Economics 121b: Intermediate Microeconomics
Problem Set 4

1. **Edgeworth Box and Pareto Efficiency** Consider the island economy with Friday and Robinson. They have agreed to share their resources and they have also agreed that the weight that Friday receives in the economy is $w^f \in (0,1)$ and the weight that Robinson receives is $w^r = 1 - w^f$. Their preferences are different. Friday likes bananas more than coconuts, and Robinson likes coconuts more than bananas (i.e. you can assume that $\alpha \in (0.5), 1))$. The utility functions are given by:

$$u^f(x^f) = \alpha \ln x_b^f + (1-\alpha)\ln x_c^f,$$

and

$$u^r(x^r) = (1-\alpha)\ln x_b^r + \alpha \ln x_c^r.$$

The endowment of the island is given by

$$e_b = 2, e_c = 2c \geq 2.$$

(a) For every weight $w_f \in (0,1)$, find the allocation which would maximize the social surplus given the weights; in other words, we are interested in finding the allocation $\left(x_b^f, x_b^r, x_c^f, x_c^r\right)$ which maximizes the sum

$$w_f u_f(x_f) + (1-w_f)\, u_r(x_r)$$

subject to the resource constraints of the economy.

<u>Solution</u>

The maximization problem is

$$\begin{aligned} \max_{x_b^f, x_b^r, x_c^f, x_c^r} \quad & w_f\left(\alpha \ln x_b^f + (1-\alpha)\ln x_c^f\right) + (1-w_f)\left((1-\alpha)\ln x_b^r + \alpha \ln x_c^r\right) \\ \text{s. t.} \quad & x_b^f + x_b^r = 2 \\ & x_c^f + x_c^r = 2c \end{aligned}$$

The first order conditions are (after eliminating the Lagrange multipliers)

$$\begin{aligned} w_f \alpha x_b r &= (1-w_f)(1-\alpha)x_b^f \\ w_f(1-\alpha)x_c^r &= (1-w_f)\alpha x_c^f \\ x_b^f + x_b^r &= 2 \\ x_c^f + x_c^r &= 2c \end{aligned}$$

Solve this system of equations to obtain

$$x_b^r = \frac{2(1-w_f)(1-\alpha)}{w_f\alpha + (1-w_f)(1-\alpha)} \quad (1)$$

$$x_b^f = \frac{2w_f\alpha}{w_f\alpha + (1-w_f)(1-\alpha)} \quad (2)$$

$$x_c^r = \frac{2c(1-w_f)\alpha}{w_f(1-\alpha) + (1-w_f)\alpha} \quad (3)$$

$$x_c^f = \frac{2cw_f(1-\alpha)}{w_f(1-\alpha) + (1-w_f)\alpha} \quad (4)$$

(b) For every weight $w_f \in (0,1)$, can you find an initial endowment of bananas and coconuts among Robinson and Friday and a pair of prices that the Pareto efficient allocation actually constitutes an equilibrium of the market. (Here we decentralize the Pareto efficient allocation via a market equilibrium.) It is sufficient to discuss the case of $c = 1$.

Solution

For every weight $w_f \in (0,1)$, let the initial endowment be given by the system of pareto efficient allocations in (1)-(4). Normalize the price of bananas to 1, i.e. $p_b = 1$. We need to find the price of coconuts that together with endowments given by (1)-(4) will constitute a competitive equilibrium.

Since (1)-(4) is a Pareto efficient allocation, the marginal rate of substitution between coconuts and bananas are the same for Robinson and Friday (verify by yourself that it is actually true). Next, recall that at a competitive equilibrium MRS equals the price ratio. We will use that property to recover the price for coconuts.

$$\begin{aligned}
MRS_{cb}(x_b^{r*}, x_c^{r*}) &= \frac{p_c}{p_b} \\
\frac{\alpha x_b^{r*}}{(1-\alpha)x_c^{r*}} &= p_c \\
\frac{\alpha \frac{2(1-w_f)(1-\alpha)}{w_f\alpha+(1-w_f)(1-\alpha)}}{(1-\alpha)\frac{2c(1-w_f)\alpha}{w_f(1-\alpha)+(1-w_f)\alpha}} &= p_c \\
\frac{w_f(1-\alpha)+(1-w_f)\alpha}{c(w_f\alpha+(1-w_f)(1-\alpha))} &= p_c
\end{aligned}$$

Thus, prices $(p_b, p_c) = \left(1, \frac{w_f(1-\alpha)+(1-w_f)\alpha}{c(w_f\alpha+(1-w_f)(1-\alpha))}\right)$ and the allocation given by (1)-(4) constitute a competitive equilibrium in an economy where initial endowments are given by (1)-(4). It will be just a simple economy where everybody consumes their own endowment at the equilibrium prices.

2. **Edgeworth Box and Competitive Equilibrium.** Consider again Robinson and Friday. Now, Robinson and Friday have the same endowment of bananas and coconuts:

$$\begin{aligned} e_b^f &= e_b^r = 1 \\ e_c^f &= e_c^r = c > 1 \end{aligned}$$

but their preferences are different.

(a) We normalize the price of bananas to be equal to one, or $p_b = 1$. Compute the demand function of Robinson and Friday as a function of the price for coconuts p_c.

Solution

Utility maximization problem for Robinson is

$$\max_{x_b^r, x_c^r} (1-\alpha) \ln x_b^r + \alpha \ln x_c^r$$
$$\text{s. t. } x_b^r + p_c x_c^r = 1 + p_c c$$

The first order conditions are

$$\begin{aligned} \frac{(1-\alpha) x_c^r}{\alpha x_b^r} &= \frac{1}{p_c} \\ x_b^r + p_c x_c^r &= 1 + p_c c \end{aligned}$$

Solve the two equations for Robinson's demand for coconuts and bananas

$$\begin{aligned} x_b^r &= (1-\alpha)(1 + p_c c) \\ x_c^r &= \alpha \frac{1 + p_c c}{p_c} \end{aligned}$$

By the symmetry of the preferences Friday's demand is

$$\begin{aligned} x_b^f &= \alpha (1 + p_c c) \\ x_c^f &= (1-\alpha) \frac{1 + p_c c}{p_c} \end{aligned}$$

(b) Find the equilibrium price of this island economy and determine the net trading quantities.

i. Start with the case of $c = 1$.
Solution
Consider the market clearing condition for bananas

$$\begin{aligned}
x_b^f + x_b^r &= 1 + 1 \\
\alpha(1 + p_c c) + (1 - \alpha)(1 + p_c c) &= 2 \\
1 + p_c &= 2 \\
p_c &= 1
\end{aligned}$$

At the equilibrium price the consumption of Robinson and Friday is

$$\begin{aligned}
x_b^r &= 2(1 - \alpha) \\
x_c^r &= 2\alpha \\
x_b^f &= 2\alpha \\
x_c^f &= 2(1 - \alpha)
\end{aligned}$$

The net trading quantities are

$$\begin{aligned}
e_b^r - x_b^r &= 2\alpha - 1 \\
e_c^r - x_c^r &= 1 - 2\alpha \\
e_b^f - x_b^f &= 1 - 2\alpha \\
e_c^f - x_c^f &= 2\alpha - 1
\end{aligned}$$

ii. *Continue to describe the equilibrium for general $c > 1$ and describe the intuition.*
Consider the market clearing condition for bananas

$$\begin{aligned}
x_b^f + x_b^r &= 1 + 1 \\
\alpha(1 + p_c c) + (1 - \alpha)(1 + p_c c) &= 2 \\
1 + p_c c &= 2 \\
p_c &= \frac{1}{c}
\end{aligned}$$

Observe that the price of coconuts is declining in the in the number of coconuts available in the economy representing the idea that as a good becomes less scares its price declines.
The net trading quantities are

$$
\begin{aligned}
e_b^r - x_b^r &= 2\alpha - 1 \\
e_c^r - x_c^r &= c(1 - 2\alpha) \\
e_b^f - x_b^f &= 1 - 2\alpha \\
e_c^f - x_c^f &= c(2\alpha - 1)
\end{aligned}
$$

Notice that the trade and consumption of bananas is independent of the number of coconuts in the economy. The intuition is that each person spends a constant fraction of his income on coconuts, and that each person controls a constant fraction of the world coconut supply, which means that each person's proceeds from selling his endowment of coconuts is a constant fraction of his total proceeds from selling his endowment. But this means that each person controls a constant fraction of total world income, and since each person spends a constant fraction of income on bananas, world demand for bananas is constant. Since supply of bananas is also constant, trade and consumption of bananas do not change with the number of coconuts in the economy.