Integration Review: This knowledge is INTEGRAL to your success.

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Basics: Antiderivative: $\int a x^{n-1}\,dx = a\int x^{n-1}dx = \frac{a}{n}x^n$

Trig functions - consult packet!

U-substitution: frequently used when derivative of u can replace other terms in the eqn.

$\int \frac{1}{x}dx = \ln x$

Ie: $\int 2x(x^2+3)\,dx$, $u = x^2+3$, $du = 2x\,dx$

$$\int 2x(u)\cdot\frac{1}{2x}du = \int u\,du = \frac{1}{2}u^2 = \frac{1}{2}(x^2+3)^2$$

Partial Fraction Decomp:

$$\frac{1}{x^2(x^2+4)(x^2-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4} + \frac{E}{x+1} + \frac{F}{x-1}.$$

Therefore, integration leads to decomp, which is simple to solve.

Integration by Parts

① $\int u\,dv = uv - \int v\,du$

IE: $\int x\ln x$, $dv = x$, $v = \frac{1}{2}x^2$, $u = \ln x$, $du = \frac{1}{x}$

$= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2\cdot\frac{1}{x}$

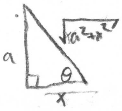
$= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$

Rapid integration: for $\int u\,dv$

derive ↓ u, u', u'' ⋮ ; dv ↓ integrate, v, $\int v$ ⋮ (+, −, +, −)

Trig Sub: in form of $\int \frac{1}{\sqrt{x^2 \pm a^2}}dx$

① Use right triangle: ie,



find x in terms of θ. $\tan\theta = \frac{a}{x}$

② Derive: ie, $x = a\cot\theta$, $dx = -a\csc\theta\,d\theta$

③ Substitute x for θ in integral and solve. Don't forget to sub $dx \to d\theta$

④ Integrate the trig function

⑤ Replace w/ x.

Completing the Square: in form of $\int \frac{x}{\sqrt{Ax^2+Bx+C}}dx$ where $Ax^2+Bx+C \neq (Dx+E)^2$

① Typically, results in $\int \frac{x}{\sqrt{(\sqrt{a}x+\frac{B}{2})^2 \pm (C-(\frac{B}{2})^2)}}$

Afterwards, use u sub to make of form $\int \frac{1}{\sqrt{u^2 \pm D^2}}$

IE: $\int \frac{x}{\sqrt{3-2x-x^2}}dx = \int \frac{x}{\sqrt{(x-1)^2+2}}dx$, $(x-1) = u$, $du = dx$

$= \int \frac{u+1}{\sqrt{u^2+2}}du = \int \frac{u}{\sqrt{u^2+2}}du + \int \frac{1}{\sqrt{u^2+2}}$ ← trig sub.

Strategies - look for derivatives

Multiply by 1 $\left(\frac{\sin\theta}{\sin\theta}\right)$

Use trig subs: $\sin^2+\cos^2 = 1$, $1+\sec^2 = \tan^2$

Sequences + Series - I seriesly hope you do well!

$\boxed{\text{Sequence}}$ $\{U_n\}$, individual #s. Series is $\sum_{i=1}^{N}\{U_i\}$

Infinite sequence: $\lim_{n\to\infty} U_N$. Ie: $U_n = \frac{1}{n^2}$, $\lim_{n\to\infty}\frac{1}{n^2} = 0$

$U_n = \frac{2n^2}{n^2+n+1}$, $\lim_{n\to\infty}\frac{2\frac{n^2}{n^2}}{\frac{n^2}{n^2}+\frac{n}{n^2}+\frac{1}{n^2}} = \lim_{n\to\infty}\frac{2}{1+\frac{1}{n}+\frac{1}{n^2}} = 2$

L'Hospital: $\lim_{n\to\infty}\frac{f(n)}{g(n)}$ = undef, $\lim_{n\to\infty}\frac{f'(n)}{g'(n)}$.

$U_n = \frac{3n^3}{2n^2+4n}$, $\lim_{n\to\infty}\frac{3n}{2/n+4/n^2} = \infty$

Every bounded monotonically decreasing sequence will converge.

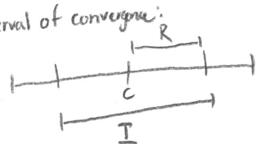
$\boxed{\text{Infinite Series}}$ - Tests for convergence / Divergence.

- consult table... quick summary.

Power Series $\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + \dots a_n(x-c)^n + \dots$

↑ Power series centered at C.

Interval of convergence:



Determine I by ratio test. Use sub to find end points.

IE $\sum_{n=1}^{\infty}\frac{x^n}{n}$, $\lim_{n\to\infty}\left|\frac{x\cdot x^n}{n+1}\cdot\frac{n}{x^n}\right| = \lim_{n\to\infty}|x|\cdot\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = |x|$

∴, $|x| < 1$ allows $\frac{x^n}{n}$ to converge, $r = I$

If $x = 1$, p-series - divergent

$x = -1$, alt harmonic, converges

Such that I: $x \in [-1, 1[$

Geo series: if $U_n = (r)^n$, $|r| < 1$, $\lim_{n\to\infty} U_n = 0$; $|r| > 1$, $\lim_{n\to\infty} = \infty$

$|r| = 1$, $\lim_{n\to\infty} = 1$ (duh)

Abs value theorem: $\lim_{n\to\infty}|U_N| = \lim_{n\to\infty}U_N$

Squeeze Theorem: If $A_n \le U_N \le B_N$ and $\lim_{n\to\infty}A_n = \lim_{n\to\infty}B_n = C$, $\lim_{n\to\infty}U_n = C$

All found in table!

(1.5) Geo series: if $|r| < 1$, $\sum_{n=1}^{\infty}a_0 r^n = \frac{a_0}{1-r}$

① If $\{U_N\} \neq 0$, $\sum_{n=1}^{\infty}U_N$ diverge.

② P-series: $\sum_{n=1}^{\infty}\frac{1}{n^p}$, if $p \le 1$, diverge; $p > 1$, converge.

③ Integral Test: iff $\lim_{a\to\infty}\int_1^a f(x)$ diverge, then $\sum_{x=1}^{\infty}f(x)$ diverge.

④ Comparison test: if $U_N < V_N$, if V_N converge, U_N converg; if U_N diverge, V_N diverge.

⑤ Limit Comparison test: if $\lim_{n\to\infty}\frac{U_N}{V_N} = L$, both series have same behavior.

" " $= 0$, if V_N converge, U_N converge

" " $= \infty$, if V_N diverge, U_N diverge.

⑥ Ratio Test: if $\lim_{n\to\infty}\left|\frac{U_{N+1}}{U_N}\right| < 1$, converge; $= 1$, inconclusive.

" " > 1, diverge

⑦ Alternating Series: if $\lim_{n\to\infty}U_N = 0$, $U_{N+1} \le U_N$ for all N, $(-1)^n U_N$ converges.

Remainder after n^{th} term $= U_{N+1}$

If U_N also converges, absolute convergence

If U_N diverges, conditional convergence

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Taylor and Maclaurin Series

↳ forms to model all sorts of functions.

Taylor: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}c}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \dots$

Maclaurin: $f(x) = \frac{f^n(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots \frac{f^n(0)}{n!}x^n$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^{(n+1)}}{(n+1)!}$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \frac{(-1)^{n+1}x^{n+1}}{n+1}$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots \frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!}$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!}$

$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \dots + \frac{(-1)^n x^{2n-1}}{(2n-1)}$

You can stack functions inside Taylor series.

Watch out for telescoping series to simplify!

Adding/dividing series can lead to very interesting results.

Differential Equations

Ordinary differential equations -

Order = highest derivative

Linearity: if coefficients of derivatives of y are at most dependent on x.

how to solve: seperate your variables! Afterwards, integrate both sides.

Homogenous: if $f(tx, ty) = t^n(f(x,y))$

Homogenous First-order Diffy Q:

$m(x,y)\,dx + N(x,y)\,dy = 0$

This will be solved by $y = ux$

$dy = u\,dx + x\,du$

IE: $(x^2 - y^2)\,dx + 3xy\,dy = 0$

$(x^2 - x^2u^2)\,dx + 3x(ux)(u\,dx + x\,du) = 0$

↳ solve and separate!

First Order Differential Equations using Integrating Factor.

① $\frac{dy}{dx} + P(x)y = f(x)$ → Standard Form

② Integrating factor $= e^{\int P(x)dx}$

③ Multiply ①·②. The LHS = ②·y

so that ②·$y = \int f(x) \cdot$ ② dx

Solve. Don't forget about $+\underline{C}$! ← May need to find exact values.

Euler's Method - fill out this table!

n	x_n	y_n	$hF(x_n, y_n)$	(x_n+h) x_{n+1}	y_{n+1}
0					
1					
2					
3					
4					
5					
6					
⋮					