Dirk Bergemann
Department of Economics
Yale University

Microeconomic Theory (501b)

Problem Set 9. Auctions

4/8/14

This problem set is due on Tuesday, 4/15/14.

1. **All Pay Auction. Complete Information.**The rules of the all pay auction are: (i) the highest bid receives the object, (ii) each bidder pays his bid, independent of whether he wins or loses the object. If two bidders offer exactly the same bid, then each bidder receives the object with equal probability

   (a) Consider the complete information version of the all pay auction. There are two bidders, and each bidder values the object at $v > 0$.

      i. Argue first that there cannot be a pure strategy equilibrium in this game.

      ii. Argue next that there cannot be a mixed strategy equilibrium where any specific bid receives a strictly positive probability.

      iii. Argue next that the support of the equilibrium bidding strategy must form an interval, that is, it is convex and does not display any gaps.

      iv. Now compute the mixed strategy Nash equilibrium of the all pay auction and argue that the expected net utility has to be equal to zero.

   (b) Consider the complete information version of the all pay auction. There are two bidders, but now each bidder values the object differently, namely $v_1 > v_2 > 0$. Compute the mixed strategy Nash equilibrium of the all pay auction.

      i. Verify that the losing bidder may now place an atom, a positive probability on the lower bound, of the support of his strategy.

      ii. Now compute the mixed strategy Nash equilibrium of the all pay auction and identify the expected net utility of each bidder.

      iii. Is the resulting equilibrium leading to an efficient allocation of the object.

2. **All Pay Auction. Incomplete Information.** The rules of the all pay auction are: (i) the highest bid receives the object, (ii) each bidder pays his bid, independent of whether he wins or loses the object.

(a) Characterize the equilibrium of the all-pay auction in the symmetric environment with a *uniform distribution on the unit interval* for $I = 2$ bidders. (Hint: Guess that the equilibrium bidding function is an increasing and quadratic function.)

(b) Characterize the equilibrium of the all-pay auction in the symmetric environment with a *continuously differentiable distribution function.* (You may either proceed similarly to the method we proceeded in class and/or directly the assume the validity of the revenue equivalence theorem.)

3. **First Price Auction.** Consider the first price auction in a symmetric environment with binary valuations, i.e. the value of bidder i is given by $v_i \in \{v_l, v_h\}$ with $0 \leq v_l < v_h < \infty$. It is sufficient to consider the case of $i = 1, 2$. (You may assume an efficient tie-breaking rule; i.e. if the there are two bidders, then the bidder with the higher value receives the object, if they have the same value, then the probability of receiving the object is the same.)

   (a) The prior probability is given by $\Pr(v_i = v_h) = \alpha$ for all i. Characterize the equilibrium in the first price auction. (Hint: Can you find a pure strategy Bayesian Nash equilibrium?)

   (b) The prior probability is now given by $\Pr(v_i = v_h) = \alpha_i$ with $0 < \alpha_1 < \alpha_2 < 1$. Characterize the equilibrium in the first price auction. (Hint: Can you find a pure strategy Bayesian Nash equilibrium?)

   (c) Does the revenue equivalence result between the first and the second price auction still hold with the binary payoff types.

4. **Bilateral Trading.** Suppose there is a *continuum* of buyers and sellers (with quasilinear preferences). Each seller initially has one unit of indivisible good and each buyer initially has none. A seller's valuation for consumption of the good is $\theta_1 \in [\underline{\theta}_1, \overline{\theta}_1]$, which is independently and identically drawn from distribution $\Phi_1(\cdot)$ with associated strictly positive density $\phi_1(\cdot)$. A buyer's valuation from consumption of the good is $\theta_2 \in [\underline{\theta}_2, \overline{\theta}_2]$, which is independently and identically drawn from distribution $\Phi_2(\cdot)$ with association strictly positive density $\phi_2(\cdot)$.

   (a) Characterize the trading rule in an ex post efficient social choice function. Which buyers and sellers end up with a unit of the good?

   (b) Exhibit a social choice function that has the trading rule you identified in **(a)**, is Bayesian incentive compatible, and is the individually rational. [Hint: Think of a "competitive" mechanism.] Conclude that the inefficiency identified in the Myerson-Satterthwaite theorem goes away as the number of buyers and sellers grows large.

5. **Single Unit Auction.** Suppose the valuation of agent $i = 1, 2$, and $j \neq i$, for the object is given by

$$u_i(\theta_i, \theta_j) = \theta_i + \gamma \theta_j$$

with $0 < \gamma < 1$. The type θ_i is private information of agent i and as the valuation of the object by agent i also depends on the type of his competitor j, we are in a world of interdependent rather than private values.

   (a) Find a transfer rule t^* such that truthtelling is an ex post equilibrium in the direct revelation game and such that the efficient allocation is realized and such that the transfer of each agent only depends on the announcement of the other agent and the allocation decision, but not on the announcement of agent i.

   (b) Given the transfer rule, is truthtelling also an equilibrium in dominant strategies?

Reading MWG: 23, S (=Salanie) 2 and 3