Dirk Bergemann and Johannes Hoerner
Department of Economics
Yale University

Final Exam (Solution)
Economics 501b
Microeconomic Theory

May 2010

This is a **closed-book** exam. The exam lasts for **180** minutes. **Please write clearly and legibly.** Be especially careful in the definition of the game, the payoff function and the equilibrium notions. The allocated points are also a good indicator for your time budget. Please record the answers $(1, 2)$, $(3, 4)$, and $(5, 6)$ in separate bluebooks.

1. **(30)** Recall the definition of a signalling game. Nature draws player 1's type $\theta \in \Theta$, a finite set, according to some distribution p on Θ. Player 1 knows θ, player 2 does not. Player 1 takes an action $\alpha_1 \in \triangle A_1$, where A_1 is a finite set of actions. Player 2 observes the realization a_1 of α_1 and chooses then $\alpha_2 \in \triangle A_2$, where A_2 is a finite set of actions. This is the end of the game. Payoffs are given by $u_i(a, \theta)$, $i = 1, 2$.

   (a) For this signalling game, define the notion of a weak perfect Bayesian equilibrium.
   [Solution] A weak PBE in the signaling game is an assessment $\{\sigma^*, \mu^*\}$ such that

$$(1)\ \forall \theta, \sigma_1^*(.|\theta) \in \arg\max_{\alpha_1 \in \Delta A_1} u_1(a_1, \sigma_2^*, \theta);$$

$$(2)\ \forall a_1, \sigma_2^*(.|a_1) \in \arg\max_{\alpha_2 \in \Delta A_2} \sum_{\theta} \mu^*(\theta|a_1)\, u_2(a_1, \alpha_2, \theta);$$

$$(3; \text{Weak Consistency})\ \mu^*(\theta|a_1) = \frac{p(\theta)\,\sigma_1^*(a_1|\theta)}{\sum_{\theta'} p(\theta')\,\sigma_1^*(a_1|\theta')} \text{whenever} \sum_{\theta'} p(\theta')\,\sigma_1^*(a_1|\theta') > 0;$$

   and (trivially)

$$(4)\ \mu^*(\theta|a_1) \text{ is a probability distribution on } \Theta \text{ for } \sum_{\theta'} p(\theta')\,\sigma_1^*(a_1|\theta') = 0;$$

   (b) For this signalling game, define the notion of a sequential equilibrium.
   [Solution] A sequential equilibrium is an assessment $\{\sigma^*, \mu^*\}$ that satisfies (1) to (4) above (hence must be a PBE). In addition, it puts further restriction on (4) by requiring consistency. That is, $\exists\,(\sigma^m, \mu^m)$ such that

$$\{\sigma^*, \mu\} = \lim_{m \to \infty} (\sigma^m, \mu^m)$$

   for some totally mixed strategy $\sigma^m \in \Sigma^0$ (set of totally mixed strategies as defined in notes) such that μ^m for all m is derived from (3) above [notice since $\sigma^m \in \Sigma^0$, $\sum_{\theta'} p(\theta')\,\sigma_1^*(a_1|\theta') > 0$ for $\forall a_1 \in A_1$].

   (c) Prove, or provide a counterexample for, the following claim: in a signalling game, weak perfect Bayesian equilibria and sequential equilibria coincide.
   [Solution] Fudenberg and Tirole (1991) shows that PBE and SE coincide in a multi-stage game of incomplete information with independent types if each player has at most two possible types or if there are just two periods [see 346 of their textbook]. We show here that the additional condition in (b) is always satisfied.

   Since SEs are PBEs. we just need to establish that PBEs are SEs for the signaling game. Take any PBE $\{\sigma^*, \mu^*\}$. The idea is to construct a totally mixed strategy profile σ^m that generates $\mu^m \to \mu^*$. Consider the following σ^m:

$$\sigma_1^m(a_1|\theta) = \left\{ \begin{array}{c} \frac{m-1}{m}\left(\frac{m-1}{m}\right)\sigma_1^*(a_1|\theta), \text{ if } \sigma_1^*(a_1|\theta) > 0 \text{ and } \sum_{\theta'} p(\theta')\,\sigma_1^*(a_1|\theta') > 0 \\ \frac{m-1}{m}\left(\frac{1}{mN(\theta)}\right), \text{ if } \sigma_1^*(a_1|\theta) = 0 \text{ and } \sum_{\theta'} p(\theta')\,\sigma_1^*(a_1|\theta') > 0 \\ \frac{1}{m}\left(\frac{m-1}{m}\frac{\mu^*(\theta|a_1)p(a_1)}{p(\theta)}\right), \text{ if } \mu(\theta|a_1) > 0 \text{ and } \sum_{\theta'} p(\theta')\,\sigma_1^*(a_1|\theta') = 0 \end{array} \right\},$$

   where $N(\theta) := \#\left\{a_1 \in A_1 | \sigma_1^*(a_1|\theta) = 0 \text{ and } \sum_{\theta'} p(\theta')\,\sigma_1^*(a_1|\theta') > 0\right\}$;

   Otherwise (if $\mu(\theta|a_1) = 0$ and $\sum_{\theta'} p(\theta')\,\sigma_1^*(a_1|\theta') = 0$),

$$\sigma_1^m(a_1|\theta) = \frac{1}{m}\left[1 - \sum_{\substack{\forall a_1' \in A_1 \\ \text{s.t.}\mu(\theta|a_1')>0 \\ \& \ \sum_{\theta'} p(\theta')\sigma_1^*(a_1'|\theta')=0}} \left(\frac{m-1}{m}\frac{\mu^*(\theta|a_1')\,p(a_1')}{p(\theta)}\right)\right].$$

By construction $\sigma_1^m(a_1|\theta) > 0\ \forall m, \forall a_1 \in A_1$, and $\sum_{\forall a_1 \in A_1} \sigma_1^m(a_1|\theta) = 1$. So $\sigma_1^m \in \Sigma_1^0$. Moreover, $\sigma_1^m(a_1|\theta) \to \sigma_1^*(a_1, \theta)$ as $m \to \infty$.

Now for σ_2^m, simply set it to play the strategies specified in σ_2^* with probability $1 - \frac{1}{m}$, and to play every strategy in A_2 with probability $\frac{1}{m}\frac{1}{\#A_2}$. Then clearly $\sigma_2^m \to \sigma_2^*$ as $m \to \infty$.

For belief μ^m, only σ_1^m is relevant. It is easy to check that by construction for $\forall a_1 \in A_1$ and $\theta \in \Theta$,

$$\mu^m(\theta|a_1) = \frac{p(\theta)\,\sigma_1^m(a_1|\theta)}{\sum_{\theta'} p(\theta')\,\sigma_1^m(a_1|\theta')} \to \mu^*(\theta|a_1).$$

2. **(30)** Consider a finite normal form game with complete information.

   (a) Define the notion of a strictly dominated strategy, first a strictly dominated pure strategy, second a strictly dominated mixed strategy.

   [Solution] A pure strategy $a_i \in A_i$ is strictly dominated if there exists some $\sigma_i' \in \Delta A_i$ such that

   $$\forall \sigma_{-i} \in \times_{j \neq i} \Delta A_j \text{ , we must have } u_i(\sigma_i', \sigma_{-i}) > u_i(a_i, \sigma_{-i}).$$

   Equivalently, the above can be written as the following:

   $$\forall a_{-i} \in \times_{j \neq i} A_j, \text{ we must have } u_i(\sigma_i', s_{-i}) > u_i(a_i, s_{-i}).$$

   The definition for a strictly dominated mixed strategy simply replaces a_i with $\sigma_i \in \Delta A_i$ in the definition provided above.

   (b) Then prove or disprove: a mixed strategy α_i is strictly dominated if and only if it assigns positive probability to a pure action a_i that is itself strictly dominated.

   [Solution] 1. The forward direction (only if part) is false. Consider the following game, in which we only specify the row player's payoffs:

U	3	3
M	4	0
D	0	4

   The mixed strategy $\frac{1}{2}M + \frac{1}{2}D$ is strictly dominated by the pure strategy U. Yet, neither M nor D is is strictly dominated by any strategy.

   2. The backward direction (if part) is true. Suppose a mixed strategy σ_i assigns positive probability p to a pure action a_i that is strictly dominated by some σ_i'. Then construct a new mixed strategy $\tilde{\sigma}_i$ by replacing action a_i from σ_i with σ_i' played with probability p. Since $p > 0$ and σ_i' strictly dominates a_i, then by construction $\tilde{\sigma}_i$ strictly dominates σ_i.

3. **(40)** Consider the following game between two sellers and a continuum of identical buyers. The sellers produce a homogenous good that the buyers value at v. The sellers contact the buyers by making simultaneously a price offer of p_i for each firm $i \in \{1, 2\}$. The buyers observe the offers and buys from the seller with the lower price if that price does not exceed v. If the two prices are equal, then the market is equally split between the buyers.

   (a) Find the pure strategy Nash equilibria of this pricing game.

   [Solution] Let (p_1^*, p_2^*) be a pure strategy NE.

   Suppose $p_1^* < p_2^*$. If $p_1^* < v$, seller 1 can increase p_1 to $p_1^* + \varepsilon$ to improve his payoff while still winning the object. Hence, we consider $p_1^* \geq v$. In this case, seller 1 makes a payoff of 0. However, by decreasing his price to $v - \varepsilon$, seller 1 wins the object and makes a payoff of $v - \varepsilon > 0$. Hence, with $p_1^* < p_2^*$, (p_1^*, p_2^*) cannot be a pure strategy NE Similarly, $p_2^* > p_1^*$ will not give an NE by symmetry. Hence, we just need to consider $p_1^* = p_2^*$.

   If $p_1^* = p_2^* \in (v, \infty)$, then both sellers receive a payoff of 0: anyone of them can improve by setting $p_1 = v$ to win the object and gain a payoff of $v > 0$.

If $p_1^* = p_2^* \in (0, v]$, then any seller can improve by decreasing his price just a little bit to gain the entire market, which improves his payoff.

Hence, we are left with $p_1^* = p_2^* = 0$, which is indeed the unique pure strategy NE. Given $p_j^* = 0$, seller i cannot improve his payoff since setting $p_i^* = 0$ gives a payoff of 0 whereas setting $p_i > 0$ results in his losing the market and gaining the same payoff of 0.

In conclusion, the (unique) pure strategy NE is $p_1^* = p_2^* = 0$.

(b) Suppose next that it costs $c < v$ to make the price offer. The buyers can purchase only from a firm that has made an offer. Again, the buyers buy from the firm with the lowest offer and if the prices are equal, then the market is split between the firms. Are there pure strategy Nash equilibria in the game?

[Solution] No pure strategy NE. Suppose there exists a pure strategy NE.

If the pure strategy NE involves one seller (say seller i) opting out, then the other seller (seller j) should opt in and set the price $p_j = v$ to gain the highest payoff of $v - c > 0$. But given that, seller i's opting out is not a BR since by opting in and setting price $p_i = v - \varepsilon$, seller i can gain the entire market and make a profit of $v - \varepsilon - c > 0$. Hence, any pure strategy NE must involve both sellers opting in.

Now suppose the pure strategy NE involves both sellers opting in with prices p_1^* and p_2^*. If $p_1^* < p_2^*$, then seller 1 loses for sure with payoff $-c$, and he is strictly better off opting out. Hence $p_1^* \neq p_2^*$ cannot be a pure strategy equilibrium. Hence, we just need to consider $p_1^* = p_2^* = p^*$. If $p^* > v$ or if $p^* < 2c$, then both sellers strictly make losses; each can deviate to opt out to improve his payoff strictly. Hence, we are left with $p^* \in [2c, v]$. If $2c > v$, then we are done showing that there is no pure strategy equilibrium. So suppose $2c \leq v$. Then $p_1^* = p_2^* = p^*$ still cannot constitute an equilibrium because either seller can simply decrease price a little to $p^* - \varepsilon$ to gain the whole market with a payoff of $p^* - \varepsilon - c > \frac{p^*}{2} - c$.

The argument above exhausts all possible pure strategies equilibrium candidates. We thus conclude that there is no pure strategy NE of this game.

(c) Suppose now that there is no cost of making the price offer, but firm i incurs a privately observed transportation cost of d_i that is uniformly distributed on $[0, v]$. Hence firm i makes a profit of $p_i - d_i$ on the units that it sells. Consider again the game where the firms announce prices simultaneously.

i. What is the relevant solution concept for such a game? Define the strategy and the equilibrium for this game.
[Solution] The relevant solution concept is Bayesian Nash Equilibrium (BNE). Seller i's (pure) strategy is a mapping $p_i : [0, v]$ to $[0, \infty)$, for $i = 1, 2$. The strategy profile (p_1^*, p_2^*) is a BNE if $\forall i \in \{1, 2\}$, $j \in \{1, 2\} \setminus i$,

$$E_{d_j}\left[u_i\left(p_i^*(d_i), p_j^*(d_j), d_i\right)\right] \geq E_{d_j}\left[u_i\left(p_i'(d_i), p_j^*(d_j), d_i\right)\right], \ \forall d_i \in [0, v], \forall p_i' : [0, v] \to [0, \infty).$$

ii. Find the equilibrium of this game.
[Solution] We look for a symmetric equilibrium with a linear form. Suppose seller j plays $p_j^*(d_j) = \beta d_j + \gamma$ with $\beta \neq 0$.Then, given d_i, seller i chooses $p_i^*(d_i)$ to solve

$$\begin{aligned}
& \max_{p_i} E_{d_j}\left[u_i\left(p_i, p_j^*(d_j), d_i\right)\right] \\
= & \max_{p_i} E_{d_j}\left[(p_i - d_i) I\left\{p_i < p_j^*(d_j)\right\} + \frac{p_i - d_i}{2} I\left\{p_i = p_j^*(d_j)\right\}\right] \\
= & \max_{p_i} E_{d_j}\left[(p_i - d_i) I\left\{p_i < \beta d_j + \gamma\right\} + \frac{p_i - d_i}{2} I\left\{p_i = \beta d_j + \gamma\right\}\right] \\
= & \max_{p_i} \int_{\frac{p_i - \gamma}{\beta}}^{v} (p_i - d_i) \frac{1}{v} dd_j + 0 \text{ (since } p_i = \beta d_j + \gamma \text{ occurs with probability 0)} \\
= & \max_{p_i} (p_i - d_i) \frac{1}{v}\left[v - \frac{p_i - \gamma}{\beta}\right].
\end{aligned}$$

FOC yields

$$\frac{1}{v}\left[v-\frac{p_i^*(d_i)-\gamma}{\beta}\right]-\frac{1}{\beta}\left(p_i^*(d_i)-d_i\right)=0$$
$$\Leftrightarrow\quad p_i^*(d_i)=\frac{v}{(1+v)}d_i+\frac{\beta v+\gamma}{(1+v)}.$$

Matching this to $p_i^*(d_i)=\beta d_i+\gamma$, we get

$$\beta=\frac{v}{(1+v)}\neq 0,\text{ and }\frac{\beta v+\gamma}{(1+v)}=\gamma$$

equivalent to

$$\beta=\frac{v}{1+v}\text{ and }\gamma=\frac{v}{1+v}.$$

Hence, a BNE of the game is for each seller i to bid $p_i^*(d_i)=\frac{v}{1+v}d_i+\frac{v}{1+v}$.
Comment: Notice that as $v\to 0$, $p_i^*(d_i)\to 0$ for $i=1,2$ as in Question (a).

4. **(40)** Consider the following Bayesian game. Player 1 is interested in selling his watch to player 2. The cost of selling the watch is zero and his utility is p if he sells the watch at price p. Player 2 has value v for the watch and would obtain utility $v-p$ if he buys the watch for price p. Only player 2 knows the value of v. It is common-knowledge that player 1 believes that v is uniformly distributed on the interval $[0,1]$. They will bargain over the price through a mediator. Each player i will submit a number b_i in a sealed envelope to the mediator. The mediator unseals the envelopes and implements a trade according to the following rules. If $b_1>b_2$ then there will be no sale. If $b_1\leq b_2$ then the watch will be sold to bidder 1 at price $p=b_1$.

   (a) What are the sets of pure strategies in this Bayesian game for players 1 and 2?

   [Solution] Player 1's set of pure strategy is the interval $[0,\infty)$. Player 2's set of pure strategy is all the functions that map the interval $[0,1]$ to $[0,\infty)$.

   (b) For what set of values b_1 does there exists a pure-strategy Bayesian Nash equilibrium in which player 1 submits b_1? Explain carefully how you reach your conclusion.

   [Solution] In a pure strategy BNE, if player 1 chooses b_1^*, then given v, player 2's pure strategy best response $b_2^*(v)$ must solve

$$\max_{b_2}\left(v-b_1^*\right)I\left\{b_1^*\leq b_2\right\}.$$

   Thereofre b_2^* must necessarily satisfy the following:

$$b_2^*(v)=\left\{\begin{array}{c}\geq b_1^*\text{ if }v>b_1^*;\\ <b_i^*\text{ if }v<b_1^*;\\ \in[0,\infty)\text{ if }v=b_1^*.\end{array}\right\}\ (1).$$

   To make b_1^* an optimal choice, it must solve

$$\max_{b_1}E_v\left[b_1I\left\{b_1\leq b_2^*(v)\right\}\right]=\max_{b_1}\int_0^1 b_1I\left\{b_1\leq b_2^*(v)\right\}dv\ (2).$$

   $(b_1^*,b_2^*(.))$ is a pure stratege BNE iff it satisfies both (1) and (2).

   (i) It is clear that any $b_1^*\in[1,\infty)$ with $b_2^*(v)=0$ is a pure strategy BNE, since they satisfy (1) and the expression in (2) is 0 regardless of what b_1 is.

   (ii) Consider $b_1^*\in(0,1)$. Examine

$$b_2^*(v)=\left\{\begin{array}{c}b_1^*\text{ if }v\geq b_1^*;\\ 0\text{ if }v<b_1^*.\end{array}\right\}.$$

Obviously (1) is satisfies. We need to show b_1^* solves (2). Now

$$\int_0^1 b_1^* I\left\{b_1^* \leq b_2^*(v)\right\} dv = \int_{b_1^*}^1 b_1^* dv = (1-b_1^*)\, b_1^* > 0.$$

If player 1 deviates to $b_1 > b_1^*$, then

$$\int_0^1 b_1 I\left\{b_1 \leq b_2^*(v)\right\} dv = 0 \text{ since } b_2^*(v) \leq b_1^* \ \forall v \in [0,1], \text{ so } b_1 > b_2^*(v), \forall v \in [0,1].$$

On the other hand, if player 1 deviates to $0 < b_1 < b_1^*$,

$$\int_0^1 b_1 I\left\{b_1 \leq b_2^*(v)\right\} dv = b_1(1-b_1^*) < b_1^*(1-b_1^*).$$

Finally, if player 1 deviates to $b_1 = 0$,

$$\int_0^1 b_1 I\left\{b_1 \leq b_2^*(v)\right\} dv = 0.$$

Hence, we've demonstrated that $b_1 = b_1^*$ indeed maximizes $\int_0^1 b_1 I\left\{b_1 \leq b_2^*(v)\right\} dv$.

(iii) Lastly, it is clear that $b_1^* = 0$ and $b_2^*(v) = 0, \forall v \in [0,1]$ constitute a pure strategy BNE. Obviously (1) is satisfied. Given $b_2^*(v) = 0, \forall v \in [0,1]$, $\int_0^1 b_1 I\left\{b_1 \leq b_2^*(v)\right\} = 0$ for all $b_1 \in [0,\infty)$. So (2) is trivially satisfied.

Hence, any $b_1^* \in [0,\infty)$ can be part of a pure strategy BNE.

(c) Give an example of a strategy for player 2 which is weakly dominated. (It makes no difference here or in the next part whether you use the *ex ante* or *interim* versions of weak dominance.)

[Solution] Consider $b_2(v) = 2 > 1 \ \forall v$. This strategy gives player 2 an ante payoff of

$$\int_0^1 (v-b_1)\, I\left\{b_1 \leq b_2(v) = 2\right\} dv.$$

We show that $b_2'(v) = 1$ weakly dominates it: for all $b_1 \in [0,\infty)$,

$$\begin{aligned}
& \int_0^1 (v-b_1)\, I\left\{b_1 \leq b_2'(v) = 1\right\} dv - \int_0^1 (v-b_1)\, I\left\{b_1 \leq b_2(v) = 2\right\} dv \\
= & \int_0^1 (v-b_1)\left[I\left\{b_1 \leq 1\right\} - I\left\{b_1 \leq 2\right\}\right] dv \\
= & -\int_0^1 (v-b_1)\, I\left\{1 < b_1 \leq 2\right\} dv \geq 0, \text{ since } v - b_1 \leq 0 \text{ for all } b_1 \in (1,2].
\end{aligned}$$

We just need to establish strict inequality

$$\int_0^1 (v-b_1)\, I\left\{b_1 \leq b_2'(v) = 1\right\} dv > \int_0^1 (v-b_1)\, I\left\{b_1 \leq b_2(v) = 2\right\} dv,$$

for some b_1. Consider $b_1 = 1.5$. Then

$$\int_0^1 (v-1.5)\, I\left\{1.5 \leq b_2'(v) = 1\right\} dv = 0$$

whereas

$$\int_0^1 (v-1.5)\, I\left\{1.5 \leq 2\right\} dv = \int_0^1 (v-1.5)\, dv = \frac{1}{2} - 1.5 = -1 < 0.$$

(d) What is the set of Bayesian Nash equilibria in which player 2 does not use a weakly dominated strategy? Explain carefully how you reach your conclusion.

[Solution] We consider the interim version of weak dominance. Given $v \in [0,1]$, player 2, choosing $b_2(v)$, receives a payoff of

$$(v - b_1) I \{b_1 \leq b_2(v)\}.$$

We show that $b_2(v) = v$ is a weakly dominant strategy. Examine the table below, which shows player 2's payoff for different b_1 values given $b_2(v) < v, b_2(v) = v$, and $b_2(v) > v$:

payoff to 2	$b_2(v) < v$	$b_2(v) = v$	$b_2(v) > v$
$b_1 < v$	$v - b_1 > 0$ or 0[1]	$v - b_1 > 0$	$v - b_1 > 0$
$b_1 = v$	0	0	0
$b_1 > v$	0	0	$v - b_1 < 0$ or 0[2]

By observing the table above, we can see that clearly $b_2(v) = v$ gives player 2 weakly better payoff than any $b_2(v) \neq v$ for all $b_1 \in [0, \infty)$ (and strictly better payoff for some values of b_1).

Hence, we conclude that $b_2(v) = v$ is the unique weakly dominant strategy. So the set of BNE in which player 2 does not use a weakly dominated strategy is simply the singleton $\{b_2(v) = v, \forall v \in [0,1]\}$.

5. **(40)** Consumers buy insurance from a competitive insurance industry. Each consumer is an expected utility maximizer with utility function $u(x) = \sqrt{x}$ for final consumption x. Each consumer has initial wealth \$20. There are two types of consumers, high risk and low risk. High risk consumers have a probability $\frac{2}{3}$ of losing \$10, while low risk consumers have a probability $\frac{1}{3}$ of losing \$10. The insurance industry is perfectly competitive and risk-neutral, that is each insurance company is maximizing expected profit but because of perfect competition each firm earns zero expected profit from the insurance policy that it sells.

(a) First, suppose that insurance companies can distinguish high and low risk consumers,

i. What premiums do they charge to the two types of consumers in a free entry competitive equilibrium? Give an argument supporting your claim.

[Solution] A high risk consumer without insurance is facing a consumption plan:

$$\left(20, \frac{1}{3}; 10, \frac{2}{3}\right),$$

which reads "he gets \$20 with prob. $\frac{1}{3}$ and \$10 with prob. $\frac{2}{3}$."

A high risk consumer with an insurance contract (α, β) is facing a consumption plan:

$$\left(20 - \alpha, \frac{1}{3}; 10 + \beta, \frac{2}{3}\right),$$

where α is the premium he pays to the insurance company in return for which he will be paid $\alpha + \beta$ in the case of accident.

Likewise, a low risk consumer without insurance is facing a consumption plan:

$$\left(20, \frac{2}{3}; 10, \frac{1}{3}\right).$$

A low risk consumer with an insurance contract (α, β) is facing a consumption plan:

$$\left(20 - \alpha, \frac{2}{3}; 10 + \beta, \frac{1}{3}\right).$$

[1] $v - b_1 > 0$ if $b_1 \leq b_2(v)$; otherwise 0.Notice that both cases are feasible since $b_1 = 0$ will always give $v - b_1 > 0$.

[2] $v - b_1 < 0$ if $b_1 \leq b_2(v)$; otherwise 0. Notice that both cases are feasible since for any $b_1 > v$, we can find some $b_2(v) > b_1$.

Since insurance companies can differentiate the two types of consumers, it can provide two different insurances to the two types of consumers (high/low insurance to the high/low risk type).
For the high risk consumers only. Its expected profit is $\frac{1}{3}\alpha^H - \frac{2}{3}\beta^H$. Since consumers are free to accept the service, his expected utility should be no less than the level if he opts out of the insurance:

$$\frac{1}{3}\sqrt{20-\alpha^H} + \frac{2}{3}\sqrt{10+\beta^H} \geq \frac{1}{3}\sqrt{20} + \frac{2}{3}\sqrt{10} \approx 3.60.$$

Competitiveness requires that the profit should be equal to zero, so $\frac{1}{3}\alpha^H - \frac{2}{3}\beta^H = 0 \Rightarrow \alpha^H = 2\beta^H$. Second, it also requires the firm to offer a contract that maximizes the consumer's expected utility.

$$\left(\alpha^H, \beta^H\right) \in \max_{\alpha,\beta} \frac{1}{3}\sqrt{20-\alpha} + \frac{2}{3}\sqrt{10+\beta}, s.t. \text{zero profit condition or } \alpha^H = 2\beta^H.$$

The problem is then equivalent to

$$\beta^H \in \max_{\beta} \frac{1}{3}\sqrt{20-2\beta} + \frac{2}{3}\sqrt{10+\beta},$$

the FOC of which requires

$$-\frac{2}{6}\left(20-2\beta^H\right)^{-\frac{1}{2}} + \frac{1}{3}\left(10+\beta^H\right)^{-\frac{1}{2}} = 0$$

$$\Leftrightarrow \beta^H = \frac{10}{3},$$

which implies full insurance: the consumer is guaranteed the same amount of consumption under different states.

Hence,

$$\alpha^H = \frac{20}{3}, \beta^H = \frac{10}{3}.$$

Given this, the expected utility for a high risk consumer is: $\sqrt{\frac{40}{3}} \approx 3.65 > 3.60$, the level such a consumer receives if not buying insurance. Hence, the high risk consumer is better off through the purchase of such a contract.

In a similar fashion, if the insurance companies face low risk consumers, they will set $\frac{2}{3}\alpha^L = \frac{1}{3}\beta^L \Leftrightarrow 2\alpha^L = \beta^L$ and solve

$$\alpha^L \in \max_{\alpha} \frac{2}{3}\sqrt{20-\alpha} + \frac{1}{3}\sqrt{10+2\alpha}.$$

FOC yields

$$-\frac{1}{3}\left(20-\alpha^L\right)^{-\frac{1}{2}} + \frac{1}{3}\left(10+2\alpha^L\right)^{-\frac{1}{2}} = 0.$$

Solving, we obtain $\left(\alpha^L = \frac{10}{3}, \beta^L = \frac{20}{3}\right)$. This also assures full insurance in the consumption plan. The corresponding expected utility is given by $\sqrt{\frac{50}{3}} \approx 4.08$, which is strictly higher than $\frac{2}{3}\sqrt{20} + \frac{1}{3}\sqrt{10} \approx 4.04$, the level that the low-risk consumer receives without the insurance.

ii. How much insurance do the two types of consumers buy in a free entry competitive equilibrium? Give an argument supporting your claim.
[Solution] As argued above, since each type of consumers is strictly better off with his corresponding insurance, all customers will demand insurances, with supply matched due to perfect entry competition.

(b) Now suppose that insurance companies cannot distinguish high and low risk consumers.

i. What problem would arise if insurance tried to sell insurance to consumers as described in part (a)?

[Solution] A high risk consumer would prefer $\left(\alpha^L = \frac{10}{3}, \beta^L = \frac{20}{3}\right)$:

$$\frac{1}{3}\sqrt{20 - \frac{10}{3}} + \frac{2}{3}\sqrt{10 + \frac{20}{3}} = \sqrt{\frac{50}{3}} > \sqrt{\frac{40}{3}}.$$

The expected profit that a company earns from a high risk consumer who chooses a contract for a low risk consumer turns out to be negative:

$$\frac{1}{3} \times \frac{10}{3} + \frac{2}{3} \times \left(-\frac{20}{3}\right) = -\frac{10}{3} < 0.$$

In a similar fashion, we can show that the low-type will NOT deviate to choose the high insurance.

If some or all high risk consumers choose the contract for the low risk consumers, the insurance company's expected profit is actually negative (since what the company can earn from a low risk consumer choosing a low insurance is zero) – this will thus result in the insurance companies exiting the market.

ii. Suppose there were a "separating equilibrium," where different types of consumers bought different contracts. Describe what the equilibrium must graphically and/or algebraically. (Hint: You do not have solve for the equilibrium solution analytically. It suffices to describe the properties of the equilibrium allocation from the conditions of incentive compatibility and zero profit/free entry condition of the insurance firms.)

[Solution] Suppose an insurance company offers two contract $\left(\alpha^L, \beta^L\right)$ and $\left(\alpha^H, \beta^H\right)$. In a separating equilibrium, low-risk consumers must choose the former and high risk consumers must choose the latter. This translates to the following IC conditions:

$$\begin{aligned}
(IC_H) \quad &: \quad \frac{1}{3}\sqrt{20 - \alpha^H} + \frac{2}{3}\sqrt{10 + \beta^H} \geq \frac{1}{3}\sqrt{20 - \alpha^L} + \frac{2}{3}\sqrt{10 + \beta^L}, \\
(IC_L) \quad &: \quad \frac{2}{3}\sqrt{20 - \alpha^L} + \frac{1}{3}\sqrt{10 + \beta^L} \geq \frac{2}{3}\sqrt{20 - \alpha^H} + \frac{1}{3}\sqrt{10 + \beta^H}.
\end{aligned}$$

Moreover, individual rationality for each type requires:

$$\begin{aligned}
(IR_H) \quad &: \quad \frac{1}{3}\sqrt{20 - \alpha^H} + \frac{2}{3}\sqrt{10 + \beta^H} \geq \frac{1}{3}\sqrt{20} + \frac{2}{3}\sqrt{10} \approx 3.60, \\
(IR_L) \quad &: \quad \frac{2}{3}\sqrt{20 - \alpha^L} + \frac{1}{3}\sqrt{10 + \beta^L} \geq \frac{2}{3}\sqrt{20} + \frac{1}{3}\sqrt{10} \approx 4.04.
\end{aligned}$$

(i) We first argue that due to the assumption of free entry/ competition the expected payoff from EACH contract should be zero: i.e., we must have $\alpha^H = 2\beta^H$ and $2\alpha^L = \beta^L$.

(ii) We now argue that only (IC_H) matters while (IC_L) will be satisfied automatically. Verbally, this means that low risk consumers have no incentive to pretend themselves to be of high risk.

To see this, summing up the two self-selection conditions,

$$(\bigstar) : \frac{1}{3}\left(\sqrt{20 - \alpha^L} - \sqrt{10 + \beta^L}\right) \geq \frac{1}{3}\left(\sqrt{20 - \alpha^H} - \sqrt{10 + \beta^H}\right).$$

We can now show that (IC_L) will be satisfied by $(\bigstar)$:

$$\begin{aligned}
\frac{2}{3}\sqrt{20 - \alpha^L} + \frac{1}{3}\sqrt{10 + \beta^L} \quad &= \quad \left(\frac{1}{3}\sqrt{20 - \alpha^L} + \frac{2}{3}\sqrt{10 + \beta^L}\right) + \frac{1}{3}\left(\sqrt{20 - \alpha^L} - \sqrt{10 + \beta^L}\right) \\
&\underset{(\bigstar)}{\geq} \left(\frac{1}{3}\sqrt{20 - \alpha^L} + \frac{2}{3}\sqrt{10 + \beta^L}\right) \\
&\quad + \frac{1}{3}\left(\sqrt{20 - \alpha^H} - \sqrt{10 + \beta^H}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{3}\sqrt{20-\alpha^L} - \sqrt{10+\beta^L}\right) + \sqrt{10+\beta^L} \\
&\quad + \frac{1}{3}\left(\sqrt{20-\alpha^H} - \sqrt{10+\beta^H}\right) \\
&= \frac{2}{3}\sqrt{20-\alpha^L} + \frac{1}{3}\sqrt{10+\beta^L}
\end{aligned}
$$

(iii) Given this observation, it involves full insurance for high risk consumers since there is no reason to distort their choice of insurance as low risk consumers will not mimick high risk consumers. In this sense, $\left(\alpha^H = \frac{20}{3}, \beta^H = \frac{10}{3}\right)$ is the equilibrium contract.

(iv) To pin down the contract for low risk, note that $\left(\alpha^L, \beta^L\right)$ should be the solution to:

$$
\begin{aligned}
& \max_{(\alpha^L,\beta^L)} \frac{2}{3}\sqrt{20-\alpha^L} + \frac{1}{3}\sqrt{10+\beta^L} \\
s.t.\ 2\alpha^L &= \beta^L \text{ (non-negative profit), } (\bigstar) \\
(IC_H) &: \sqrt{\frac{40}{3}} \geq \frac{1}{3}\sqrt{20-\alpha^L} + \frac{2}{3}\sqrt{10+\beta^L}.
\end{aligned}
$$

This translates to

$$
\begin{aligned}
& \max_{\alpha^L} \frac{2}{3}\sqrt{20-\alpha^L} + \frac{1}{3}\sqrt{10+2\alpha^L} \\
\text{s.t. } \sqrt{\frac{40}{3}} &\geq \frac{1}{3}\sqrt{20-\alpha^L} + \frac{2}{3}\sqrt{10+2\alpha^L}.
\end{aligned}
$$

The constraint must be binding for otherwise we are back to the situation in b(i). So we just need to solve

$$\sqrt{\frac{40}{3}} = \frac{1}{3}\sqrt{20-\alpha^L} + \frac{2}{3}\sqrt{10+2\alpha^L}.$$

The solution is $\alpha^L = \frac{20}{27}\left(11 - 4\sqrt{7}\right) = .309$. This gives a payoff of

$$\frac{2}{3}\sqrt{20-\alpha^L} + \frac{1}{3}\sqrt{10+2\alpha^L} = 4.04.$$

A key result here is the absence of full insurance for low risk consumers: $20 - \alpha^L > 10 + 2\alpha^L$. Low risk consumers have to shoulder some degree of risk.

(c) Suppose that a proportion π of consumers were high risk. For which values of π would another firm be able to enter and earn positive profits selling to both high and low risk consumers relative to the "separating equilibrium"?

[Solution] Suppose that an entrant offers a pooling contract (α, β) which consumers of both types would like to accept. The probability of accident is $\pi\frac{2}{3} + (1-\pi)\frac{1}{3} = \frac{1}{3}(1+\pi)$. The probability of no accident is $1 - \frac{1}{3}(1+\pi) = \frac{2}{3} - \frac{1}{3}\pi$. The expected profit from this contract is

$$-\beta \times \frac{1}{3}(1+\pi) + \alpha \times \left(\frac{2}{3} - \frac{1}{3}\pi\right) = \frac{1}{3}\left[2\alpha - \beta - (\alpha+\beta)\pi\right].$$

We see the profit is positive iff $2\alpha - \beta \geq (\alpha+\beta)\pi$ iff

$$\pi \leq \frac{2\alpha-\beta}{\alpha+\beta} = 2 - \frac{3\beta}{\alpha+\beta} = \frac{3\alpha}{\alpha+\beta} - 1,$$

decreasing in β and increasing in α. We want to find the upper bound for the π. In other words, we want to minimize $\frac{3\beta}{\alpha+\beta}$ or equivalent maximize $\frac{3\alpha}{\alpha+\beta}$ subject to the following:

(a) The high risk consumers would prefer this contract iff:

$$\frac{1}{3}\sqrt{20-\alpha}+\frac{2}{3}\sqrt{10+\beta}\geq\frac{1}{3}\sqrt{20-\alpha^H}+\frac{2}{3}\sqrt{10+\beta^H}=\sqrt{\frac{40}{3}};$$

and (b) The low risk consumers would prefer this contract iff:

$$\frac{2}{3}\sqrt{20-\alpha}+\frac{1}{3}\sqrt{10+\beta}\geq\frac{2}{3}\sqrt{20-\alpha^L}+\frac{1}{3}\sqrt{10+\beta^L}=4.04.$$