Dirk Bergemann
Department of Economics
Yale University

Microeconomic Theory (501b)

Problem Set 8. Mechanism Design

4/1/14

This problem set is due on Tuesday, 4/8/14.

1. **(Global Game)** We consider the same game considered in the last problem set. A large, that is a continuum, population with unit mass (so you can index player $i \in [0,1]$), must choose an action, "invest" or "not invest". There is a cost y to investing and assume $0 < y < 1$. There is a benefit 1 of investing (investment "succeeds") if at least proportion $1-\theta$ of the population invests (i.e., at most θ do not invest), and as before state $\theta \in \mathbb{R}$ represents "fundamentals" of the economy. We suppose that the players only have private and noisy information, that is we assume that

   $$\theta \sim \mathcal{U}\left[-M, M\right]$$

   for large $M > 0$, but each player only observe a signal x_i

   $$x_i = \theta + \frac{1}{z}\varepsilon_i, \quad \varepsilon_i \sim \mathcal{U}\left[-\frac{1}{2}, \frac{1}{2}\right]$$

   and so z represent the accuracy of the signal x_i.

   We will show the following proposition. For all $\tilde{\theta} \in [-M, M] \backslash \{y\}$, there exists a z small enough, such that in every Bayes Nash equilibrium of the incomplete information game, players invest when the realized state is $\theta = \tilde{\theta}$ if and only if $\tilde{\theta} \geq y$.

   (a) Define formally a strategy s for a player in this game.

   (b) Suppose an investment succeeds if $\theta > \bar{\theta}$, find the critical type $\bar{x}$ that finds it a dominant strategy to invest. Similarly, suppose an investment does not succeed if $\theta < \underline{\theta}$, find the critical type $\underline{x}$ that finds it a dominant strategy not to invest.

   (c) Suppose an agent invests if he gets a signal $x > \bar{x}$, find the critical state $\bar{\theta}$ such that the investment surely succeeds for $\theta > \bar{\theta}$. Similarly, suppose an agent does not invest if he gets a signal $x < \underline{x}$, find the critical state $\underline{\theta}$ such that the investment surely does not succeed for $\theta < \underline{\theta}$.

(d) Define the following sets,

$$\mathcal{S} \triangleq \{ \text{ the set of all available strategies for any given player}\}$$

$$\mathcal{S}_n \triangleq \left\{ \begin{array}{c} \text{the set of strategies that are not strictly dominated for player } i, \\ \text{when all other players choose } s \in \mathcal{S}_{n-1} \end{array} \right\}.$$

:

$$\bar{\theta}_n = \min\{ \ \theta : \text{investment succeeds, given that agents choose } s \in \mathcal{S}_{n-1}\}$$

$$\bar{\theta}_n = \max\{ \ \theta : \text{investment does not succeed, given that agents choose } s \in \mathcal{S}_{n-1}\}.$$

Using parts 1b and 1c, provide a characterization of $\bar{\theta}_n$ and $\underline{\theta}_n$ in terms of $\mathcal{S}_{n-1}$ and a characterization of $\mathcal{S}_n$ in terms of $\bar{\theta}_n$ and $\underline{\theta}_n$.

(e) Characterize S_∞.

(f) Conclude by arguing that for all z, the investment succeeds if and only if $\theta \geq y$. Moreover, as $z \to \infty$ in any BNE agents invest if and only if the state is greater than y.

2. **Optimal Taxation.** Exercise 3.1 in Salanie. This question explores the famous model of optimal taxation of Mirrlees (1971).

3. **Revelation Principle.** In class we stated the revelation principle for a single agent. Now, state and proof the revelation principle for many agents with:

   (a) for pure strategy equilibrium in dominant strategies;

   (b) for pure strategy Bayesian Nash equilibrium;

   (c) what, if any are differences in the proof of the revelation principle for dominant and Bayesian Nash equilibrium.

4. **First Price Auction with Private Values.** Consider a first-price sealed-bid auction of an object between two risk-neutral bidders. Each bidder i (for $i = 1, 2$) simultaneously submits a bid $b_i \geq 0$. The bidder who submits the highest bid receives the object and pays his bid; both bidders win with equal probability in case they submit the same bid. Before the auction takes place, each bidder i privately observes the realization of a random variable t_i that is drawn independently from a uniform distribution over the interval $[0, 1]$. The actual valuation of the object to bidder i is equal to $t_i + 0.5$. Therefore, the payoff of bidder i is given by

$$u_i = \begin{cases} t_i + 0.5 - b_i & \text{if } b_i > b_j \\ \frac{1}{2}(t_i + 0.5 - b_i) & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}.$$

(a) Derive the symmetric linear Bayesian Nash equilibrium for this game (i.e., each bidder uses an equilibrium strategy of the form $b_i = \alpha t_i + \beta$).

(b) What is the conditionally expected payoff of bidder i with type t_i in this equilibrium?

5. **First Price Auction with Common Values.** Consider the first-price auction in problem (4), except that the actual valuation of the object to bidder i is now equal to $t_i + t_j$ $(j \neq i)$ and therefore the payoff of bidder i now becomes

$$u_i = \begin{cases} t_i + t_j - b_i & \text{if } b_i > b_j \\ \frac{1}{2}(t_i + t_j - b_i) & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}.$$

Notice that, given his own private type t_i, the expected value of the object is $t_i + 0.5$ (i.e., the same as that in problem (4)).

(a) Derive the symmetric linear Bayesian Nash equilibrium for this game.

(b) Compare the equilibrium bid for any given type t_i in this problem to that in problem (4). Interpret your result.

6. **First Price Auction.** Consider the first price auction in a symmetric environment with binary valuations, i.e. the value of bidder i is given by $v_i \in \{v_l, v_h\}$ with $0 \leq v_l < v_h < \infty$. It is sufficient to consider the case of $i = 1, 2$.

(a) The prior probability is given by $\Pr(v_i = v_h) = \alpha$ for all i. Characterize the equilibrium in the first price auction. (Hint: Can you find a pure strategy Bayesian Nash equilibrium?)

(b) The prior probability is now given by $\Pr(v_i = v_h) = \alpha_i$ with $0 < \alpha_1 < \alpha_2 < 1$. Characterize the equilibrium in the first price auction. (Hint: Can you find a pure strategy Bayesian Nash equilibrium?)

(c) Does the revenue equivalence result between the first and the second price auction still hold with the binary payoff types.

Reading MWG: 23, S (=Salanie) 2 and 3