

# Number theory (and more with integers)

## Putnam Seminar

### Week 6

Number theory is really way too broad. But here we go. Assume all the variables in the following are integers.

**Modular arithmetic:** When trying to prove things about divisibility, primes, or non-existence of solutions, reach for *modular arithmetic*. It's great.

**Quadratic residues:** Every odd square is one more than a multiple of 8. (Other useful moduli to think on are 3, 5, et cetera?) Similarly,  $x^3$  is 0,  $\pm 1$  mod 7.

**Fundamental theorem of arithmetic:** Prime factorization is unique.

**gcd:** Fact  $\gcd(a, b) = \gcd(a - kb, b)$ . Another fact:  $\gcd(x^a - 1, x^b - 1) = x^{\gcd(a, b)} - 1$ .

**Euler's theorem:** If  $\gcd(a, n) = 1$ , then  $a^{\varphi(n)} \equiv 1 \pmod{n}$ , where  $\varphi(n) = \#\{x \leq n : \gcd(x, n) = 1\}$ .

**Chinese remainder:** If you got it, go for it. No promises it'll help.

**“Simon’s favorite factoring trick”**  $xy + ax + by = (x + b)(y + a) - ab$

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- Find all integer solutions to  $2b + 3c = 5bc$
  - Prove that for any integer  $n > 1$ , the quantity  $2^n - 1$  is not a multiple of  $n$ .
1. *Twin primes* are primes of the form  $p$  and  $p + 2$ . *Conjoined* primes are of the form  $p$  and  $p + 1$ . *Prime triplets* are primes of the form  $p$ ,  $p + 2$ , and  $p + 4$ . Prove there are finitely many conjoined primes. Prove there are finitely many prime triplets.<sup>1</sup>
  2. Find all primes  $p$  such that  $p^2 + 2$  is also prime.
  3. How many zeroes does  $1000!$  end with?
  4. Find all integer solutions to  $mn + 3m - 8n = 59$ .
  5. (2015-A2) Let  $a_0 = 1$ ,  $a_1 = 2$ , and  $a_n = 4a_{n-1} - a_{n-2}$  for  $n \geq 2$ . Find an odd prime factor of  $a_{2015}$ .
  6. Find all primes that are one less than a perfect cube.
  7. Prove  $\lceil(\sqrt{8n - 1} + 1)/2\rceil = \lceil 1/2 + \sqrt{2n} \rceil$ , where  $\lceil x \rceil$  is the least integer greater than or equal to  $x$ .

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<sup>1</sup>I made up exactly one of these phrases.

8. Suppose  $N$  is some fixed positive integer, and suppose there are exactly 2005 ordered pairs  $(x, y)$  of positive integers satisfying

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}.$$

Prove that  $N$  is a perfect square.

9. (1989-A1) How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1's and 0's beginning and ending with 1?
10. (2001-B1) Let  $n$  be an even positive integer. Write the numbers  $1, 2, \dots, n^2$  in the squares of an  $n \times n$  grid so that the  $k^{\text{th}}$  row, from right to left is

$$(k-1)n+1, (k-1)n+2, \dots, (k-1)n+n.$$

Color the squares of the grid so that half the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each such coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

11. (2015-A2) Let  $a_0 = 1$ ,  $a_1 = 2$ , and  $a_n = 4a_{n-1} - a_{n-2}$  for  $n \geq 2$ . Find an odd prime factor of  $a_{2015}$ .
12. Prove that if  $n^4 + 4^n$  is prime, then the ones digit of  $n$  must be 5.
13. (1988-B1) Prove that every composite number (greater than 1) is expressible as  $xy + xz + yz + 1$  with  $x, y$ , and  $z$  positive integers.
14. (2018-A1) Find all ordered pairs  $(a, b)$  of positive integers where  $\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}$ .
15. (2011-B2) Let  $S$  be the set of ordered triples  $(p, q, r)$  of prime numbers for which at least one rational number  $x$  satisfies  $px^2 + qx + r = 0$ . Which primes appear in seven or more elements of  $S$ ?
16. (1992-A3) For a given positive integer  $m$ , find all triples  $(n, x, y)$  of positive integers, with  $n$  relatively prime to  $m$ , which satisfy  $(x^2 + y^2)^m = (xy)^n$ .
17. Prove there are infinitely many primes of the form  $4n + 3$ .
18. (2010-A4) Prove that for each positive integer  $n$ , the number

$$10^{10^{10^n}} + 10^{10^n} + 10^n - 1$$

is not prime.

19. (2003-B3) Show that for each positive integer  $n$ ,

$$n! = \prod_{i=1}^n \text{lcm}\{1, 2, \dots, \lfloor n/i \rfloor\},$$

where lcm denotes least common multiple, and  $\lfloor x \rfloor$  denotes the greatest integer  $\leq x$ .

20. (1994-B1) Find all positive integers that are within 250 of exactly 15 perfect squares.

21. (1997-B3) For each integer  $n$  write the sum  $\sum_{m=1}^n \frac{1}{m}$  in the form  $p_n/q_n$  where  $p_n$  and  $q_n$  are relatively prime positive integers. Determine all  $n$  such that 5 does not divide  $q_n$ .

22. Show there exist at least 2018 consecutive integers, each of which is divisible by the cube of some integer greater than 1.

23. (2002-A3) Let  $n \geq 2$  be an integer and  $T_n$  be the number of non-empty subsets  $S$  of  $\{1, 2, \dots, n\}$  with the property that the average of the elements of  $S$  is an integer. Prove that  $T_n - n$  is always even.

24.  $n$  prisoners are blindfolded and placed in a room. When they open their eyes, they see that every prisoner has a hat on their head, and this hat has an integer from 1 to  $n$  written on it (these need not be distinct). Each prisoner can see everybody's hat except for their own. Then they must each secretly write down a guess for what number is on their own hat, and they reveal all their guesses simultaneously. If *at least one* person guesses correctly, they all win (otherwise they all lose). They can coordinate and plan ahead of time, but they cannot communicate in any way once they see the hats. Is there a strategy that necessarily always works?

25. (1996-A5) If  $p$  is a prime number greater than 3, and  $k = \lfloor 2p/3 \rfloor$ , prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

is divisible by  $p^2$ .

26. Show that the sum of consecutive primes is never twice a prime.

27. (1998-B6) Prove that, for any integers  $a, b, c$ , there exists a positive integer  $n$  such that  $\sqrt{n^3 + an^2 + bn + c}$  is not an integer.

28. (IMO 1994) Prove that there exists a set  $A$  of positive integers with the property that for any infinite set  $S$  of primes, there exist two positive integers  $m \in A$  and  $n \notin A$  each of which is a product of  $k$  distinct elements of  $S$  for some  $k \geq 2$ .

29. Prove there are infinitely many primes of the form  $4k + 3$ .

30. For each integer positive integer  $n$ , prove that  $3^{3^n} + 1$  is the product of at least  $2n + 1$  not necessarily distinct primes.

31. Let  $p$  be a prime of the form  $p = 3k + 1$ , and let

$$\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(2k-1)2k} = \frac{m}{n},$$

for integers  $m, n$ . Prove  $p$  divides  $m$ .

32. Let  $p$  be a prime number. Prove there are infinitely many multiples of  $p$  whose last ten digits are all distinct.

33. Let  $A$  be the set of positive integers representable in the form  $a^2 + 2b^2$  for integers  $a, b$  with  $b \neq 0$ . If  $p^2 \in A$  for some prime  $p$ , show that  $p \in A$ .
34. (2006-A2) Alice and Bob play a game in which they take turns removing stones from a heap that initially has  $n$  stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many  $n$  such that Bob has a winning strategy. (For example, if  $n = 17$ , then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)
35. (2010-B2) Given that  $A$ ,  $B$ , and  $C$  are noncollinear points in the plane with integer coordinates such that the distances  $AB$ ,  $AC$ , and  $BC$  are integers, what is the smallest possible value of  $AB$ ?