

Pigeonhole principle*

Putnam Seminar

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The *pigeonhole principle* is a basic and super useful idea from combinatorics. It has three basic flavors (all called the pigeonhole principle):

- If there are n ‘pigeons’ placed into m ‘holes,’ then at least one hole has at least n/m pigeons. (and least one hole has *at most* n/m pigeons)
 - If $n > m$, then there’s a hole with at least 2 pigeons
- If there are infinitely many ‘pigeons’ placed into finitely many ‘holes,’ then at least one hole has infinitely many pigeons.

All are based on the *hyper-useful* fact (S being “how many pigeons in each hole”):

- If S is a finite set of real numbers, then $\min(S) \leq \text{average}(S) \leq \max(S)$.

The hardest part is always figuring out what the “pigeons” and “holes” should be.

Some examples:

- Suppose I pick a thirty word passage from the Count of Monte Cristo.¹ Prove that at least two words of that passage begin with the same letter.
- Prove that there are at least nine Yale students who all have the same first and last initials. (Pro-tip: Yale has at least 5700 undergrads)
- Suppose a_1, a_2, \dots, a_n is a sequence of n (not necessarily distinct) integers. There are values $k \leq l \leq n$ such that $a_k + a_{k+1} + \dots + a_l$ is divisible by n .
- (**Dirichlet’s theorem**) For all $x \in \mathbb{R}$ and for all positive integers N , there is some rational number p/q with $1 \leq q \leq N$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{qN} \leq \frac{1}{q^2}.$$

I.e., every real can be well-approximated by a rational with a small denominator. This implies for all $x \in \mathbb{R} \setminus \mathbb{Q}$, there are infinitely many rationals p/q for which $|x - p/q| < 1/q^2$. (This is deep. Google “rational approximation”)

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1. There are 17 points inside an equilateral triangle with side lengths 1. Prove there are at least two points within distance $1/4$ of each other.
 2. Given 51 distinct positive integers strictly less than 100, prove that some two of them sum to 99.

*Some material from *Putnam and Beyond* by Razvan Gelca and Titu Andreescu (section 1.3)

¹This is one of Pat’s favorite books.

3. Given any five points on a sphere, show that some four of them must lie on a closed hemisphere. (Putnam 2002)
4. Suppose every point in the plane is colored with one of three colors. Prove that there exists a rectangle in the plane such that all four of its vertices are the same color.
5. Given nine points inside the unit square, prove that some three of them form a triangle whose area does not exceed $1/8$.
6. For each integer $n \geq 1$, there exists a positive multiple of n consisting only of the digits 0 and 5.
7. Show there is a positive Fibonacci number that is divisible by 1000.
8. A circle is divided into 432 congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored Red, some 108 points are colored Green, some 108 points are colored Blue, and the remaining 108 points are colored Yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.
9. Prove that the decimal representation of any irrational number has at least two digits appearing infinitely often.
10. You have 200 monkeys placed in 101 spaceships such that each spaceship contains at least one monkey. Prove there is a subset of spaceships containing a total of exactly 100 monkeys.
11. A math student practices for the Putnam because it's fun. She does at least one problem every day, but she decided never to do more than 12 in any week. Prove that after 1000 days of this, there is some group of consecutive days in which she did exactly 20 problems.
12. Inside a circle of radius 4 are chosen 61 points. Show that among them there are two at distance at most $\sqrt{2}$ from each other.
13. Prove that every set of 10 two-digit natural numbers has two disjoint subsets with the same sum of elements. (IMO 1972)
14. (Putnam 2006) Prove that for every set $X = \{x_1, x_2, \dots, x_n\}$ of n real numbers, there exists a nonempty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$
15. Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 26, prove that M has at least one subset of four distinct elements whose product is the fourth power of an integer. (IMO 1985)
16. Prove that among any $2m+1$ distinct integers of absolute value less than or equal to $2m-1$, there are three whose sum is zero.
17. Let P_1, \dots, P_{2n} be a permutation of the vertices of a regular polygon. Prove the closed polygonal line $P_1P_2P_3 \dots P_{2n}$ contains a pair of parallel segments.