Algorithms Homework - Day 1

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(1) How many ways are there to make change for $2 using some combination of pennies, nickels, dimes, and quarters?

(2) Suppose we have a 12-sided die (numbered 1 through 12) and a six-sided die (numbered 1 through 6).

   (a) How many ways are there to roll both dice and end up getting the same number?
   (b) How many ways are there to roll both dice and end up getting prime numbers on both of them?
   (c) (Trickier!) How many ways are there to roll both dice and end up getting a higher number on the six-sided die?

(3) Some chess questions!

   (a) How many legal chess moves does white have from the opening position?
   (b) How many different ways are there to start a chess game with a legal move for white followed by a legal move for black?
   (c) (Trickier!) Try to generalize this to white's second move.

(4) State a version of the addition principle involving k types of objects (with none of the types overlapping). Prove your statement (assuming the $k = 2$ case if you want).

(5) State a version of the multiplication principle involving k choices to be made. Prove your statement (assuming the $k = 2$ case if you want).

(6) Prove the inclusion/exclusion principle stated for three sets. Try to generalize this.

(7) A cool dice problem!

   (a) You have two standard 6-sided dice, one blue and one yellow. Each die has the numbers 1, 2, 3, 4, 5, 6. Make a table for all 36 possible outcomes that can happen if we roll your two dice.
   (b) For each integer k, how many ways are there for your dice to add up to exactly k?
   (c) Pat has two weird 6-sided dice, one red and one green. The red die has the numbers 1,2,2,3,3,4. The green die has the numbers 1,3,4,5,6,8. Make a table for all 36 possible outcomes that can happen if we roll Pat's two dice.
   (d) For each integer k, how many ways are there for Pat's dice to add up to exactly k?
   (e) Try to find other pairs of six-sided dice with this same cool property. :-)

(8) You are playing a game sort of like scrabble, and you have four tiles A, B, C, D. In the game, you pick any number of your tiles and you put them down to make a "word" like BAC or perhaps CB. Luckily, in this game **everything** that's at least one letter long counts as a valid word.

   (a) If we have 4 different letters, how many k-letter "words" (including nonsense "words") can we make? (Answer this question for $k = 1, 2, 3, 4$)
   (b) How many total "words" of length at least 1 could we make given 4 different letters?
   (c) Try to generalize this.

(9) How many ways are there to write the number 6 as a sum of three positive integers? (For instance, $2 + 2 + 2$ and $1 + 3 + 2$ should both count. Let's say that $1 + 3 + 2$ is considered different from $1 + 2 + 3$.) Try to generalize this.

(10) Suppose we have a deck of 100 cards labelled $1, 2, 3, \ldots, 100$, and we flip over 3 without replacement.

   (a) How many ways are there to flip over the three cards such that all three of the cards are odd numbers?
   (b) How many ways are there to flip over the three cards such that the first card is less than the second one, which is less than the third?
   (c) Try to generalize your answers.