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Economics 121b: Intermediate Microeconomics
Problem Set 2:
Demand Functions
1/30/12

This problem set is due on Monday, 2/6/12, in class. To receive full credit, provide a complete defense of your answer.

1. In class we defined the notion of a concave function, in particular a concave utility function. We said that a function $f : \mathbb{R}^n \to \mathbb{R}$ is concave if for all $x, x' \in \mathbb{R}^n$, and all $\lambda \in (0,1)$:
$$f\left(\lambda x + (1-\lambda)\, x'\right) \geq \lambda f(x) + (1-\lambda)\, f\left(x'\right). \tag{0.1}$$
   1. Graphically represent the concavity condition (0.1) by carefully introducing and labelling all the relevant information for $x, x' \in \mathbb{R}_+$ (i.e. x is a one dimensional scalar, and not a many-dimensional vector).
   2. Remember that if f is differentiable, then the above definition of concavity, the inequality (0.1) is equivalent to requiring that $f''(x) \leq 0$. Briefly explain and illustrate why a decision maker with a concave utility function therefore has *diminishing marginal utility.*
   3. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconcave if for all $x, x' \in \mathbb{R}^n$, and for all $\lambda \in (0,1)$:
   $$f\left(\lambda x + (1-\lambda)\, x'\right) \geq \min\left\{f(x), f\left(x'\right)\right\}. \tag{0.2}$$
   Graphically represent the quasiconcavity condition (0.2) by carefully introducing and labelling all the relevant information for $x, x' \in \mathbb{R}_+$ (i.e. x is a one dimensional scalar, and not a many-dimensional vector).
   4. By using the condition (0.1) and (0.2), show that a function that is quasiconcave is also concave. By a graphical illustration, give an example of a quasiconcave function that is not a concave function.

2. An equivalent description of a quasiconcave function is the following definition: $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconcave if for all y, the upper contour set U, defined as follows
$$U(y) = \{x \in X \,|\, f(x) \geq y\} \tag{0.3}$$
is a convex set.

1. Recall a set S is convex if $x, x' \in S$, then for all $\lambda \in (0,1)$, $\lambda x + (1-\lambda)\, x' \in S$. Draw a set S in $\mathbb{R}^2$ that is convex and one that is not convex.
2. Graphically represent the quasiconcavity condition (0.3) by carefully introducing and labelling all the relevant information for $x, x' \in \mathbb{R}_+$ (i.e. x is a one dimensional scalar, and not a many-dimensional vector).
3. We said that a consumer has convex preferences if $x \sim y$ and $x \neq y$, then $\lambda x + (1-\lambda)\, y \succsim x$ and $\lambda x + (1-\lambda)\, y \succsim y$ for all x, y and $\lambda \in (0,1)$ (i.e. a preference for moderation). Now using, the language of (0.2) and/or (0.3), show that if a utility function represents an agent with convex preferences, then it must be a quasiconcave utility function.

3. A consumer's preferences are represented by the following utility function:

$$u(x, y) = lnx + 2lny.$$

   1. Which of the two bundles $(x_A, y_A) = (1, 4)$ or $(x_B, y_B) = (4, 1)$ does the consumer prefer?
   2. Take as given for now that this utility function represents a consumer with convex preferences. Use this information and your answer to part (a) to determine which of the two bundles $(x_C, y_C) = (2.5, 2.5)$ or $(x_B, y_B) = (4, 1)$ the consumer prefers. Verify your answer.
   3. Derive an equation for the indifference through the bundle $(x_B, y_B) = (4, 1)$.
   4. Derive an equation for the marginal rate of substitution between x and y for an individual with these preferences. Interpret the MRS. What is the MRS at the point $(x_C, y_C) = (2.5, 2.5)$?

4. Consider the following function

$$f(x) = a - (x - b)^2 \tag{0.4}$$

with $a, b > 0$. Graphically display the function, label a and b on the respective axis. Solve analytically for the unconstrained maximum of this function. In particular, verify the second order conditions.

5. Consider the same function (0.4), and suppose now that the choice of the optimal x is constrained by $x \leq \overline{x}$, where $\overline{x}$ is an arbitrary number, satisfying

$$0 < \overline{x} < b. \tag{0.5}$$

   1. Graphically display the nature of the new constraint relative to the optimization problem and the function f.

2. Solve via the Lagrangian method for the optimal solution.
3. **Lagrange multiplier.** Suppose that we increase the bound $\overline{x}$ by dx. What is the marginal effect this has on the value of the function to be optimized in terms of a and b. How does it compare to the value of the Lagrangian multiplier which you just computed.
4. **Interpretation of Lagrangian multiplier.** Imagine now that we cannot insist that the decision maker respect the constraint but that we can ask a penalty, say λ, for every unit of x over and above $\overline{x}$. What is the price that we would have to charge so that the decision maker would just be happy to exactly choose $x^* = \overline{x}$ and thus in fact respect the constraint even so did not face the constraint directly.

   In the process we have replaced the strict constraint with a price for the constraint (equal to the Lagrange multiplier). At the optimum the price is equal to the marginal value of the objective function. For this reason we refer to it as the *shadow price* of the constraint. It is the smallest price we can associate to the constraint so that the decision maker, facing this price would respect the constraint.

6. **Comparative Statics.** Consider the same function (0.4) and constraint (0.5) as in (5). The value of the function f at the optimal solution x^* is given by $y^* = f(x^*)$. Suppose now that we are interested as to how the value y^* of the problem depends on the exogenous variables of the problem, a or b. For concreteness, let us focus on b and we could like to know how $y^*(b)$ varies with b, in other words we are interested in

$$\frac{dy^*(b)}{db}.$$

   For the problem given by (0.4) and constraint (0.5), compute $dy^*(b)/db$ and relate your result to the implicit function theorem and the envelope theorem.

7. Consider the following utility function (often referred to as quasi-linear utility function as it is linear in the second element):

$$u(x,y) = \ln(x) + y;$$

   with prices and income given by: $p_x = 1, p_y \in \mathbb{R}_+$ and $I \in \mathbb{R}_+$.

   1. For special case of $p_y = 2$ and $I = 1$, show analytically and graphically that there is no positive consumption choice $(x^*, y^*) \geqq 0$ which satisfies the tangency condition

$$\frac{\frac{\partial u(x,y)}{\partial x}}{\frac{\partial u(x,y)}{\partial y}} = \frac{p_x}{p_y}.$$

Briefly describe where the failure of the tangency condition stems from and what does it mean.

2. If we insist that $(x^*, y^*) \geqq 0$, then describe, analytically and graphically, the optimal consumption choice for $p_y = 2$ and $I = 1$.

3. Now solve by the method of substitution (using $p_x x + p_y y = I$) and do not insist on positive values of consumption of either x or y.

   1. For which values of p_y and I do you get a nonsensical solution in the sense that $y^* < 0$. (Can you give an economic interpretation where y^* would actually make some sense ?) How is this related to the fact that the method by substitution only required:

   $$p_1 x_1 + p_2 x_2 = I \text{ ?}$$

   2. Now consider the Lagrangean of the following optimization problem:

   $$\max_{x,y} \{\ln(x) + y\}$$

   subject to the following constraints:

   $$\begin{aligned} I - p_x x - p_y y &\geq 0 \quad (\lambda_I) \\ x &\geq 0 \quad (\lambda_x) \\ y &\geq 0 \quad (\lambda_y) \end{aligned}$$

   Notice that we now introduce new inequality constraints, namely that the consumption choices are nonnegative. The Lagrangean then formally becomes

   $$L(x, y, \lambda_I, \lambda_x, \lambda_y) = \ln(x) + y + \lambda_I \cdot (I - p_x x - p_y y) + \lambda_x \cdot x + \lambda_y \cdot y.$$

   Show that the optimality condition of the Lagrangean, first order conditions, and complementary slackness condition, where the later ones are written as:

   $$\begin{aligned} \lambda_I^* \cdot (I - p_x x^* - p_y y^*) &= 0 \\ \lambda_x^* \cdot x^* &= 0 \\ \lambda_y^* \cdot y^* &= 0 \end{aligned}$$

   are either of the form:

   $$\begin{aligned} \lambda_I^* &> 0, \ (I - p_x x^* - p_y y^*) = 0 \\ \lambda_x^* &= 0, \ x^* > 0 \\ \lambda_y^* &= 0, \ y^* > 0, \end{aligned}$$

or

$$
\begin{aligned}
\lambda_I^* &> 0,\ (I - p_x x^* - p_y y^*) = 0 \\
\lambda_x^* &= 0,\ x^* > 0 \\
\lambda_y^* &> 0,\ y^* = 0.
\end{aligned}
\tag{0.6}
$$

3. Show that the later set of conditions, (0.6), hold if and only if the method of substitution would have given you $y^* < 0$ as an answer. What is a possible interpretation of λ_y^* in this case ?
4. Assume $p_x = 2$ and $p_y = 1$ but I varies. Find the income expansion path, i.e. compute how the optimal consumption (x, y) would change as the income increases and graph these curves.

4. Compute the own and cross price elasticity for the two goods at $p_y = 2$ and $I = 10$.