1. (a) Graphically, the demand functions look like this:

\[ q = \frac{k_a}{p - a} \]

\[ q = \frac{k_R}{p - 3a} \]

Elasticity is defined as \( \varepsilon = \frac{dq}{dp} \) so \( \varepsilon_A = -\frac{pA}{b \cdot q_A} = -\frac{pA}{a - pA} \) and \( \varepsilon_R = -\frac{3pR}{b \cdot q_R} = -\frac{3pR}{a - 3pR} \). It is the case that for any \( p < \frac{a}{3} \), \( |\varepsilon_R| > |\varepsilon_A| \). Namely, the demand of the retirees is more elastic than that of the adults.

(b) Graphically, the aggregate demand can be found by taking a horizontal sum of the two segment demands.

The resulting demand function is kinked where \( p = \frac{a}{3} \) because for prices lower than that, the retirees demand something but demand
nothing for prices that are higher. Algebraically, to find the aggregate demand function we would take the inverse demand functions that are given to us and invert them into demand functions by solving for $q$ and then sum them up. So our demand functions are $q_A = \frac{a - p}{b}$ and $q_R = \frac{a - 3p}{b}$ and so our aggregate demand is $q_{A+R} = \frac{2a - 4p}{b}$ for $p < \frac{a}{3}$ because for prices higher than that, this aggregate demand function allows $q_R$ to be negative. For $p \geq \frac{a}{3}$, $q_{A+B} = q_A = \frac{a - p}{b}$.

So we have two pieces to our aggregate demand function so let us start by assuming that the optimal choice for the monopolist is on the $q_{A+R} = \frac{2a - 4p}{b}$ portion. Then we want to solve $\max_p p \frac{2a - 4p}{b}$, which yields the FOC $2a - 8p = 0$ or $p = \frac{a}{4} < \frac{a}{3}$ so the demand function that we assumed is in fact relevant for the price we found. At this price $q_A = \frac{3a}{2b}$, $q_R = 0$ and $q_{A+B} = \frac{a}{4}$. This implies that $\pi = \frac{a^2}{16}$.

Alternatively, we could assume that the optimal choice for the monopolist is on the $q_{A+B} = q_A = \frac{a - p}{b}$ portion of the aggregate demand function. Then we want to solve $\max_p p \frac{a - p}{b}$, which yields the FOC $a - 2p = 0$ or $p = \frac{a}{2} \geq \frac{a}{3}$ so again the demand function that we assumed is in fact relevant for the price we found. At this price $q_A = \frac{a}{2b}$, $q_R = 0$ and $q_{A+B} = \frac{a}{2b}$. This also implies that $\pi = \frac{a^2}{8}$. Since the profits are equal, we have two optimal choices for the monopolist. One where it sets a low price and sells to both adults and retirees, though adults make up three quarters of demand, and one where it sets a high price and sells only to the adults.

(c) In the above question, when we assumed that the aggregate demand is given by $q_{A+B} = q_A = \frac{a - p}{b}$, in effect we found the optimal price that the monopolist should charge to the adults, which was $p_A = \frac{a}{2}$. To find the optimal price for the retirees, we want to solve $\max_p p \frac{a - 3p}{b}$, which yields the FOC $a - 6p = 0$ or $p = \frac{a}{6}$.

(d) Consumer surplus is defined as the area between the demand curve and the price line for all the quantities that are consumed. Mathematically, $CS = \int_0^{q^*} p(q) - p^* dq$, where $q^*$ and $p^*$ denote the quantity demanded and market price, respectively. But since we have linear demand functions, the area between the demand curve and price line is a right triangle with a width of $q^*$ and a height of $p(0) - p^*$. So we can calculate consumer surplus by finding the area of this triangle, which is given by $q^*(p(0) - p^*)/2$.

Under a common price of $\frac{a}{4}$, $CS_A = \frac{3a}{6b}(a - \frac{a}{4})/2 = \frac{9a^2}{32b}$, $CS_R = \frac{a}{2b}(\frac{a}{3} - \frac{a}{4})/2 = \frac{a^2}{24b}$ and $\pi = \frac{a^2}{18}$. Under a common price of $\frac{a}{2}$, $CS_A = \frac{a}{2b}(a - \frac{a}{2})/2 = \frac{a^2}{8b}$, $CS_R = 0$ and $\pi = \frac{a^2}{16}$. Under third degree price discrimination, the adults face a price of $\frac{a}{2}$ so $CS_A = \frac{a^2}{8b}$. The retirees face a price of $\frac{a}{6}$ and demand $\frac{a}{2b}$ so $CS_R = \frac{a}{2b}(\frac{a}{3} - \frac{a}{6})/2 = \frac{a^2}{24b}$ and $\pi = \frac{a}{2b} + \frac{a}{6} \frac{a}{2b} = \frac{a^2}{3b}$.
So going from a common price of $\frac{a}{2}$ to third degree price discrimination makes the adults worse off and the retirees and firm better off. This is because the price that adults face increases but the price for retirees decreases. Allowing for price discrimination generally helps the firm because it allows them to extract more surplus from consumers. Going from a common price of $\frac{a}{2}$ to third degree price discrimination makes the adults no better or worse off and the retirees and firm better off. This is because the price for the adults remains the same but the price for retirees decreases to something they can afford.

2. Under collusion, the $I$ firms act together as if they were a single monopolist. Hence, their profit maximization problem is

$$\max_q (a - bq) q - cq,$$

where $q$ is total supply, $q = \sum_{i=1}^I q_i$. The first-order condition is

$$a - 2bq - c = 0.$$ 

This leads to the profit-maximizing quantity

$$q^{\text{mono}} = \frac{a - c}{2b}.$$ 

Assuming that each firm produces the same quantity under the collusive agreement, each firm produces

$$q_i^{\text{coll}} = \frac{1}{I} \frac{a - c}{2b}.$$ 

For $I > 1$, $q_i^{\text{coll}}$ is smaller than the optimal quantity in the Cournot game with $I$ firms:

$$q_i^{\text{Cour}} = \frac{a - c}{(I + 1)b}.$$ 

When the firms are able to collude, they can keep supply down in order to increase the price. This is the same reasoning as in the monopolist’s problem. In the Cournot game, however, firms cannot commit to a quantity that is different from their best response.

3. (a) We derive the best response function by maximizing firm $i$’s profit

$$\max_{q_i} [a - b (q_i + q_j)] q_i - cq_i,$$

and taking first-order conditions

$$a - b (q_i + q_j) - bq_i - c = 0.$$ 

Rearranging, this yields the best response function

$$BR_i(q_j) = \frac{a - bq_j - c}{2b}.$$ 

3
(b) i. See Figure 1.

ii. To obtain the limit of the two firm’s best response functions, we start with the first few elements of the series. The first element is simply two arbitrary quantities:

\[(q_1^0, q_2^0).\]

The second element is obtained by plugging the first element into the best response functions:

\[ (q_1^1, q_2^1) = \left( \frac{a - c}{2b} - \frac{q_2^1}{2}, \frac{a - c}{2b} - \frac{q_1^1}{2} \right). \]

For the third element, we plug in again and simplify:

\[ (q_1^2, q_2^2) = \left( \frac{a - c}{2b} - \frac{q_2^1}{2}, \frac{a - c}{2b} - \frac{q_1^1}{2} \right) = \left( \frac{a - c}{2b} - \frac{1}{2} \left( \frac{a - c}{2b} - \frac{q_1^0}{2} \right), \frac{a - c}{2b} - \frac{1}{2} \left( \frac{a - c}{2b} - \frac{q_2^0}{2} \right) \right) = \left( \frac{a - c}{4b} + \frac{q_1^0}{4}, \frac{a - c}{4b} + \frac{q_2^0}{4} \right). \]
Similarly, we get the next two elements of the series:

\[(q_1^3, q_2^3) = \left( \frac{3a - c}{8} - \frac{q_2^0}{8}, \frac{3a - c}{8} - \frac{q_2^0}{8} \right)\]

and

\[(q_1^4, q_2^4) = \left( \frac{5}{16} \frac{a - c}{b} + \frac{q_1^0}{16}, \frac{5}{16} \frac{a - c}{b} + \frac{q_1^0}{16} \right).

Now, we see the pattern:

\[(q_k^1, q_k^2) = \begin{cases} 
\left( A \frac{a - c}{b} + \frac{q_0^k}{B}, A \frac{a - c}{b} + \frac{q_0^k}{B} \right) & \text{if } k \text{ is even} \\
\left( C \frac{a - c}{b} - \frac{q_0^k}{D}, C \frac{a - c}{b} - \frac{q_0^k}{D} \right) & \text{if } k \text{ is odd}
\end{cases},
\]

where \(A\) is the series \(\frac{1}{4}, \frac{1}{4} + \frac{1}{16}, \frac{1}{4} + \frac{1}{16} + \frac{1}{32}, \ldots\) and \(C\) is the series \(\frac{1}{2}, \frac{1}{2} - \frac{1}{8}, \frac{1}{2} - \frac{1}{8} - \frac{1}{32}, \ldots\), both of which converge to \(\frac{1}{3}\). Finally, \(B = D = 2^k\). Hence, we have shown that

\[
\lim_{k \to \infty} (q_k^1, q_k^2) = \left( \frac{1}{3} \frac{a - c}{b}, \frac{1}{3} \frac{a - c}{b} \right).
\]

4. (a) As the demand function is not continuous, calculus is not helpful in solving the problem. We will analyze the problem by considering how firm 1 maximizes its profit given \(p_2\), and then do the same for firm 2 given \(p_1\). First consider a small \(p_2\) such that \(p_2 < c\). If firm 1 sets \(p_1 > p_2\), it will not sell any goods and earns a profit of 0. However, if the firm sets \(p_1 \leq p_2\), it would sell a positive quantity of goods but its profit would be negative. Therefore the best response is to set any price \(p_1 > p_2\). If \(p_2 = c\), by a similar argument, the best response is to set some price \(p_1 \geq p_2 = c\).

Now we turn to the case where \(p_2 > c\). If \(p_1 > p_2\), then firm 1 has no sales and thus no profit. However, as \(p_2 > c\), firm 1 can always find a price \(p_1 \leq p_2\) that earns a positive profit. Provided firm 2 does not charge above the monopoly price, notice that if firm 1 undercut \(p_2\) by small amount \(\varepsilon > 0\), it can capture the entire market. Therefore the profit maximizing response for firm 1 given \(p_2 > c\) is to charge the monopoly price \(p^M = \frac{A - c}{2B}\) if \(p_2 > p^M\), and otherwise to undercut \(p_2\) by a little bit.

In this model, firm 2 is symmetric to firm 1, so firm 2 has a symmetric best response correspondence as firm 1. The only point where the prices of the firms are mutual best responses to each other is at \(p_1 = p_2 = c\).

(b) The duopoly model where firms set their quantity is called the Cournot model. The above model where firms set their prices is known as the Bertrand model. The technical distinction between the models is that a firm in the Cournot model faces a continuous demand curve.
whereas a firm in the Bertrand model faces a discontinuous demand curve (with an infinite price elasticity at the point where both firms charge equal price). The economic distinction is that in the Cournot model, the market outcome approaches the competitive outcome only in the limit as the number of firms gets arbitrarily large, while in the Bertrand model for any number of firms larger than two the market gives the competitive price and quantity. Hence, when appearing before the Justice Department, you might be tempted to use the Bertrand model if you were working for two firms who wanted to merge, arguing that as long as there is still at least one other firm in the industry, the market will still give a competitive outcome and hence Justice has no reason to object to the merger. If you worked for the Justice Department, you would be tempted to use the Cournot model, arguing that any reduction in the number of firms pushes the market further from the competitive outcome. To address this difference, you would have to know more about the particular market in question, whose actual behavior is likely to lie somewhere between the extremes captured by the Cournot and Bertrand models.