Pareto Efficiency, Competitive Equilibrium and the Two Fundamental Welfare Theorems
• many individuals

\[ i = 1, \ldots, I \]

• each individual has an endowment:

\[ e^i = (e^i_1, \ldots, e^i_K) \]

• each individual \( i \) has a budget constraint:

\[
\sum_{k=1}^K p_k x^i_k \leq \sum_{k=1}^K p_k e^i_k = I
\]
• a competitive equilibrium \((p, x)\) consist of a pair of prices:

\[ p = (p_1, \ldots, p_K) \]

and allocations

\[ x = (x^1, \ldots, x^i, \ldots, x^l) \]

• we say that markets clear if for all commodities \(k\):

\[ \sum_{i=1}^{l} x^i_k \leq \sum_{i=1}^{l} e^i_k. \]
Definition of Competitive Equilibrium

Definition

A pair \((p^*, x^*)\) is a competitive equilibrium if

1. for each individual \(i\), \(x^{i*}(p^*, e^i)\) solves the consumer maximization problem:

\[
x^{i*}(p^*, e^i) \in \arg \max_{x \in \mathbb{R}^n} u^i(x^i) \text{ subject to } \sum_{k=1}^{K} x_k^i p_k \leq \sum_{k=1}^{K} e_k^i p_k;
\]

2. for each commodity \(k\), the market clears:

\[
\sum_{i=1}^{l} x_k^i \leq \sum_{i=1}^{l} e_k^i.
\]
The Competitive Equilibrium with Ann and Bob

- Ann and Bob share the same preferences:
  \[ u^i (x_1^i, x_2^i) = \ln x_1^i + \ln x_2^i \]
- but their endowments are not necessarily identical:
  \[ e_1^A \neq e_1^B, \quad e_2^A \neq e_2^B, \]
- the budget constraint, as an equality, is
  \[ p_1 x_1^i + p_2 x_2^i = p_1 e_1^i + p_2 e_2^i \]
  or
  \[ p_1 (x_1^i - e_1^i) + p_2 (x_2^i - e_2^i) = 0 \]
  or using the relative prices \( p_1 / p_2 \)
  \[ (x_2^i - e_2^i) = -\frac{p_1}{p_2} (x_1^i - e_1^i) \]
• excess demand

\[ z_k^i = x_k^i - e_k^i \]

• the budget constraint, expressed in excess demand:

\[ p_1 z_1^i + p_2 z_2^i = 0 \iff z_2^i = -\frac{p_1}{p_2} z_1^i \]
we have the conditions for optimal choice:

\[
\frac{x^i_2}{x^i_1} = \frac{p_1}{p_2}
\]

we have the balanced budget condition:

\[
(x^i_2 - e^i_2) = -\frac{p_1}{p_2} (x^i_1 - e^i_1)
\]

we can compute optimal demands:

\[
\left( x^i_1 \frac{p_1}{p_2} - e^i_2 \right) = -\frac{p_1}{p_2} (x^i_1 - e^i_1) \iff
\]

\[
x^i_1 = \frac{1}{2} \left( \frac{p_2}{p_1} e^i_2 + e^i_1 \right)
\]
CE (1): Optimal Choices of Ann and Bob

- and then (symmetrically) for good 2:

\[
\begin{align*}
(x_2^i - e_2^i) &= -\frac{p_1}{p_2} \left( \frac{1}{2} \left( \frac{p_2}{p_1} e_2^i + e_1^i \right) - e_1^i \right) \\
x_2^i &= \frac{1}{2} \left( e_2^2 + e_1^1 \frac{p_1}{p_2} \right)
\end{align*}
\]

- remember and compare to the solution with given income:

\[
x_k^i = \frac{\alpha_i}{p_k} = \frac{1}{2} \left( \frac{p_1 e_1^i + p_2 e_1^i}{p_k} \right)
\]
we have the conditions for market clearing for good 1:

\[ x_A^1 + x_B^1 = \frac{1}{2} \left( \frac{p_2}{p_1} e_2^A + e_1^A \right) + \frac{1}{2} \left( \frac{p_2}{p_1} e_2^B + e_1^B \right) = e_1^A + e_1^B \]

and symmetrically good 2:

\[ x_A^2 + x_B^2 = \frac{1}{2} \left( \frac{p_1}{p_2} e_1^A + e_2^A \right) + \frac{1}{2} \left( \frac{p_1}{p_2} e_1^B + e_2^B \right) = e_2^A + e_2^B \]

we observe that only the ratio of prices matters:

\[ \frac{p_2}{p_1} \]

and normalize \( p_1 = 1 \)

we then have

\[ \frac{1}{2} \left( p_2 e_2 + e_1 \right) = e_1 \iff \]

\[ p_2^* = \frac{e_1}{e_2} \]

prices only depend on aggregate endowment and express relative scarcity
• the competitive equilibrium \((p^*, x_1^*, x_2^*)\) is formed by the relative price:

\[ p_2^* = \frac{e_1}{e_2} \]

• the resulting demands are:

\[ x_1^* = \frac{1}{2} \left( \frac{e_1}{e_2} e_2^i + e_1^i \right), \quad x_2^* = \frac{1}{2} \left( e_2^i + e_1^i \frac{e_2}{e_1} \right) \]

• the net trading quantities are:

\[ z_1^* = \frac{1}{2} \left( \frac{e_1}{e_2} e_2^i + e_1^i \right) - e_1^i = \frac{1}{2} \left( \frac{e_1}{e_2} e_2^i - e_1^i \right) \]
Walras Law and Price Normalization

- Walras law: given that $K - 1$ markets clear, the $K$–th market clears as well
- obtains from the interaction of market clearing and budget balance

\[
\sum_{k=1}^{K-1} \sum_{i=1}^{l} p_k (x_k^i - e_k^i) = -\sum_{i=1}^{l} p_K (x_K^i - e_K^i) \iff \\
\sum_{k=1}^{K} \sum_{i=1}^{l} \frac{p_k}{p_K} (x_k^i - e_k^i) = -\sum_{i=1}^{l} (x_K^i - e_K^i),
\]

but now market clearing for every market $k < K$, means that for all $k < K$

\[
\sum_{i=1}^{l} (x_k^i - e_k^i) = 0,
\]

and hence the rhs has to equal zero as well
Welfare Theorems

- competitive equilibrium leads to a Pareto efficient allocation

Theorem (First Welfare Theorem)

*Every competitive equilibrium \((p^*, x^*)\) supports a Pareto efficient allocation \(x^*\).*

- the notion of decentralization: can we achieve an arbitrary efficient allocation through the market rather than another economic system (as in planning, command, organization)

Theorem (Second Welfare Theorem)

*Every Pareto efficient allocation \(x^*\) can be decentralized via a competitive equilibrium \((p^*, x^*)\) (and an endowment assignment or a taxation policy).*
The Limits of the Welfare Theorems

- every agent is small, and takes the price as given
- every agent has the same information, symmetric and complete
- absence of externalities: agent \( i \)’s utility only depends on agent \( i \)’s consumption
Pareto Efficiency versus Competitive Equilibrium

- optimal demand (marginal rate of substitution equal to relative prices) vs equalized marginal rate of substitution
- market clearing (no excess demand) versus feasibility.

Now:

- we can always normalize one price, say $p_1 = 1$.
- Walras law: if $K - 1$ markets clear, then $K$ markets clear
Ann and Bob are Different!

- an example with Ann:

\[ u^A \left( x_1^A, x_2^A \right) = \alpha \ln x_1^A + (1 - \alpha) \ln x_2^A \]

and Bob:

\[ u^B \left( x_1^B, x_2^B \right) = \beta \ln x_1^B + (1 - \beta) \ln x_2^B \]

- for optimal demand

\[ \frac{\partial u^A(\cdot)}{\partial x_1^A} = \frac{\partial u^B(\cdot)}{\partial x_1^B} = \frac{p_1}{p_2} \]

\[ \frac{\partial u^A(\cdot)}{\partial x_2^A} = \frac{\partial u^B(\cdot)}{\partial x_2^B} \]

and thus

\[ \frac{\alpha}{1 - \alpha} \frac{x_2^A}{x_1^A} = \frac{\beta}{1 - \beta} \frac{x_2^B}{x_1^B} = \frac{p_1}{p_2} \]

- for market clearing

\[ x_1^A + x_2^A = e_1^A + e_2^A; \quad x_1^B + x_2^B = e_1^B + e_2^B. \]
• Ann and Bob do not share the same preferences: $\alpha \neq \beta = \frac{1}{2}$
• but their endowments are identical:

$$e_1^A = e_1^B = e_2^A = e_2^B = \frac{1}{2}$$
The Competitive Equilibrium with Heterogeneous Consumers

- we have the conditions for optimal choice:
  \[ \frac{\alpha}{1 - \alpha} \frac{x_2^A}{x_1^A} = \frac{1}{p} \]
  \[ \frac{x_2^B}{x_1^B} = \frac{1}{p} \]

- we have the balanced budget condition:
  \[ x_1^A + x_2^A = \frac{1}{2} + \frac{1}{2}p \]
  \[ x_1^B + x_2^B = \frac{1}{2} + \frac{1}{2}p \]

- we have the market clearing condition:
  \[ x_1^A + x_1^B = e_1^A + e_1^B = 1 \]
Solving the Competitive Equilibrium Equations

• five unknowns, five equations:

\[
\frac{\alpha}{1 - \alpha} \frac{x_2^A}{x_1^A} = \frac{1}{p}, \quad \frac{x_2^B}{x_1^B} = \frac{1}{p}
\]

• we have the balanced budget conditions:

\[
x_1^A + x_2^A = \frac{1}{2} + \frac{1}{2}p
\]
\[
x_1^B + x_2^B = \frac{1}{2} + \frac{1}{2}p
\]

• we have the market clearing condition:

\[
x_1^A + x_1^B = e_1^A + e_1^B = 1
\]

• with the solutions given by:

\[
x_1^A = \frac{2\alpha}{1 + 2\alpha}, \quad x_1^B = \frac{1}{1 + 2\alpha},
\]
\[
x_2^A = \frac{2 (1 - \alpha)}{1 + 2 (1 - \alpha)}, \quad x_2^B = \frac{1}{1 + 2 (1 - \alpha)}, \quad p = \frac{1 + 2 (1 - \alpha)}{1 + 2\alpha}
\]
• consider Bob’s utility function

\[ U(\alpha) = \frac{1}{1 + 2\alpha} \frac{1}{1 + 2(1 - \alpha)} \]

and hence

\[ U'(\alpha) = 0 \iff \alpha = \frac{1}{2} \]

• strong preference of Ann puts Bob into a better trading position