1. (a) The equation of the indifference curve is given by,

\[(x_1 + 1)(x_2) = U\]

\[\Rightarrow x_2 = \frac{U}{x_1 + 1}\]

The MRS between goods 1 and 2 is given by,

\[MRS_{1,2} = \frac{MU_1}{MU_2} = \frac{x_2}{x_1 + 1}\]

(b) See Figure 1.

![Figure 1: Indifference Curve](image)

(c) The expenditure function is given by,

\[E(p_1, p_2, U) = p_1 h_1(p_1, p_2, U) + p_2 h_2(p_1, p_2, U)\]

Hence, by envelope theorem,

\[\frac{\partial E}{\partial p_1} = h_1 > 0\]

\[\frac{\partial E}{\partial p_2} = h_2 > 0\]
(d) The expenditure minimization problem is given by,

\[ \min_{x_1, x_2} p_1 x_1 + p_2 x_2 \]

subject to \((x_1 + 1)(x_2) \geq U\)

Here the endogenous variables are \(x_1\) and \(x_2\) and exogenous variables are \(p_1, p_2\) and \(U\).

(e) We have noted in part (c) of this question that the expenditure function is strictly increasing. Hence increasing the consumption of any of the goods raises the expenditure of the consumer. Hence given any floor utility level consumer will never go beyond it while minimizing expenditure. If he does then he can always reduce the purchase any one of the goods by a small amount such that he still the minimum utility level but the expenditure would be reduced. Hence we can restrict attention to the equality constraint.

2. (a) First, we shall calculate the first-order condition through the method of substitution.

We solve the utility function in terms of \(x_2\) and substitute it into the cost function. We have now gone from a constrained two-variable optimization problem to an unconstrained one-variable optimization problem. The consumer now seeks to minimize the following:

\[ e(x_1) = p_1 x_1 + p_2 \left( \frac{U}{x_1 + 1} \right) . \]

To find the minimum, we take the first derivative and set it to zero. This is the first-order condition:

\[ \frac{de}{dx_1} = p_1 + p_2 \left( - \frac{U}{(x_1 + 1)^2} \right) = 0. \]

Second, we shall calculate the first-order condition through the Lagrangean method.

This time, we change the problem from a constrained two-variable optimization to an unconstrained three-variable optimization problem, where the cost function is augmented with a penalty for violating the utility constraint:

\[ L(x_1, x_2, \lambda) = p_1 x_1 + p_2 x_2 + \lambda(U - (x_1 + 1)(x_2)). \]

We seek to minimize the augmented cost function \(L(x_1, x_2, \lambda)\) with respect to \(x_1\) and \(x_2\) while maximizing it with respect to \(\lambda\). To do so, we take the partial first derivatives with respect to each variable
and set them to zero. These are the first-order conditions:
\[
\frac{\partial L}{\partial x_1} = p_1 - \lambda x_2 = 0,
\]
\[
\frac{\partial L}{\partial x_2} = p_2 - \lambda(x_1 + 1) = 0,
\]
\[
\frac{\partial L}{\partial \lambda} = U - (x_1 + 1)(x_2) = 0.
\]

(b) See Figure (2).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Expenditure minimization}
\end{figure}

(c) Using the method of substitution:
\[
p_1 + p_2 \left(-\frac{U}{(h_1 + 1)^2}\right) = 0
\]
\[
\Rightarrow (h_1 + 1)^2 = \frac{p_2}{p_1} U
\]
\[
\Rightarrow h_1 = \sqrt{\frac{p_2}{p_1} U} - 1
\]
\[
\Rightarrow h_2 = \frac{U}{x_1 + 1} = \sqrt{\frac{p_1}{p_2} U}.
\]
Using the Lagrangean method:

\[
\begin{align*}
  p_1 &= \lambda h_2 \\
  p_2 &= \lambda (h_1 + 1) \\
  \Rightarrow h_2 &= \frac{p_1 (h_1 + 1)}{p_2} \\
  \Rightarrow U - (h_1 + 1) \left( \frac{p_1 (h_1 + 1)}{p_2} \right) &= 0 \\
  \Rightarrow U &= \frac{p_1}{p_2} (h_1 + 1)^2 \\
  \Rightarrow h_1 &= \sqrt{\frac{p_2}{p_1} U} - 1 \\
  \Rightarrow h_2 &= \sqrt{\frac{p_1}{p_2} U} \\
  \Rightarrow \lambda &= \sqrt{\frac{p_1 p_2}{U}}.
\end{align*}
\]

(d) Own-price elasticity of a good \( x \) is the percentage change in the demand of \( x \) over the percentage change in the price of \( x \). In other words, own-price elasticity shows the percentage change in \( x \) given a one-percent change in the price of \( x \).

Mathematically,

\[
\epsilon_{i,i} = \frac{\partial x_i}{\partial p_i} \cdot \frac{p_i}{x_i}.
\]

For the own-price elasticity of the Hicksian demand of good 1,

\[
\epsilon_{1,1} = -\frac{1}{2} \sqrt{\frac{p_2}{(p_1)^3} U} \sqrt{\frac{p_2}{(p_1)^3} U} = -\frac{p_2}{2(p_1)^3} U.
\]

(e) The expenditure function is found by plugging Hicksian demand into the cost function:

\[\begin{align*}
E(p_1, p_2, U) &= p_1 \left( \sqrt{\frac{p_2}{p_1} U} - 1 \right) + p_2 \left( \sqrt{\frac{p_1}{p_2} U} \right) \\
&= 2 \sqrt{p_1 p_2 U} - p_1.
\end{align*}\]

By the envelope theorem, \( \frac{\partial E}{\partial U} = \frac{\partial L}{\partial U} \) evaluated at \((h_1, h_2)\). Remember that the Hicksian demand is treated as fixed in this case. Therefore,

\[\frac{\partial E}{\partial U} = \lambda = \sqrt{\frac{p_1 p_2}{U}}.\]
(a) The Slutsky equation is

$$\frac{\partial x_i^*}{\partial p_j} = \frac{\partial h_i^*}{\partial p_j} - \frac{\partial x_i^*}{\partial I} x_j,$$

where the total (uncompensated) price effect appears on the LHS and the RHS consists of the compensated price effect (substitution effect, SE) and the income effect (IE).

(b) Differentiate

$$x_i^* (p, E(p, U)) = h_i^* (p, U),$$

on both sides with respect to $p_j$ using the chain rule:

$$\frac{\partial x_i^*}{\partial p_j} + \frac{\partial x_i^*}{\partial I} \frac{\partial E}{\partial p_j} = \frac{\partial h_i^*}{\partial p_j}.$$

Using the fact that $\frac{\partial E}{\partial p_j} = x_j$ and rearranging we get

$$\frac{\partial x_i^*}{\partial p_j} = \frac{\partial h_i^*}{\partial p_j} - \frac{\partial x_i^*}{\partial I} x_j,$$

(c) See Figure (3).
Figure 3: Slutsky equation

Substitution effect and income effect. The pivot gives the substitution effect, and the shift gives the income effect.