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Microeconomic Theory (501b)

Final Exam

This is a **closed-book** exam. The exam lasts for **180** minutes. **Please write clearly and legibly.** Be especially careful in the definition of the game, the payoff function and the equilibrium notions. The allocated points are also a good indicator for your time budget. Please record the answers $(1, 2, 3)$ and 4 in two separate bluebooks.

1. **(40)** Consider simple case of an indivisible public project that has value S for the consumers. A single firm (monopolist) can realize the project. Its cost function is

$$c = c(e, \beta) = \beta - e, \tag{1}$$

where β is a known efficiency parameter and e is the managers' effort. If the firm exerts effort level e, it decreases the (monetary) cost of the project by e and incurs a disutility (in monetary units)of $\psi(e)$. This disutility displays $\psi', \psi'' > 0$, and satisfies $\psi(0) = 0$ and $\lim_{e \to \beta} \psi(e) = +\infty$. The firm's utility level is:

$$U = t - (\beta - e) - \psi(e)$$

The "reservation utility" of the firm is normalized to 0. Let $\lambda > 0$ denote the shadow cost of public funds. That is taxation inflicts a disutility \$ $(1 + \lambda)$ on taxpayers in order to levy \$ 1 for the state. The net surplus of consumers/taxpayers if the project is realized is $S - (1+\lambda)t$. Hint: You may assume that it is socially optimal to realize the project).

(a) Assume first that cost and in particular effort is observable by the regulator. Describe the optimal solution $\{e^*, t^*(e^*)\}$ for a utilitarian regulator, whose ex-post social welfare can be described by

$$S + U - (1 + \lambda) t$$

who has to respect the participation constraint by the firm. Briefly describe the intuition of your result.

(b) Show that the optimal solution can be implemented through fixed price contract such that $t^*(e) = t^*$ for all e provided that the project is realized.

(c) Suppose now that the firm could either be efficient β_l or inefficient β_h with $\beta_l < \beta_h$. The prior probability of each type is given by p_l and p_h. The regulator only observes the realized cost c as defined in (1) and can make a transfer t to the firm. However, he does not observe β and e separately. A contract based on the observables t and c specifies a transfer-cost pair for each type of firm, namely $\{t(\beta_l), c(\beta_l)\}$ for type β_l and $\{t(\beta_h), c(\beta_h)\}$ for type β_h. Define the optimization program for the regulator who wants to maximize social welfare and would like to make separate offers to low and high cost types of the firm. Hint: First write the utility of an agent of type β_i that chooses contract (t_j, c_j) so that (verify for yourself (!)):

$$V(\beta_i, c_j, t_j) = t_j - c_j - \psi(\beta_i - c_j).$$

(d) Derive the effort levels under the optimal regulation scheme. (Hint: First argue briefly which of the participation and incentive constraints are going to bind. After that it suffices to give a clean version of the first order conditions.)

(e) What can you say about efficiency and information rent for the firms.

2. **(40)** Consider a second price auction with two bidders and an entrance fee f. The valuation for each bidder is uniformly distributed in $[0, 1]$. The timing is as follows. First the seller offers a second price auction with an entrance fee $f > 0$ which is a fee that the bidders have to pay if they participate in the auction, independently of the outcome of the auction. Second, given the auction format, the bidders have to simultaneously decide whether or not to participate, and if so, how much to bid. If they don't bid at all (or equivalently bid zero) then they don't have to pay the fee but will also never get the object.

   (a) Describe the payoff function of each bidder as a function of an arbitrary pair of bids.

   (b) Define the notion of a strategy and the notion of a Bayesian Nash equilibrium for the auction game with the entry fee. Describe the expected payoff of the auctioneer as a function of the bidding strategies and the entry fee.

   (c) Does the imposition of an entrance fee change the bidding strategy of the agents?

   (d) Compute the symmetric Bayesian Nash equilibrium.

   (e) Compute the seller's expected revenue from the auction with an arbitrary entrance fee $f > 0$.

   (f) Which entrance fee maximizes the expected revenue of the auctioneer? Is the resulting allocation ex-post efficient (assume the seller's valuation of the good is zero)?

3. **(40)** Consider a firm that can invest an amount I in a project generating a high observable cash flow $C > 0$ with probability θ and 0 otherwise. This probability depends on the firm's efficiency (type) $0 < \theta_l < \theta_h < 1$. Let $\Pr(\theta = \theta_l) = \alpha$. The firm needs to raise I from external investors who do not observe the value of θ. Assume that

$$\theta_h C - I > 0 > \theta_l C - I.$$

The external investors accept any contract that yields nonnegative profit in expectations. (The external investors are "perfectly competitive").

   (a) Suppose that firms can only promise to repay an amount R chosen by the firm (with $0 \leq R \leq C$) when cash flow is C and 0 otherwise.

      i. For this game carefully define the notion of a Perfect Bayesian Equilibrium.

      ii. Can you give conditions under which there is a pure strategy separating PBE in which the good firm signal its type and receive funding?

      iii. Can you give conditions under which there is a pure strategy pooling PBE in which both firms receive funding.

   (b) Suppose now that the firm can promise to repay an amount R chosen by the firm (with $0 \leq R \leq C$) when cash flow is C, but also has the possibility of pledging some other assets as a collateral for the loan. Should a "default" occur (the firm being unable to repay R), an asset of value K to the firm is transferred to the creditor whose valuation is xK with given and fixed $x : 0 < x < 1$. The size of the collateral K is a choice variable of the entrepreneur.

      i. For this extended game, define the notion of a Perfect Bayesian Equilibrium.

      ii. Give a necessary and sufficient condition for the least cost separating equilibrium to exist. How does it depend on α and x?

4. **(60)** Consider an economy with n consumers and two goods. Each consumer i is endowed with 1 unit of good 1 and no good 2, and has utility function $u_i(x_{i1}, y)$, where x_{i1} is i's consumption of good 1 and y is i's consumption of good 2. Utility functions are differentiable, strictly increasing in both arguments, and quasi-concave. Good 2 is a public good, and every consumer consumes the same quantity y of good 2.

   (a) Each consumer decides how much of her endowment of good 1 to consume and how much to contribute to the production of the public good. The amount of public good produced is given by $y = f(\sum_{i=1}^{n}(1-x_{i1}))$, and hence is a function of the total contribution to the public good. The function f is increasing, concave, and differentiable. Characterize the Pareto efficient allocations of this economy. If you find it helpful in simplifying the notation, it is fine to consider the case $n = 2$. (Hint: As the public good generates utility for all agents, the necessary and and sufficient conditions for Pareto optimality can be represented by two equations, one expressing aggregate feasibility and the other involving the sum of marginal rates of substitution across agents.) Interpret your conditions.

   (b) Find conditions characterizing the equilibrium allocation in this economy. Show that this equilibrium allocation is inefficient. What is the nature of the inefficiency?

   (c) Now let us introduce prices. The price of good 1 is 1. We introduce n prices for good 2, consisting of a personalized price for each consumer, with p_i being the price at which consumer i can buy the public good. An equilibrium for this economy is a specification of the n prices $p_1, \ldots, p_n$ and a quantity Y of the public good with the properties that (i) setting $y = Y$ maximizes the profits $\sum_{i=1}^{n} p_i y - f^{-1} y$ of single firm that produces the public good, (ii) each consumer i's utility $u_i(x_{i1}, y)$ is maximized by buying $y = Y$ of the public good, given the price p_i faced by the consumer and given the budget constraint $x_{i1} + p_i y \leq 1 + s_i \pi$, where s_i is the share of the firm owned by consumer i and π is the equilibrium profit level of the firm, and (iii) markets clear, meaning that $\sum_{i=1}^{n} x_{i1} + f^{-1}(Y) \leq n$ (where n is the total endowment of good 1). Show that an equilibrium of this economy (called a Lindahl equilibrium) is Pareto efficient. (Hint: As in (a), the first order conditions for optimal consumer choice can be aggregated (i.e. summed up) over the individuals and thus the profit shares, while formally part of the set-up, may eventually drop out of the argument.)

   (d) Your answer to the previous question suggests that competitive equilibria would give efficient provision of public goods if only we had enough prices. How many prices do we need? In light of this, what are the weaknesses of this solution to the public goods problem?