Dirk Bergemann
Department of Economics
Yale University

Economics 121b: Intermediate Microeconomics
Problem Set 3:
Welfare and Taxation
2/13/12

This problem set is due on Monday, 2/20/12, in class. To receive full credit, provide a complete defense of your answer.

1. **Compensating Variation**. In class we discussed commodity taxation versus lump sum taxation (using the notion of compensating variation). Suppose the individual has income $I$ and to obtain the utility level $U$ and prices $p_x, p_y$ the expenditures necessary are given by

$$E(p_x, p_y, U).$$

We fix the price $p_y$ of commodity $y$ and vary the price of commodity $x$ by commodity taxation. The tax rate is $t_x > 0$ and hence the new price is

$$P_x = (1 + t_x) p_x.$$

   (a) **Commodity Taxation.** Using the notion of compensating variation to describe analytically and graphically how much more the consumer needs to expend to reach the utility level $U$ at the new prices $(P_x, p_y)$. (Use the graph with $p_x$ on the $x-$axis and $h_x(p_x, p_y, U)$ on the $y-$axis). Describe analytically and graphically how much of the additional expenditure is captured by the tax revenue $R = t_x p_x h_x(p_x, p_y, U)$.

   (b) **Lump Sum Taxation.** If the price for commodity $x$ where instead reduced from $P_x$ to $p_x$, by how much would the necessary expenditure of the consumer be reduced, use again the notion of compensating variation to describe analytically and graphically your result. Call the difference $T = E(P_x, p_y, U) - E(p_x, p_y, U)$, and let this be the lump sum tax that the government could raise from the consumer.

   (c) **Comparison.** Now argue that the consumer has as much total expenditure in the lump sum tax system than in the commodity tax system, and that he reaches exactly the same utility, but that $T - R > 0$, i.e. the lump sum taxes are a more efficient way to raise tax revenue. Use again the notion of compensating variation to describe analytically and graphically your result and in particular $T - R$.

2. There are two goods, food and clothing, whose quantities are denoted by $x$ and $y$ and prices by $p_x$ and $p_y$, respectively. There is a consumer whose income is denoted by $I$ and utility by $U$. His utility function is

$$u(x, y) = x^{1/2} y^{1/2}.$$

   (a) Find his Hicksian (compensated) and Marshallian (uncompensated) demand functions. Find the expenditure function and the indirect utility function.

   (b) Initially $I = 100$, $p_x = 1$, and $p_y = 1$. What quantities does the consumer buy, and what is his resulting utility?

   (c) Now the price of food rises to $p_x = 1.21$, while income and the price of clothing are as before. What quantities does the consumer buy and what is his resulting utility?

   (d) Suppose the increase in the price of food was caused by the government levying a tax of 21 percent on food. What is the government's revenue from this tax?

   (e) If the government wants to compensate the consumer by giving him some extra income, how much extra income would be needed to restore him to the old utility level (i.e., the "Hicksian compensation" or "compensation variation" for the price change)? Is the government's revenue from the tax on food itself sufficient to provide this compensation? What is the economic intuition for your answer?

   (f) If the government tries to compensate the consumer by giving him enough extra income to enable him to purchase the same quantities as he did at the original income and prices of part (b), how much extra income would the government have to give him (the "Slutsky" compensation)? With this income and the new prices, what quantities will the consumer actually buy? What will be his resulting utility?

   (g) In a graph with $x$ on the horizontal axis and $p_x$ on the vertical axis, show the consumer's choices with the income $I = 100$ and the two prices $p_x = 1$ and $p_x = 1.21$. Calculate the reduction of "consumer surplus" measured by the area under the Marshallian demand curve. Compare this to the Hicksian compensation (your answer to part (e)). If the two are different, explain the difference.

3. **Expenditure Function.** We defined the expenditure function through the expenditure minimization problem:

$$\min \sum_{i=1}^{n} p_i x_i$$

subject to

$$u(x_1, ..., x_n) \geq U$$

for some fixed utility level $U > 0$. The optimal solution to the expenditure minimization problem delivered the compensated (or Hicksian) demands:

$$h^*(p, U) = (h_1^*(p, U), ...h_n^*(p, U))$$

where $p$ is the price vector $p = (p_1, ..., p_n)$ and the associated expenditure function

$$E(p, U) = \sum_{i=1}^{n} p_i h_i^*(p, U).$$

(a) State the expenditure minimization problem as a constrained optimization problem (using the Lagrange multiplier.)

(b) Using the envelope theorem carefully show and argue that

$$\frac{\partial E(p, U)}{p_i} = h_i^*(p, U). \tag{1}$$

(c) For simplicity, consider now an environment with two goods only, 1 and 2. Suppose the price of good 1 is held constant and we consider how the expenditure changes with increases in the price of good 2.

i. Draw an expenditure function that is (strictly) concave in the price of good 2, holding the price $p_1$ constant.

ii. In particular, pick three (arbitrary price levels) $0 < p_2 < p_2' < p_2''$ and draw the tangent line to the expenditure function at $p_2, p_2'$, and $p_2''$. Argue that the tangent line represent the behavior of the expenditure function around $p_2, p_2'$, and $p_2''$, if there were no substitution at all in the presence of the increasing price.

(d) Now remember the definition of a concave function, i.e. a function $f$ is concave if for all $x, x' \in X$ and all $\lambda \in (0, 1)$,

$$f(\lambda x + (1-\lambda) x') \geq \lambda f(x) + (1-\lambda) f(x')$$

and argue that every expenditure function is concave. Hint: Fix a utility level $U$ and let $p'' = \lambda p + (1-\lambda) p'$ be the price vector that comes about by the convex combination of $p$ and $p'$ and let $h''$ be the optimal demand under $p''$ and $U$. Now you have to argue that:

$$E(p'', U) \geq \lambda E(p, U) + (1-\lambda) E(p', U),$$

and since

$$E(p'', U) = h'' \cdot p''$$

all you have to show is that the following inequality holds:

$$h'' \cdot p'' \geq \lambda E(p, U) + (1-\lambda) E(p', U).$$

(e) Now, take the concavity of the expenditure function given and hence that

$$\frac{\partial^2 E(p, U)}{\partial p_i^2} \leq 0. \tag{2}$$

Use the concavity, (2) to argue that the own price effect of the compensated demand is always negative (the "law of demand")

$$\frac{\partial h_i(p, U)}{\partial p_i} \leq 0.$$