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Final Exam
Economics 501b
Microeconomic Theory

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This is a **closed-book** exam. The exam lasts for **180** minutes. **Please write clearly and legibly.** Be especially careful in the definition of the game, the payoff function and the equilibrium notions. The allocated points are also a good indicator for your time budget. Please record the answers $(1, 2)$, $(3, 4)$, and $(5, 6)$ in separate bluebooks.

1. **(40)** Consider the job market signalling model with many types (productivity) of the worker given by:

$$a \in \{a_0, a_1, a_2, ..., a_k, ..., a_K\} = \left\{1, 1 + \frac{1}{K}, ..., 1 + \frac{K-1}{K}, 1 + \frac{K}{K}\right\}.$$

The worker has productivity a_k with prior probability $p(a_k) = p_k$. The worker can choose an educational level $e \in \mathbb{R}_+$. The worker's utility is

$$u(w, e, a) = w - \frac{e}{a}.$$

The objective function of the firm is given by

$$\min \sum_{a_k \in A} (a_k - w)^2 \, p(a_k \,|e),$$

and hence offers competitive wages given its beliefs $p(a_k \,|e)$.

(a) Define the notion of separating perfect bayesian equilibrium for the job market signalling game.

(b) Characterize the least cost separating equilibrium for the finite number of types. In particular, what is the education level provided by each type k?

(c) Can you construct a semi-separating equilibria where the types $k = 1, ..., l$ pool and all the types $k = l+1, ..., K$ separate from the pool and from each other.

2. **(20)** Each of two players receives a ticket on which there is a number $\left\{0, \frac{1}{N}, ..., \frac{N}{N}\right\}$ with some given finite N. The number on a player's ticket is the size of a prize that he may receive. The two prizes are identically and independently distributed with distribution function F. Each player is asked independently and simultaneously whether he wants to exchange his prize for the other player's prize. If both players agree then the prizes are exchanged, otherwise each player receives his own prize. Each player's objective is to maximize his expected payoff.

(a) Model this situation carefully and completely as a Bayesian game and define the notion of a pure and mixed strategy equilibrium for this game.

(b) Characterize the Bayesian Nash equilibria of this game. What properties do all strategies and what properties do all equilibrium outcomes share across the equilibria?

3. **(25)** A monopolistic seller faces a unit mass of consumers which have preferences given by:

$$u(\theta, q, t) = \theta v(q) - t.$$

The consumption q can take a value of 0, 1, or 2. The value of consumption as represented by v is given by:

$$v(0) = 0, \quad v(1) = 1, \quad v(2) = \frac{7}{4}.$$

The production cost is $c = \frac{3}{4}$ whatever the (strictly positive) size of the item. Each buyer can engage in personal arbitrage, i.e. rather than buy one item with $q = 2$, buy two items each with $q = 1$.
There are two types of consumers: $\theta_1 = 1$ (in proportion λ) and $\theta_2 = 2$ (in proportion $1 - \lambda$). The willingness-to-pay θ is private information to each buyer. The seller maximize the expected net profit, the expected revenue minus the expected cost. The outside utility of each buyer is zero.

(a) The seller can either offer a single item or a menu of items for sale. Describe the optimal policy of the seller as a function of the proportion λ of low value buyers. In particular, does the seller ever find it optimal to only offer the bundle $q = 2$, i.e. to engage in pure commodity bundling.

(b) Suppose now, that for technological reasons, the monopolist must choose to produce the good in either size $q = 1$ or size $q = 2$. Describe the optimal policy of the seller as a function of the proportion λ of low value buyers. Does your answer differ in (3a) and (3b), and if so why?

4. **(35)** Consider the moral hazard problem with three possible effort levels $E = \{e_1, e_2, e_3\}$ and two possible revenue levels $\pi_h = 10$ and $\pi_l = 0$. Let the probability of π_h given e be:

$$p(\pi_h | e) = \begin{cases} 2/3 & \text{if } e = e_1, \\ 1/2 & \text{if } e = e_2, \\ 1/3 & \text{if } e = e_3. \end{cases}$$

The agent's cost of effort is

$$c(e) = \begin{cases} 5/3 & \text{if } e = e_1, \\ 8/5 & \text{if } e = e_2, \\ 4/3 & \text{if } e = e_3. \end{cases}$$

The agent's utility of wealth function is $v(w) = \sqrt{w}$. His reservation utility $\overline{u}$ is zero. The principal is risk-neutral and seeks to maximize the expected profit $\pi - w$, i.e. revenue minus wages.

(a) What is the optimal contract if effort is observable?

(b) Show that if effort is not observable, then it is not possible to induce effort level e_2.

(c) What is the optimal contract when effort is not observable?

5. **(45)** Consider the following game:

	L	R
T	$2,3$	$0,2$
B	$3,0$	$1,1$

(a) Find the set of rationalizable strategies and find the Nash equilibria of this game.

(b) Suppose now that we have an extensive-form version of this game in which player 1 first chooses T or B, then player 2 observes player 1's action, and then player 2 chooses L or R. Identify the subgame equilibria of this game, and present a Nash equilibrium that is not subgame perfect. Which players gain from player 1 moving first?

(c) Let us now make the order of moves endogenous by assuming that there are two stages. Each player can choose an action only once, but can do so in either the first stage or the second. Players move simultaneously within a stage (if they both choose to move within that stage), and first stage choice (if any), are observed before the second-stage actions (if any) are chosen. Payoffs are received only once, after the end of the second stage. Find the subgame perfect equilibria of this game. Find the Nash equilibria that do not involve dominated strategies. How do the two equilibria compare?

(d) Now return to the case in which the order of moves is fixed exogenously. Suppose that player 1 moves first, but instead of observing player 1's action, player 2 observes only an imperfect signal of that action. In particular, 2 observes either t or b, with the following distribution,

	t	b
T	$1-\epsilon$	ϵ
B	ϵ	$1-\epsilon$

For example, if 1 plays T, then 2 observes t with probability $1-\epsilon$ and b with probability ϵ. Find the pure-strategy Nash, subgame-perfect, and sequential equilibria of this game.

(e) Argue that in any mixed equilibrium of this game, player 1 must mix. Draw a graph showing the probability player 2 attaches to 1 having played T, given that t is observed, as a function of the probability with which 1 plays T. Do the same for the probability player 2 attaches to 1 having played T, given that b is observed, as a function of the probability with which 1 plays T

(f) Find the mixed equilibria of this game. What are the limits of these mixed equilibria as ϵ approaches zero?

(g) In light of the answer to your previous question, how do you assess the subgame-perfect equilibrium of the game in which player 1 moves first, and has an action that is perfectly observed? Is it robust to perturbations in 2's ability to observe 1's action?

6. **(15)** Consider once again the game:

	L	R
T	$2, 3$	$0, 2$
B	$3, 0$	$1, 1$

(a) Suppose that the game is infinitely repeated. What is the lowest discount factor that will support an equilibrium in which (T, L) is played in every period?

(b) Now suppose the game is infinitely repeated, but in each period, player 2 observes not player 1's action from the previous period, but either the signal t or b, according to the distribution

	t	b
T	$1 - \epsilon$	ϵ
B	ϵ	$1 - \epsilon$

For example, if 1 played T *in the previous period* , then 2 observes t in the current with probability $1 - \epsilon$ and b with probability ϵ. (Notice that this is similar to the imperfect monitoring we considered when the game is played only once, but not quite the same.) Find a discount factor and an equilibrium that gives the players payoffs larger than $(1, 1)$. As the discount factor approaches 1, can we approach payoff $(2, 3)$? Why or why not?