Algorithms - Day 1

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Overview

Combinatorics—or more specifically *enumerative combinatorics*—is the mathematics of counting.

Example 1 Pat has 7 different colored shirts, and he has 5 different pairs of shorts. How many articles of clothing does he own? How many outfits can he make?

Or as perhaps a slightly more interesting example, consider the following problem.

Example 2 Pat has 3 red shirts and 4 green shirts, and he has 5 pairs of shorts. How many articles of clothing does he own? How many outfits can he make?

The last two examples already show us many of the key ideas in enumerative combinatorics, but perhaps the most important is the following.

Whenever you answer "how many different ways can we pick..." **you always need to know:**

(1) **how exactly are we picking our things?** (e.g., can we pick two shirts for the same outfit? can we use the same shirt in two different outfits?)

(2) **what counts as "different"?** (e.g., if two outfits look the same but they use different shirts, are they different? How close do colors have to be to call them "the same color"?)

Based on how you choose to answer these these, you'll get very different answers to very different counting problems! All of these questions give us interesting math problems, and we'll always go out of our way to say exactly what we mean.

Example 3 The previous example was too vague and could be interpreted in many different ways. Try to imagine an interpretation that makes the answer as small as possible and other that makes the answer as big as possible.

Two more key ideas that we've already seen from our examples are:

The Addition Principle: If we have a objects of type I and b objects of type II, and if none of the objects fits in both categories, then we have $a + b$ objects total.

The Multiplication Principle: If we first want to pick from among a total options and then we want to make a choice from among b total options, there are a total of $a \times b$ ways to do this.

Said in a mathier way, these are

- If two sets A and B have no elements in common, then $|A \cup B| = |A| + |B|$.
- For any two sets A and B, we have $|A \times B| = |A| \cdot |B|$.

Example 4 A pizza shop has 3 types of meat for toppings, and it also has 4 types of veggies.

(i) How many total toppings does the pizza shop have?

(ii) How many different types of pizza can be made that have one meat topping and one veggie topping?

Note! The addition and multiplication principles were stated above for only *two* sets, but they both generalize to any number of sets.[1]

Example 5 Pat has three cats and five different hats. He wants to give each cat one hat.

(a) How many ways can he do this if it's ok for some of the cats to share the same hat?

(b) How many ways can he do this if he must give a different hat to each cat?

[1]Hmm... How might you formally state versions of these principles for any number of sets? And how could you prove those more general versions...?

Example 6 Suppose a New Jersey license plate consists of two letters (A-Z) followed by 4 numbers (0-9).

(i) How many total possible license plates are there? (letters and numbers can be reused)

(ii) How many possible license plates are there that don't contain a repeated letter?

(iii) How many possible license plates are there that do contain a repeated letter?

Example 7 Given a standard deck of 52 cards, how many ways are there to pick …

(i) …a red card?

(ii) …a king?

(iii) …a king or a queen?

(iv) …a king or a red card?

The last example shows a weakness of the addition principle: the **categories involved cannot have anything in common**. But what if they do overlap?

Example 8 Pat has a lot of shirts in a lot of colors. Some of them have writing, and some don't.

- Pat has 100 total shirts
- Pat has 20 green shirts
- Pat has 15 shirts with writing
- Pat has 3 green shirts with writing on them

How many shirts does Pat have that are green OR have writing [or both]?

This leads to the following generalization of the addition principle.

Principle of Inclusion/Exclusion: For any sets A, B, we have $|A \cup B| = |A| + |B| - |A \cap B|$.

Proof:

There is also a generalization for more than two sets. For instance:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Question 9 How does the principle of inclusion/exclusion relate to the addition principle?

Usually there's more than one way to count the same thing, and that can be really fun!

Example 10 How many subsets are there of the set $\{1, 2, \ldots, n\}$?