of integration is perpendicular to the vector-field and the integrand in (4.2) vanishes. For each $v = \gamma(s)$ on the third line segment, $q_\tau^t(s) = 0$ for $\tau \neq \theta_j$, and $\gamma_{\theta_j}'(v) = 0$. Hence, the integrand vanishes as well. On the second line segment, only the θ_j^{th} components of q^t and γ' are non-zero. Therefore, with a simple change of variables, (4.2) can be simplified to

$$U^t(v_j) = U^t(0) + \int_{\hat{v}_{\theta_j}(0)}^{v_j^{\theta_j}} q_{\theta_j}^t(0, \dots, v, 0, \dots) dv. \tag{4.3}$$

For given $v \in [0, \overline{v}]$, $q_{\theta_j}^t(0, \dots, v, 0, \dots)$ is equal to the probability conditional on h_t^{-j}, that η_{-j} belongs to the set

$$\begin{aligned}
\Omega_{\theta_j}^t(v) := \Big\{ \tilde{\eta}_{-j} \,\Big|\, & \tilde{\eta}_t(I_t^{-j}) \leq \hat{v}_t(\tilde{\eta}_t(I_t^{-j}), \dots, \max\{\tilde{\eta}_{\theta_j}(I_t^{-j}), v\}, \dots, \tilde{\eta}_T(I_t^{-j})), \\
& \dots \\
& \tilde{\eta}_{\theta_j - 1}(I_{\theta_j - 1}^{-j}) \leq \hat{v}_{\theta_j - 1}(\tilde{\eta}_{\theta_j - 1}(I_{\theta_j - 1}^{-j}), \max\{\tilde{\eta}_{\theta_j}(I_{\theta_j - 1}^{-j}), v\}, \dots, \tilde{\eta}_T(I_{\theta_j - 1}^{-j})), \\
& v > \tilde{\eta}_{\theta_j}(I_{\theta_j}^{-j}), \\
& v > \hat{v}_{\theta_j}(\tilde{\eta}(I_{\theta_j}^{-j})) \Big\},
\end{aligned}$$

$$
\begin{aligned}
= \Big\{ \tilde{\eta}_{-j} \;\Big|\; & \tilde{\eta}_t(I_t^{-j}) \le \hat{v}_t(0,\dots,v,\dots,0), \\
& \dots \\
& \tilde{\eta}_{\theta_j-1}(I_{\theta_j-1}^{-j}) \le \hat{v}_{\theta_j-1}(0,v,0\dots,0), \\
& v > \tilde{\eta}_{\theta_j}(I_{\theta_j}^{-j}), \\
& v > \hat{v}_{\theta_j}(\tilde{\eta}(I_{\theta_j}^{-j}))\Big\},
\end{aligned} \tag{4.4}
$$

where $I_\tau^{-j} = I_\tau \backslash \{j\}$. In the first expression, the last inequality ensures that in period θ_j, none of the other buyers' valuations for future periods are high enough to lift the option value of waiting in period θ_j above v. The second last inequality ensures that no other buyer has a valuation for period θ_j that exceeds v. Together these two inequalities ensure that the object is allocated to buyer j with valuation $v_j^{\theta_j} = v$ if the object is still available in period θ_j. The remaining inequalities ensure that j is also the tentative winner in periods $t,\dots,\theta_j-1$. The fact that j is the tentative winner in all periods $t,\dots,\theta_j-1$ implies that the option values in all but the last inequality only depend on buyer j's type. Therefore the definition of the set simplifies to the second expression in (4.4). With this definition, (4.3) becomes

$$
U^t(v_j) = U^t(0) + \int_{\hat{v}_{\theta_j}(0)}^{v_j^{\theta_j}} \int_{\Omega^t_{\theta_j}(v)} dG^t(\eta_{-j} \mid h_t^{-j}) dv,
$$

where $G^t(\,.\mid h_t^{-j})$ shall denote the distribution function of η_{-j}, conditional on h_t^{-j}.

The inequalities defining $\Omega^t_{\theta_j}(v)$ in (4.4) are lower bounds for v. Therefore, $\Omega^t_{\theta_j}(v') \supseteq \Omega^t_{\theta_j}(v)$ if $v' \ge v$. We can rewrite the expected payoff and use Fubini's theorem as follows:

$$
\begin{aligned}
U^t(v_j) &= U^t(0) + \int_{\hat{v}_{\theta_j}(0)}^{v_j^{\theta_j}} \int_{\Omega^t_{\theta_j}(v_j^{\theta_j})} \mathbf{1}_{\{\eta_{-j}\in\Omega^t_{\theta_j}(v)\}}\, dG^t(\eta_{-j} \mid h_t^{-j})\, dv, \\
&= U^t(0) + \int_{\Omega^t_{\theta_j}(v_j^{\theta_j})} \int_{\hat{v}_{\theta_j}(0)}^{v_j^{\theta_j}} \mathbf{1}_{\{\eta_{-j}\in\Omega^t_{\theta_j}(v)\}}\, dv\, dG^t(\eta_{-j} \mid h_t^{-j}).
\end{aligned}
$$

Finally, we define

$$
\underline{v}^\tau(\theta_j,\eta_{-j}) := \inf\Big\{ v \;\Big|\; \eta_{-j} \in \Omega^\tau_{\theta_j}(v) \Big\}. \tag{4.5}
$$

$\underline{v}^\tau(\theta_j,\eta_{-j})$ is the *critical type* of the winning bidder. It is the lowest valuation $v_j^{\theta_j}$, that ensures that j wins against η_{-j}. Using this, we can rewrite the expected payoff to get

$$
\begin{aligned}
U^t(v_j) &= U^t(0) + \int_{\Omega^t_{\theta_j}(v_j^{\theta_j})} v_j^{\theta_j} - \underline{v}^t(\theta_j,\eta_{-j})\, dG^t(\eta_{-j} \mid h_t^{-j}) \\
&= U^t(0) + \underbrace{\int_{\Omega^t_{\theta_j}(v_j^{\theta_j})} v_j^{\theta_j}\, dG^t(\eta_{-j} \mid h_t^{-j})}_{v_j^{\theta_j}\cdot q_{\theta_j}(v_j)} - \underbrace{\int_{\Omega^t_{\theta_j}(v_j^{\theta_j})} \underline{v}^t(\theta_j,\eta_{-j})\, dG^t(\eta_{-j} \mid h_t^{-j})}_{p^t(v_j)}
\end{aligned} \tag{4.6}
$$

The last line shows that the expected payment is the integral over the critical type. The domain of integration is restricted to the set of profiles of other bidders for which j wins the object. Therefore, a payment rule that requires no payment from losing bidders

and a payment amounting to the critical type from the winner, implements the efficient allocation rule in periodic ex-post equilibrium. Obviously, with this definition, it is ensured that buyer j has to pay a positive amount only if she gets the object. Furthermore, the payoff of bidder j is non-negative because her valuation $v_j^{\theta_j}$ is greater than or equal to the critical value if she wins the auction. As the inequalities in (4.4) only depend on information available in period θ_j, the payment can be determined at the same time as the allocation. To summarize, we can state

Theorem 6. *Let π be a payment rule that defines the payment for bidder j as follows: when she reports type v_j in period t, and the history of reports is h_T,*

$$\pi_j(h_T) := \begin{cases} \underline{v}^t(\theta_j, \eta_{-j}) & \text{if } \eta_{-j} \in \Omega^t_{\theta_j}(v_j^{\theta_j}) \\ 0 & \text{if } \eta_{-j} \notin \Omega^t_{\theta_j}(v_j^{\theta_j}) \end{cases}. \tag{4.7}$$

Then

(i) the mechanism (S, x^*, π) is periodic ex-post incentive compatible,
(ii) payments are zero for all bidders except the winning bidder, (properties (A) and (C))
(iii) payments are completely determined in the period when the object is allocated, (property (D))
(iv) and the mechanism is ex-post individually rational. (property (B))

Remarks:

(1) With these payments, $U^t(0) = 0$.
(2) The critical value $\underline{v}^t(\theta_j, \eta_{-j})$, can be computed by setting the inequalities in the definition of $\Omega^t_\theta(v)$ equal (see (4.4)) and solving each equality for v. The critical value is the maximum of these solutions. There is at least one condition that holds with equality at this maximum. If the last inequality is binding we have $\underline{v}^t(\theta_j, \eta_{-j}) = \hat{v}_{\theta_j}(\eta(I^{-j}_{\theta_j}))$: j has to pay the option value of retaining the object to period $\theta_j + 1$. If the penultimate equality is fulfilled we have $\underline{v}^t(\theta_j, \eta_{-j}) = \eta_{\theta_j}(I^{-j}_{\theta_j})$: j has to pay the second highest valuation for period θ_j. If one of the other equalities is fulfilled, the auction would have ended earlier than θ_j if j had not made her report. Suppose for example, that the inequality for $\tilde{\eta}_{t'}$ for some $t' \in \{t, \dots, \theta_j - 1\}$ bind. Then, without j's report, the auction would have stopped in period t'. Solving $\tilde{\eta}_{t'}(I^{-j}_{t'}) = \hat{v}_{t'}(0, \dots, v, 0, \dots, 0)$ for v yields $\underline{v}^t(\theta_j, \eta_{-j})$. The solution is lower than the highest valuation in period t'. Hence, the winner has to pay less that the valuation of the bidder who would have won without j in this case.
(3) As in the static Vickrey auction, payments of bidder j do not depend directly on her report.
(4) Payments are defined as a function of the history in the final period, h_T. We know from Corollary 4, that in each period, only the tentative winner has a positive probability to win the object in the future. This implies, that for all bidders except the tentative winner, payments are determined as zero immediately after they have made their reports. For a tentative winner, the payment is determined if she wins the object or if another bidder becomes tentative winner. In the latter case, the payment is zero, in the former case, it can be made at the same time as the allocation of the object. Therefore, all payments can be made online.

5. A Generalized Ascending Auction

Theorem 6 defines a direct mechanism that generalizes the static second price auction. The ascending auction can also be generalized to the dynamic setting. In this construction, one property of the efficient allocation rule is crucial. We can define an order on the type-space such that efficient allocation rule always selects the bidder that ranks highest in this order as tentative winner and allocates to her, if her potential winning period is reached.

Definition 7. For any bidder j with arrival period t, type $v_j \in S_t$, and potential winning period θ_j, we define the *comparison price* π_j^T by

$$v_j^{\theta_j} = \hat{v}_{\theta_j}(0, \ldots, 0, \pi_j^T).$$

The following lemma shows that comparison types can be used to identify the tentative winner in every state.

Lemma 8. *Let (h_t, a_t) be a state in which the object is still available ($a_t = 1$) and suppose that $j \in I_t$ is the tentative winner. Then*

$$j \in \arg\max_{i \in I_t} \pi_i^T.$$

Proof. By Corollary 4, it suffices to prove the Lemma for the case that I_t has only two elements. The general case follows by induction on the size of I_t. Therefore, let $I_t = \{i, j\}$ and suppose without loss of generality that $\theta_i \geq \theta_j$. For $t \geq T - 1$ the Lemma is trivial. So suppose the Lemma holds for all $\tau > t$. By Corollary 4, we only have to consider the case that $\theta_j = t$. If $\theta_i = \theta_j = t$, i is the tentative winner if and only if $v_i^t \geq v_j^t$ which is equivalent to $\pi_i^T \geq \pi_j^T$.[11] If $\theta_i > \theta_j$, buyer i is the tentative winner if

$$v_j^{\theta_j} \leq \hat{v}_{\theta_j}(0, \ldots, 0, v_i^{\theta_i}, 0, \ldots, 0). \tag{5.1}$$

By the induction hypothesis (induction over t), $\hat{v}_{\theta_j}(0, \ldots, 0, v_i^{\theta_i}, 0, \ldots, 0) = \hat{v}_{\theta_j}(0, \ldots, 0, \pi_i^T)$. To see this notice that in period $\theta_j + 1 (= t + 1)$, i is the tentative winner, if and only if i has the highest comparison type. In particular, the expected value of have i as tentative winner or having a tentative winner with potential winning period T and $v^T = \pi_i^T$ is the same. Therefore, $\hat{v}_{\theta_j}(0, \ldots, 0, v_i^{\theta_i}, 0, \ldots, 0) = \hat{v}_{\theta_j}(0, \ldots, 0, \pi_i^T)$. Using this and the definition of the comparison type (5.1) can be rewritten as

$$\hat{v}_{\theta_j}(0, \ldots, 0, \pi_j^T) \leq \hat{v}_{\theta_j}(0, \ldots, 0, \pi_i^T),$$

which implies the result. □

Intuitively, the expected value of the efficient allocation is the same if the tentative winner is j or if the tentative winner has potential winning period T with a valuation for period T equal to j's comparison type. Therefore, buyers types can be compared even if their potential winning periods are different.

We will now use this order of types to define a dynamic ascending auction. In each period t, there are $T - t + 1$ price clocks that show prices $\pi^t, \ldots, \pi^T$ for buying the object in periods $t, \ldots, T$. All prices π^τ, $\tau < T$ are linked to π^T by

$$\pi^\tau = \hat{v}_\tau(0, \ldots, 0, \pi^T). \tag{5.2}$$

[11] For simplicity, we ignore ties in the proof.

In the first period, π^T starts at zero. In all periods $t > 1$, π^T starts at the value where it stopped in period $t-1$. The other prices are set such that they satisfy (5.2). In each period, the auction has two phases, the clock phase and the buying phase. Before the clock phase, buyers can chose to become active. In the clock phase, π^T is raised continuously and the other prices are updated such that (5.2) is satisfied. Bidders are free to drop out at any time. A bidder who has dropped out cannot become active again. If all bidders but one have dropped out, or if all remaining bidders decide to drop out at the same time, the clock stops immediately. The remaining bidder, or a random bidder from the group of drop-outs if all remaining bidders dropped out simultaneously, enters the buying phase.[12] In the buying phase, she can either buy immediately for the current price π^t, or she can wait. In the former case, the auction ends with a sale, in the latter case, the auction proceeds to the next period.

A bidder j is said to bid *truthfully* in the dynamic ascending auction if she uses the following strategy:

- Before the clock phase of any period τ: become active if and only if $\exists s \geq \tau : v_j^s > \pi^s$.
- In the clock phase of any period τ: drop out if and only if $v_j^s \leq \pi^s$ for all $s \geq \tau$.
- In the buying phase of any period τ: Buy if and only if $\tau = \theta_j$.

Theorem 9. *(i) If all bidders bid truthfully, the outcome of the dynamic ascending auction coincides with the outcome of the dynamic Vickrey auction with probability one.*

(ii) Truthful bidding is a periodic ex-post equilibrium in the dynamic ascending auction.

Proof. See Appendix A. □

The main steps of the proof are as follows. Under truthful bidding, the ascending auction selects the buyer with the highest comparison price and allocates the object to him in his potential winning period. Hence, the allocation rule is the same as in the dynamic Vickrey auction. Next, we show that the price for the winning period, at which the last competing bidder drops out is equal to the winner's critical type. Therefore, the payment rule implemented under truthful bidding is also identical to the payment rule in the dynamic Vickrey auction.

To show that truth-telling is an equilibrium we rule out several deviations that lead to non-positive expected payoffs and show that the remaining strategies yield the same expected payoffs as certain reports in the dynamic Vickrey auction. Truthful bidding corresponds to a truthful report in the dynamic Vickrey auction. Incentive compatibility of the latter therefore implies that truthful bidding is a periodic ex-post equilibrium of the dynamic ascending auction.

6. Conclusion

This paper shows how the payments in a dynamic mechanism can be distributed over different states of the world such that (i) expected payments ensure incentive compatibility

[12]Here, for simplicity, we use a different tie-breaking rule than in the dynamic Vickrey auction. The probability that this affects the outcome is zero.

and (ii) non-winning bidders do not make or receive a transaction, ex-post participation constraints are satisfied and payments can be made online. The result is a generalized Vickrey auction in which the winning bidder pays her critical type, i.e., she pays the lowest valuation for the winning period that would suffice to win against the other bids. The crucial step in the construction of the payment rule was to show that for each type, there is a unique potential winning period. This reduces types to essentially one dimension. Furthermore, it was shown that the efficient allocation rule allows to define a tentative winner in each period. There is an order of the type-space and the tentative winner is the highest bidder in that order. The results have been used to generalize the ascending clock auction to the dynamic framework.

The model is restrictive in at least two ways. Firstly, the allocation of a single object is studied. The case of multiple objects is left for future research. However, if more than one object is at sale, it is possible to construct simple examples where bidders can win in different periods for different profiles of competing bidders' types. Therefore, future research will have to concentrate on the generalization of the weaker result of corollaries 4 and 5.

Secondly, more general allocation problems could be studied. In this case, as in the case of multiple objects, a reduction of types to essentially one dimension may not be possible. It should be noted, however, that Theorem 3 provides much more structure than is needed for the construction of payments in Section 4. It would suffice that for each type v_j, there exists a path from the origin to v_j such that the allocation to bidder j is monotonic along this path. Depending on the choice of these paths, the payment rule may look significantly different from Vickrey payments, but nevertheless it would be possible to define payment rules that require transfers only from winning bidders.

Dynamic revenue maximization is an important question that has not been studied extensively in models with private information about time preferences.[13] The results of the present paper do not characterize the allocation rule that maximizes revenue. They can, however, be generalized to other allocation rules that are the solution to a recursive dynamic program in which valuations are replaced by some increasing function of valuations. Given the dynamic structure of the model, it is possible that the revenue-maximizing allocation rule belongs to this class. In this case, the expected payments fixed by the allocation rule (via payoff equivalence), could be distributed over different states in the same way as in this paper.

Appendix A. Omitted Proofs

Proof of Theorem 3. The result is proven by induction. For $T = 1$ the result is trivial. Assume that the theorem is true for allocation problems with $T-1$ periods. The statement for T is shown in four steps.

Step 1: If a buyer $j \in I_1$ gets the object in period one, $v_j^1 > \hat{v}_1(\eta(I_1)) \geq \hat{v}_1(v_j)$. Therefore $\theta_j = 1$.

Step 2: If it is not efficient to allocate in period one, we can consider the allocation of the retained object in periods $\{2, \ldots, T\}$ as a new allocation problem with $T-1$ periods.

[13]See Pai and Vohra (2008) and Mierendorff (2009) for exceptions. See the references there for an overview of the literature on dynamic revenue maximization.

We only have to relabel $\tilde{I}_1 = I_2, \ldots, \tilde{I}_{T-1} = I_T$ and delete the first elements from the type vectors of bidders in I_1. The decisions in periods $\{2, \ldots, T\}$ of the original problem only depend on the buyers that are present in these periods and their valuations. Therefore, the identity of the winning buyer is the same in the new problem and the original problem. The time of allocation is shifted by one. By the induction hypothesis, his implies, that buyer j with potential winning period $\theta_j^{\text{new}} \in \{1, \ldots, T-1\}$ in the new problem, can only win in period one or period $\theta'_j = \theta_j^{\text{new}} + 1$ of the original problem. Furthermore, as θ_j^{new} is characterized by condition (3.1) with $\tau \in \{1, \ldots, \theta_j^{\text{new}} - 1\}$, θ'_j is characterized by (3.1) with $\tau \in \{2, \ldots, \theta'_j - 1\}$. It therefore remains to show that $v_j^1 \leq \hat{v}_1(v_j)$, if j wins in θ'_j in the original problem.

Step 3: There is only one buyer from I_1 that can win the object in the original problem if it is retained for period two. To see this, consider again the new problem with $T-1$ periods. Define $A = I_1$ and $B = \tilde{I}_1 \backslash I_1$. Then $\tilde{I}_1 = A \cup B$. Assume $B \neq \emptyset$. By Corollary 4.ii, there are elements $a \in A$ and $b \in B$ such that the tentative winner in period one of the new problem is in $\{a, b\}$. Hence, buyer $a \in A = I_1$ is the only buyer in I_1 that can win the object in the original problem if it is retained for period two. If $B = \emptyset$ the argument is trivial.

Step 4: If $j \in I_1$ gets the object in $\theta'_j \neq 1$, we must have $\eta_1(I_1) \leq \hat{v}_1(\eta(I_1))$. By step 3 we know that j is the only bidder in I_1 that can win in periods $\{2, \ldots, T\}$. Therefore the option value of retaining the object only depends on her valuation: $\hat{v}_1(\eta(I_1)) = \hat{v}_1(v_j)$. As $v_j^1 \leq \eta_1(I_1)$ we have $v_j^1 \leq \hat{v}_1(v_j)$ as desired. □

Proof of Theorem 9. (i) If all bidders bid truthfully a bidder j drops out if $\pi^T = \pi_j^T$. Therefore, in each period, a buyer with the highest comparison price enters the buying phase. As buyers buy in their potential winning periods, and only in this period, if they bid truthfully, the allocation coincides with the efficient allocation rule of the dynamic Vickrey auction, except for the case of ties, that occur with zero probability.

Now suppose that bidder j arrives in period t and wins the object in period θ_j. We show that the price π^{θ_j} at which the last competing bidder dropped out equals the critical type of bidder j. For each $i \in I_{\theta_j}$, define $\pi_i^{\theta_j} := \hat{v}_{\theta_j}(0, \ldots, 0, \pi_i^T)$. Then, j has to pay

$$\max_{i \in I_{\theta_j}} \pi_i^{\theta_j} = \max \left\{ \max_{i \in I_{\theta_j}, \theta_i < \theta_j} \pi_i^{\theta_j}, \max_{i \in I_{\theta_j}, \theta_i = \theta_j} \pi_i^{\theta_j}, \max_{i \in I_{\theta_j}, \theta_i > \theta_j} \pi_i^{\theta_j} \right\}. \tag{A.1}$$

If $\theta_i < \theta_j$, then for all $\tau < \theta_j$, $\hat{v}_\tau(0, \ldots, 0, \pi_i^{\theta_j}, 0, \ldots, 0) = \hat{v}_\tau(0, \ldots, 0, \pi_i^T) \geq v_i^\tau$. Hence,

$$\begin{aligned} \max_{i \in I_{\theta_j}, \theta_i < \theta_j} \pi_i^{\theta_j} \quad = \quad \max \Big\{ v^{\theta_j} \Big| \exists \tau \in \{t, \ldots, \theta_j - 1\}, i \in I_\tau : \theta_i < \theta_j \ . \\ \text{and } v_i^\tau = \hat{v}_\tau(0, \ldots, 0, v^{\theta_j}, 0, \ldots, 0) \Big\} \end{aligned} \tag{A.2}$$

If $\theta_j = \theta_i$, then $\pi_i^{\theta_j} = \hat{v}_{\theta_j}(0, \ldots, 0, \pi_i^T) = \hat{v}_{\theta_i}(0, \ldots, 0, \pi_i^T) = v_i^{\theta_i} = v_i^{\theta_j}$. Hence

$$\max_{i \in I_{\theta_j}, \theta_i = \theta_j} \pi_i^{\theta_j} = \max_{i \in I_{\theta_j}, \theta_i = \theta_j} v_i^{\theta_j}. \tag{A.3}$$

Finally if $\theta_i > \theta_j$, then $\pi_i^{\theta_j} = \hat{v}_{\theta_j}(0,\dots,0,\pi_i^T) = \hat{v}_{\theta_j}(0,\dots,0,v_i^{\theta_i},0,\dots,0)$. Hence

$$\max_{i\in I_{\theta_j},\theta_i>\theta_j} \pi_i^{\theta_j} = \max_{i\in I_{\theta_j},\theta_i>\theta_j} \hat{v}_{\theta_j}(v_i^{\theta_j},\dots,v_i^T). \tag{A.4}$$

We now compare (A.2)–(A.4) to the values defining the critical type of bidder j from (4.4). Define ξ_1 as the minimal value of v that satisfies all but the last two inequalities in (4.4), ξ_2 as the infimal value of v that satisfies the second-last inequality in (4.4), and ξ_3 as the infimal value of v that satisfies the last inequality in (4.4). In general ξ_1 is greater or equal than (A.2), ξ_2 is greater or equal than (A.3) and ξ_3 is greater or equal than (A.4) as the maximizations in (A.2)–(A.4) are restricted to the sets of bidders with $\theta_i < \theta_j$, $\theta_i = \theta_j$ and $\theta_i > \theta_j$, respectively. Now suppose that the last bidder i who dropped out before j won the auction has $\theta_i < \theta_j$. Then the price clock for period θ_i stopped at $\pi^{\theta_i} = v_i^{\theta_i}$. Let π^{θ_j} be the value at which the price clock for period θ_j stopped. As i is the last drop-out, $v_i^{\theta_i} = \eta_{\theta_i}(I_{\theta_i}^{-j}) = \hat{v}_{\theta_i}(0,\dots,0,\pi^{\theta_j},0,\dots,0)$ and $\eta_\tau(I_\tau^{-j}) \le \hat{v}_\tau(0,\dots,0,\pi^{\theta_i},0,\dots 0)$ for all $\tau = t,\dots,\theta_j - 1$. Therefore, ξ_1 satisfies

$$v_i^{\theta_i} = \hat{v}_{\theta_i}(\underbrace{0,\dots,0}_{t,\dots,\theta_j-1},\xi_1,\underbrace{0,\dots,0}_{\theta_j+1,\dots,T}) = \hat{v}_{\theta_i}(0,\dots,0,\pi^{\theta_i},0,\dots,0).$$

This implies $\xi_1 = \pi_i^{\theta_j}$. If $\theta_i = \theta_j$, then i dropped out at $\pi^{\theta_j} = v_i^{\theta_j}$. This implies $\xi_2 = \pi_i^{\theta_j}$. If $\theta_i > \theta_j$, then i dropped out if $\pi^{\theta_i} = v_i^{\theta_i}$. Therefore $\pi_i^{\theta_j} = \hat{v}_{\theta_j}(0,\dots,0,v_i^{\theta_i},0,\dots,0) = \xi_3$. In summary this implies $\max_{i\in I_{\theta_j}} \pi_i^{\theta_i} = \underline{v}^t(\theta_j,\eta_{-j})$.

(ii) Suppose that for a dynamic ascending auction of length $T-1$, truthful bidding is an ex-post equilibrium. For length one this is trivial. We show by induction that the claim is also true for T periods.

Consider bidder $j \in I_1$ and suppose that all other bidders bid truthfully. If the auction reaches period two, and bidder j has not dropped out in the first period, truthful bidding is optimal for j by hypothesis.

If j enters the buying phase in period one, we have to distinguish two cases.
Case 1: $\pi^T \ge \pi_j^T$. In this case, $v_j^t \le \pi^t$ for all $t = 1,\dots,T$. Therefore j's expected utility is non-positive regardless of the continuation strategy.
Case 2: $\pi^T < \pi_j^T$. In this case, (i) implies that buying immediately yields a payoff equal to $U(v,(v_j^1,0,\dots,0),\tilde{h}_1^{-j})$ and not buying followed by truthful bidding yields a payoff equal to $U(v,(0,v_j^2,\dots,v_j^T),\tilde{h}_1^{-j})$. $\tilde{h}_1^{-j} = h_1^{-j} \cup \{(\pi^1,\dots,\pi^T)\}$ denotes the history of types of the other bidders with the addition of an artificial bidder that has valuations equal to the prices at which the clock stopped when j entered the buying phase. $U(v,v',h)$ denotes the expected payoff from participating in the dynamic Vickrey auction. If $\theta_j = 1$

$$U(v_j,(v_j^1,0,\dots,0),\tilde{h}_1^{-j}) = U(v_j,\tilde{h}_1^{-j}) \ge U(v_j,(0,v_j^2,\dots,v_j^T),\tilde{h}_1^{-j}),$$

where the inequality follows from periodic ex-post incentive compatibility of the dynamic Vickrey auction. Similarly, if $\theta_j > 1$

$$U(v_j,(0,v_j^2,\dots,v_j^T),\tilde{h}_1^{-j}) = U(v_j,\tilde{h}_1^{-j}) \ge U(v_j,(v_j^1,0,\dots,0),\tilde{h}_1^{-j}).$$

This show that it is optimal to buy according to the truthful bidding strategy.

Finally, consider the clock phase. If $\pi^T \geq \pi_j^T$, remaining active yields a payoff of at most zero as shown before, therefore it is optimal to drop out immediately. If $\pi^T < \pi_j^T$, continuing by truthful bidding yields $U(v, \tilde{h}_1^{-j}) \geq 0$. Hence it is optimal to bid truthfully in the first period. □

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