First price auctions
with general information structures:
Implications for bidding and revenue

Dirk Bergemann
Yale

Benjamin Brooks
BFI/UChicago

Stephen Morris
Princeton

New York University Abu Dhabi

December 2016
Premises

1. Classical auction theory makes stylized assumptions about information
2. Assumptions about information are hard to test
3. Equilibrium behavior can depend a lot on how we specify information
Promises

- Goal: a theory of bidding that is robust to specification of information
- First attempt: First price auction
- Hold fixed underlying value distribution,
- Consider all specifications of information and equilibrium
- We deliver:
  - A tight lower bound on the winning bid distribution
  - A tight lower bound on revenue
  - A tight upper bound on bidder surplus
- Other results on max revenue, min bidder surplus, min efficiency
A (toy) model of a first price auction

- Two bidders
- Pure common value \( v \sim U[0, 1] \)
- Submit bids \( b_i \in \mathbb{R}_+ \)
- High bidder gets the good and pays bid
  \( \implies \) winner’s surplus is \( v - b_i \)
- Allocation of good is always efficient, total surplus 1/2
- Seller’s expected revenue is \( R = \mathbb{E}[\max\{b_1, b_2\}] \)
- Bidder surplus \( U = 1/2 - R \)
- What predictions can we make about \( U \) and \( R \) in equilibrium?
Filling in beliefs

- What do bidders know about the value?
- What do they know about what others know?
- Assume beliefs are consistent with a common prior
- Still, many possible ways to “fill in” information:
  - Bidders observe nothing;
    Unique equilibrium: \( b_1 = b_2 = R = 1/2 \)
  - Bidders observe everything;
    \( b_1 = b_2 = v, \ R = 1/2 \)
- True information structure is likely somewhere in between:
  - Bidders have some information about \( v \), but not perfect
  - But exactly how much information do they have?
Lower revenue?

  - Bidder 1 observes $v$, bidder 2 observes nothing
    - $b_1 = v/2$, $b_2 \sim U[0, 1/2]$ and independent of $v$
  - Bidder 2 is indifferent:
    With a bid of $b_2 \in [0, 1/2]$, will win whenever $v \leq 2b_2$
    Expected value is exactly $b_2$!
  - Bidder 1 wins with a bid of $b_1$ with probability $2b_1$
    Surplus is $(v - b_1)2b_1$
    $\implies$ optimal to bid $b_1 = v/2!$
  - $U_1 = \int_{v=0}^{1} v(v - v/2)dv = 1/6$, $U_2 = 0$, $R = 1/3$
How we model beliefs matters

- Welfare outcomes are sensitive to modelling of information
- Why? Optimal bid depends on distribution of others’ bids, and on correlation between others’ bids and values
- Problem: hard to say which specification is “correct”
- What welfare predictions do not depend on how we model information?
Uniform example continued

- Can we characterize minimum revenue?
- Must be greater than zero!
- But seems likely to be lower than EMW
- At min $R$, winning bids have been pushed down “as far as they can go”
- Force pushing back must be incentive to deviate to higher bids
- In EMW, informed bidder strictly prefers equilibrium bid
Towards a Bound: Winning Bid

Consider symmetric equilibria in which winning bid is an increasing and deterministic function $\beta(v)$ of true value $v$.

Which $\beta$ could be incentive compatible in equilibrium?

Consider the following uniform upward deviation to $b$: Whenever equilibrium bid, winning or not, is $b' < b$, bid $b$ instead!

Now let bids $b', b$ be winning bids for some values $x, v$ respectively:

$$b' = \beta(x) < \beta(v) = b$$
Towards a Bound: Uniform Upward Deviation

Now let bids $b'$, $b$ be winning bids for some values $x$, $v$ respectively:

$$b' = \beta(x) < \beta(v) = b$$

Bid $b'$ could have been a losing or a winning bid.

Uniform upward deviation to $b = \beta(v)$ is not attractive if

$$\frac{1}{2} \int_{x=0}^{v} (\beta(v) - \beta(x)) \, dx \geq \frac{1}{2} \int_{x=0}^{v} (x - \beta(v)) \, dx$$

- loss when would have won
- gain when would have lost

Using symmetry (1/2) and deterministic winning bid $\beta(v)$
Restrictions on $\beta$

- Uniform upward deviation to $b = \beta(v)$

$$\frac{1}{2} \int_{x=0}^{v} (\beta(v) - \beta(x)) \, dx \geq \frac{1}{2} \int_{x=0}^{v} (x - \beta(v)) \, dx$$

loss when would have won

$$\frac{1}{2} \int_{x=0}^{v} (x - \beta(v)) \, dx$$

gain when would have lost

rearranges to

$$\beta(v) \geq \frac{1}{2v} \int_{x=0}^{v} (x + \beta(x)) \, dx$$  \hspace{1cm} (IC)

- What is the smallest $\beta$ subject to (IC) and $\beta \geq 0$?
- Must solve (IC) with equality for all $v$
Minimal Winning Bid $\beta$

- uniform upward deviation solves
  \[
  \beta(v) = \frac{1}{2v} \int_{x=0}^{v} (x + \beta(x))\,dx 
  \]  
  \[\text{(IC)}\]

- $\beta$ is conditional expectation of (average of) value and $\beta$:
  \[
  \overline{\beta}(v) = \frac{1}{\sqrt{v}} \int_{x=0}^{v} x \frac{1}{2\sqrt{x}}\,dx = \frac{v}{3}
  \]

- Conditional Expectation with respect to $F(v)^{1/2} = v^{1/2}$.
- Compare to the bid $b(v) = v/2$, not even winning bid in EMW.
A lower bound on revenue

▶ Induced distribution of winning bids is $U[0, 1/3]$
▶ Revenue is $1/6$
▶ In fact, symmetry/deterministic winning bid are not needed
▶ Distribution of winning bid has to FOSD $U[0, 1/3]$ in all equilibria under any information
▶ $1/6$ is a *global* lower bound on equilibrium revenue
Bound is tight

- Can construct information/equilibrium that hits bound
- Bidders get i.i.d. signals $s_i \sim F(x) = \sqrt{x}$ on $[0, 1]$
- Value is highest signal
- Distribution of highest signal is $U[0, 1]$
- Equilibrium bid: $\sigma_i(s_i) = s_i/3 \ (= \beta(s_i))$
- Defer proof until general results
Beyond the example

- Argument generalizes to:
  - Any common value distribution!
    - Any number of bidders!
    - Arbitrarily correlated values!!!
- Assume symmetry of value distribution for some results
- Minimum bidding is characterized by a *deterministic winning bid* given the true values
- In general model, only depends on a one-dimensional statistic of the value profile
- Bound is characterized by binding *uniform upward incentive constraints*
The plan

- Detailed exposition of minimum bidding
- Maximum revenue/minimum bidder surplus
- Restrictions on information
- Other directions in welfare space (e.g., efficiency)
General model

- $N$ bidders
- Distribution of values: $P(dv_1, \ldots, dv_N)$
- Support of marginals $V = [\underline{v}, \overline{v}] \subseteq \mathbb{R}_+$
- An information structure $S$ consists of
  - A measurable space $S_i$ of signals for each player $i$, $S = \times_{i=1}^N S_i$
  - A conditional probability measure

$$\pi : V^N \rightarrow \Delta(S)$$
Equilibrium

- Bidders' strategies map signals to distributions over bids in $[0, \bar{v}]$

$$\sigma_i : S_i \rightarrow \Delta(B)$$

- Assume "weakly undominated strategies": bidder $i$ never bids strictly above the support of first-order beliefs about $v_i$

- Bidder $i$'s payoff given strategy profile $\sigma = (\sigma_1, \ldots, \sigma_N)$:

$$U_i(\sigma, S) = \int_{v \in V} \int_{s \in S} \int_{b \in B^N} (v_i - b_i) \left\{ \frac{\mathbb{1}_{\{b_i \geq b_j, \forall j\}}}{\arg \max_j b_j} \right\} \sigma(db|s)\pi(ds|v)P(dv)$$

- $\sigma$ is a Bayes Nash equilibrium if

$$U_i(\sigma, S) \geq U_i(\sigma'_i, \sigma_{-i}, S) \forall i, \sigma'_i$$
Other welfare outcomes

Bidder surplus: \( U(\sigma, S) = \sum_{i=1}^{N} U_i(\sigma, S) \)

Revenue: \( R(\sigma, S) = \int_{v \in V^N} \int_{s \in S} \int_{b \in B^N} \max_i b_i \sigma(b|s) \pi(ds|v) P(dv) \)

Total surplus: \( T(\sigma, S) = R(\sigma, S) + U(\sigma, S) \)

Efficient surplus: \( \overline{T} = \int_{v \in V} \max_i v_i P(dv) \)
General common values

As we generalize, minimum bidding continues to be characterized by a *deterministic winning bid* given values: $eta(v_1, \ldots, v_N)$

- $eta$ has an explicit formula
- Consider pure common values with $v \sim P \in \Delta([v, \overline{v}])$
- Minimum winning bid generalizes to

$$
\beta(v) = \frac{1}{\sqrt{P(v)}} \int_x^v x \frac{P(dx)}{2\sqrt{P(x)}}
$$

- $P(v)^{1/2}$ generalizes to $P(v)^{(N-1)/N}$ with $N$ bidders
- Minimum revenue:

$$
R = \int_{v=v}^{\overline{v}} \beta(v) P(dv)
$$
$N = 2$ and general value distributions

- Write $P(dv_1, dv_2)$ for value distribution
- Similarly, lots of binding uniform upward IC
- Incentive to deviate up depends on value when you lose
- On the whole, efficient allocation reduces gains from deviating up
- Suggests minimizing equilibrium is efficient, winning bid is constrained by loser’s (i.e., lowest) value
General bounds for $N = 2$

- Similar $\beta$, but now depends on lowest value
- $Q(dm)$ is distribution of $m = \min\{v_1, v_2\}$ (assume non-atomic)
- Minimum winning bid is
  \[
  \beta(m) = \frac{1}{\sqrt{Q(m)}} \int_{x=v}^{v} x \frac{Q(dx)}{2\sqrt{Q(x)}}
  \]
- Minimum revenue:
  \[
  R = \int_{m=v}^{\bar{v}} \beta(m) Q(dm)
  \]
Losing values when $N > 2$

- With $N > 2$, bid minimizing equilibrium should still be efficient
- Intuition: coarse information about losers’ values lowers revenue
- Consider complete information, all values are common knowledge
- High value bidder wins and pays second highest value
Average losing values

- Simple variation: Bidders only observe
  1. High value bidder’s identity
  2. Distribution of values

- Winner is still high value bidder, but losing bidders don’t know who has which value

- If prior is symmetric, believe they are equally likely to be at any point in the distribution except the highest

- In equilibrium, winner pays average of $N - 1$ lowest values:

$$\mu(v_1, \ldots, v_N) = \frac{1}{N - 1} \left( \sum_{i=1}^{N} v_i - \max_i v_i \right)$$
General bounds

- $Q(dm)$ is distribution of $m = \mu(v)$ (assume non-atomic)
- Minimum winning bid and revenue:
  $$\beta(m) = \frac{1}{Q^{\frac{N-1}{N}}(v)} \int_{x=v}^{\bar{v}} x \frac{N-1}{N} \frac{Q(dx)}{Q^{\frac{1}{N}}(x)}$$
  $$= \frac{1}{Q^{\frac{N-1}{N}}(v)} \int_{x=v}^{\bar{v}} x Q^{\frac{N-1}{N}}(dx)$$

- Minimum revenue:
  $$\bar{R} = \int_{m=v}^{\bar{v}} \beta(m) Q(dm)$$

- Let $H(b) = Q(\beta^{-1}(b))$
Main result

Theorem (Minimum Winning Bids)

1. In any equilibrium under any information structure in which the marginal distribution of values is $P$, the distribution of winning bids must first-order stochastically dominate $H$.

2. Moreover, there exists an information structure and an efficient equilibrium in which the distribution of winning bids is exactly $H$. 
Implications

Corollary (Minimum revenue)

*Minimum revenue over all information structures and equilibria is* $R$. 
Corollary (Minimum revenue)

Minimum revenue over all information structures and equilibria is $R$.

Corollary (Maximum bidder surplus)

Maximum total bidder surplus over all information structures and equilibria is $\bar{T} - R$. 
Proof methodology

1. Obtain a bound via relaxed program
2. Construct information and equilibrium that attain the bounds
   (start with #2)
Minimizing equilibrium and information

- Bidders receive independent signals $s_i \sim Q^{1/N}(s_i)$
  $\implies$ distribution of highest signal is $Q(s)$
- Signals are correlated with values s.t.
  - Highest signal is true average lowest value, i.e.,
    \[
    \mu(v_1, \ldots, v_n) = \max\{s_1, \ldots, s_n\}
    \]
  - Bidder with highest signal is also bidder with highest value, i.e.,
    \[
    \arg\max_i s_i \subseteq \arg\max_i v_i
    \]
- All bidders use the monotonic pure-strategy $\beta(s_i)$
Proof of equilibrium

- $\underline{\beta}$ is the equilibrium strategy for an “as-if” IPV model, in which $v_i = s_i$
- IC for IPV model with independent draws from $Q^{1/N}$:

$$
(s_i - \sigma(s_i))Q^{\frac{N-1}{N}}(s_i)
$$

- Local IC:

$$
(s_i - \sigma(s_i))Q^{\frac{N-1}{N}}(ds_i) - \sigma'(s_i)Q^{\frac{N-1}{N}}(s_i) = 0
$$

- Solution is precisely

$$
\sigma(s_i) = \frac{1}{Q^{\frac{N-1}{N}}(s_i)} \int_{x=v}^{s_i} x Q^{\frac{N-1}{N}}(dx) = \underline{\beta}(s_i)
$$
Proof of equilibrium

- $\beta$ is the equilibrium strategy for an “as-if” IPV model, in which $v_i = s_i$

- IC for IPV model with independent draws from $Q^{1/N}$:

$$ (s_i - \sigma(s_i)) Q^{\frac{N-1}{N}} (s_i) $$

- Local IC:

$$ (s_i - \sigma(s_i)) Q^{\frac{N-1}{N}} (ds_i) - \sigma'(s_i) Q^{\frac{N-1}{N}} (s_i) = 0 $$

- Solution is precisely

$$ \sigma(s_i) = \frac{1}{Q^{\frac{N-1}{N}} (s_i)} \int_{x=\nu}^{s_i} Q^{\frac{N-1}{N}} (dx) = \beta(s_i) $$
Proof of equilibrium

- $\beta$ is the equilibrium strategy for an “as-if” IPV model, in which $v_i = s_i$
- IC for IPV model with independent draws from $Q^{1/N}$:

$$ (s_i - \sigma(s_i))Q^{\frac{N-1}{N}}(s_i) \geq (s_i - \sigma(m))Q^{\frac{N-1}{N}}(m) $$

- Local IC:

$$ (s_i - \sigma(s_i))Q^{\frac{N-1}{N}}(ds_i) - \sigma'(s_i)Q^{\frac{N-1}{N}}(s_i) = 0 $$

- Solution is precisely

$$ \sigma(s_i) = \frac{1}{Q^{\frac{N-1}{N}}(s_i)} \int_{x=v}^{s_i} x Q^{\frac{N-1}{N}}(dx) = \beta(s_i) $$
Downward deviations

- Expectation of the bidder with the highest signal is $\tilde{v}(s_i) \geq s_i$
- Downward deviator obtains surplus

$$ (\tilde{v}(s_i) - \beta(m)) Q^{N-1 \over N} (m) $$

and

$$ (\tilde{v}(s_i) - \beta(m)) Q^{N-1 \over N} (dm) - \beta'(m) Q^{N-1 \over N} (m) \geq (s_i - \beta(m)) Q^{N-1 \over N} (dm) - \beta'(m) Q^{N-1 \over N} (m) $$

- Well-known that IPV surplus is single peaked: if $m < s_i$,

$$ \implies (s_i - \beta(m)) Q^{N-1 \over N} (dm) - \beta'(m) Q^{N-1 \over N} (dm) \geq 0 $$
Average losing values II

- Winning bids depend on avg of lowest values
  = average of losing bids (since equilibrium is efficient)

- Suppose winning bid in equilibrium is $\beta(m) > \beta(s_i)$
  $\implies \mu(v) = m$ for true values $v$

- By symmetry, all permutations of $v$ are in $\mu^{-1}(m)$ and equally likely

- If you only know that
  
  (i) you lose in equilibrium and
  (ii) $v \in \mu^{-1}(m)$,

  you expect your value to be $m$!

- By deviating up to win on this event, gain $m$ in surplus
Upward deviations

- Upward deviator’s surplus

\[(\tilde{v}(s_i) - \bar{\beta}(m))Q^{\frac{N-1}{N}}(s_i) + \int_{x=s_i}^{m} (x - \bar{\beta}(m))Q^{\frac{N-1}{N}}(dx)\]

- Derivative w.r.t. \(m\):

\[(m - \bar{\beta}(m))Q^{\frac{N-1}{N}}(dm) - \bar{\beta}(m)'Q^{\frac{N-1}{N}}(m) = 0!\]

- In effect, correlation between others bids’ and losing values induces adverse selection s.t. losing bidders are indifferent to deviating up
Towards a general bound

- Claim is that construction attains a lower bound
- Show this via relaxed program
- Minimum CDF of winning bids subject to uniform upward IC
- Key WLOG properties of solution (and minimizing equilibrium):
  1. Symmetry
  2. Winning bid depends on average losing value
  3. Efficiency
  4. Monotonicity of winning bids in losing values
  5. All uniform upward IC bind
Other directions

- We talked about max/min revenue, max/min bidder surplus
- What about weighted sums? Minimum efficiency?
- More broadly, what is the *whole set* of possible \((U, R)\) pairs?
- Solved numerically for two bidder i.i.d. \(U[0, 1]\) model
Welfare set

Note: Lower bound on efficiency
What can we do with this?

- Applications/extensions:
  - Many bidder limit
  - Impact of reserve prices/entry fees
  - Identification

Other directions in welfare space

- Context:
  - Part of a larger agenda on robust predictions and information design