1. (a) In mathematical optimization problems, Lagrange multipliers, named after Joseph Louis Lagrange, is a method for finding the local extrema of a function of several variables subject to one or more constraints. This method reduces a problem in \( n \) variables with \( k \) constraints to a solvable problem in \( n + k \) variables with no constraints. The method introduces a new unknown scalar variable, the Lagrange multiplier, for each constraint and forms a linear combination involving the multipliers as coefficients.

For example, we are looking for a solution of the following problem

\[
\max_{x \in \mathbb{R}} f(x)
\]

subject to

\[
x - \bar{x} \geq 0.
\]

We approach this problem by associating a Lagrange multiplier to the constraint, \( \lambda \in \mathbb{R}_+ \), and define a function on \( x \) and \( \lambda \), called the Lagrangian function, \( \mathcal{L} \):

\[
\mathcal{L}(x, \lambda) = f(x) + \lambda (\bar{x} - x).
\]

We now have a new function, in two \((x, \lambda)\) rather than one \((x)\) variable. We now maximize the Lagrangian with respect to \( x \) (and as it turns out also minimize it with respect to \( \lambda \)):

\[
\max_x \mathcal{L}(x, \lambda).
\]

We now look at the first order conditions of the unconstrained problem and find the first order condition

\[
f'(x^*) - \lambda x^* = 0
\]

and the auxiliary complementary slackness constraint:

\[
\lambda (\bar{x} - x^*) = 0.
\]

If \( x^* \) is a corner solution with \( x^* = \bar{x} \), then \( \lambda \geq 0 \) and we can interpret \( \lambda \) as the price (of violating the constraint).
(b) The *expected utility* of a decision maker with utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ for a lottery $L = \{c_1, \ldots, c_N; p_1, \ldots, p_N\}$ is defined by

$$U(L) = \sum_{n=1}^{N} p_n u(c_n).$$

Note the difference between the upper-case $U$ in $U(L)$ and the lower-case $u$ in $u(c_n)$; they are two different functions. We use expected utility to evaluate decisions under uncertainty. Decision makers with a given utility function choose the lottery with the highest expected utility.

2. (a) Nina’s expected utility function is:

$$Eu^n = .9 \ln(10000 + x^n_Y) + .1 \ln(10000 + x^n_M)$$

Pete’s expected utility function is:

$$Eu^p = .3 \ln(10000 + x^p_Y) + .7 \ln(10000 + x^p_M)$$

(b) Nina’s budget constraint is:

$$px^n_Y + (1 - p)x^n_M \leq 0$$

Pete’s budget constraint is:

$$px^p_Y + (1 - p)x^p_M \leq 0$$

Note that we are assuming here that the only way to buy Yankees betting slips is to sell Marlins betting slips; betting slips cannot be exchanged for dollars before the World Series happens. This assumption implies that the net supply of betting slips is 0, as required in part d. Alternatively, one could assume that it is possible to exchange dollars for betting slips, in which case one dollar is effectively equivalent to the combination of a Yankees and a Marlins betting slip. In this case, the net supply of betting slips of both types is 20000, Nina’s budget constraint is $px^n_Y + (1 - p)x^n_M \leq 10,000$, and Nina’s expected utility is $Eu^n = .9 \ln(x^n_Y) + .1 \ln(x^n_M)$, with Pete’s budget constraint and expected utility altered in a similar fashion.

(c) Setting her marginal rate of substitution between Yankees and Marlins betting slips equal to the ratio of the price of Yankees betting slips to Marlins betting slips, Nina solves:
\[ \frac{\partial E u^n}{\partial x^n_Y} = p \]
\[ \frac{\partial E u^n}{\partial x^n_M} = \frac{p}{1 - p} \]
\[ \frac{1}{10,000 + x^n_Y} = \frac{p}{1 - p} \]
\[ .9(1 - p)(10000 + x^n_Y) = .1p(10000 + x^n_Y) \]

Plugging this into the budget constraint and rearranging yields Nina’s demand functions:

\[ x^n_Y = \frac{9000}{p} - 10,000 \]
\[ x^n_M = \frac{1000}{1 - p} - 10,000 \]

Following the same steps for Pete yields Pete’s demand functions:

\[ x^p_Y = \frac{3000}{p} - 10,000 \]
\[ x^p_M = \frac{7000}{1 - p} - 10,000 \]

(d) A competitive equilibrium is a set of allocations \( \{x^n_Y, x^n_M, x^p_Y, x^p_M\} \) and a price \( p \) such that the allocations are the solutions to the demand functions above at price \( p \) and such that \( x^n_M + x^p_M = 0 \) and \( x^n_Y + x^p_Y = 0 \).

(e) Plugging the demand functions for Yankees tickets from question 2c into the zero net supply condition yields:

\[ \frac{9000}{p} - 10000 + \frac{3000}{p} - 10000 = 0 \]

Solving for \( p \) yields \( p = 3/5 \).

(f) Plugging the equilibrium price from part 2e into the demand functions from part 2c yields:

\[ x^n_Y = 5000 \]
\[ x^n_M = -7500 \]
\[ x^p_Y = -5000 \]
\[ x^p_M = 7500 \]
3. (a) An allocation \((b_A, c_A, b_B, c_B)\) is feasible if it is weakly positive and
\[
\begin{align*}
  b_A + b_B & \leq b = 2; \\
  c_A + c_B & \leq c = 1.
\end{align*}
\]

(b) See figure 1 for graphical illustration. The indifference curves are drawn under the assumption that \(\gamma = 2\).

![Edgeworth Box and Indifference Curves](image)

(c) At Pareto efficient allocation, there is no change in allocation of goods that may result in some individuals being made better off with no individual being made worse off. For this economy, an allocation \((b_A, c_A, b_B, c_B)\) is Pareto efficient if it is feasible, and there is no change of allocation that could make one individual better off without making the other one worse off.

(d) Analytically, Pareto efficient allocations and the set of welfare maximizing allocations across all possible vectors of weights are identical. For every weight \(\alpha\), we need to find the allocation which would maximize the social surplus given the weights \((\alpha, 1 - \alpha)\); in other words, we are interested in finding the allocation \((b_A, c_A, b_B, c_B)\) which maximizes the sum
\[
\alpha u_A(b_A, c_A) + (1 - \alpha)u_B(b_B, c_B)
\]
subject to the resource constraints of the economy.
When \(\alpha = 0\) or \(1\), the Pareto efficient allocation is to allocate all the goods to Bob or Ann. Suppose \(\alpha \in (0, 1)\). The maximization problem
is
\[
\max_{b_A,c_A,b_B,c_B} \quad \alpha (\ln b_A + \ln c_A) + (1 - \alpha) (\gamma \ln b_B + \ln c_B)
\]
\[
\text{s.t.} \quad b_A + b_B = 2
\]
\[
c_A + c_B = 1.
\]

The corresponding lagrange \( L(b_A, c_A, b_B, c_B, \lambda_b, \lambda_c) \) is
\[
\alpha (\ln b_A + \ln c_A) + (1 - \alpha) (\gamma \ln b_B + \ln c_B)
\]
\[
- \lambda_b(b_A + b_B - 2) - \lambda_c(c_A + c_B - 1).
\]

The first order conditions are
\[
\frac{\alpha}{b_A} = \lambda_b
\]
\[
\frac{\alpha}{c_A} = \lambda_c
\]
\[
\frac{\gamma(1 - \alpha)}{b_B} = \lambda_b
\]
\[
\frac{1 - \alpha}{c_B} = \lambda_c
\]
\[
b_A + b_B = 2
\]
\[
c_A + c_B = 1.
\]

Solving the FOCs, we get
\[
\frac{\alpha}{b_A} = \frac{\alpha}{2 - b_B} = \frac{\gamma(1 - \alpha)}{b_B}
\]
\[
\frac{\alpha}{c_A} = \frac{\alpha}{1 - c_B} = \frac{1 - \alpha}{b_B},
\]

which implies that
\[
b_B = \frac{2\gamma t}{1 + \gamma t}
\]
\[
c_B = \frac{t}{1 + t}
\]
\[
b_A = 2 - b_B
\]
\[
c_A = 1 - c_B,
\]

where \( t = \frac{1 - \alpha}{\alpha} \).

Equivalently, we can write down the Pareto efficient allocations as
\[
b_B = \frac{2\gamma c_B}{1 + (\gamma - 1)c_B}, \quad c_B \in [0, 1].
\]
\[
b_A = 2 - b_B
\]
\[
c_A = 1 - c_B.
\]
Alternatively, we can solve for Pareto efficient allocations by setting MRS of Ann to be equal to MRS of Bob. The MRS of Ann at \((b_A, c_A)\) is
\[
\frac{\partial u_A}{\partial b_A} = \frac{\partial u_A}{\partial c_A} = \frac{1}{b_A} = \frac{c_A}{b_A}.
\]
The MRS of Bob at \((b_B, c_B)\) is
\[
\frac{\partial u_B}{\partial b_B} = \frac{\partial u_B}{\partial c_B} = \frac{\gamma}{c_B} = \frac{\gamma c_B}{b_B}.
\]
Setting \(MRS_A = MRS_B\) gives the same results.

See figure 2 for graphical illustration. Again, we assume that \(\gamma = 2\) when drawing the picture.

Figure 2: Pareto Efficient Allocations

(e) If \(\alpha = 1/2\), the corresponding lagrange \(L(b_A, c_A, b_B, c_B, \lambda_b, \lambda_c)\) is
\[
\frac{1}{2} (lnb_A + lnc_A) + \frac{1}{2} (\gamma lnb_B + lnc_B) - \lambda_b(b_A + b_B - 2) - \lambda_c(c_A + c_B - 1).
\]
The Pareto allocation is
\[
(b_A, c_A, b_B, c_B) = \left(\frac{2}{1 + \gamma}, \frac{1}{1 + \gamma}, \frac{2\gamma}{1 + \gamma}, \frac{1}{2}\right).
\]
As \(\gamma\) increases, \(b_B\) rises and \(b_A\) falls. This is because more weights are put on \(b_B\) by Bob. Since the weights in front of utility functions are kept fixed, increasing \(\gamma\) leads to more bread to be allocated to Bob.