1 Lecture 12: Social Welfare

We here are trying to formalize the problem from the point of view of a social planner. The social planner has endowments given by the endowment vector \( e = (e_1, e_2, \ldots, e_K) \) and attaches weight \( \alpha^i \) to individual \( i \)'s utility. So for him the optimization problem is given by:

\[
\max_{x^1, x^2, \ldots, x^I} \sum_{i=1}^{I} \alpha^i u^i(x^i) \quad \alpha^i \geq 0, \quad \sum_{i=1}^{I} \alpha^i = 1
\]

subject to \( \sum_{i=1}^{I} x^i_k \leq e_k \forall k = 1, 2, \ldots, K \)

The Lagrange is given by,

\[
L(x, \lambda) = \sum_{i=1}^{I} \alpha^i u^i(x^i) + \sum_{k=1}^{K} \lambda_k \left( e_k - \sum_{i=1}^{I} x^i_k \right)
\]

The first order conditions for individual \( i \) and for any two goods \( k \) and \( l \) are:

\[
x^i_k : \quad \alpha^i \frac{\partial u^i(x^i)}{\partial x^i_k} - \lambda_k = 0,
\]

\[
x^i_l : \quad \alpha^i \frac{\partial u^i(x^i)}{\partial x^i_l} - \lambda_l = 0.
\]

If we consider the ratio for any two commodities, we get for all \( i \) and for any pair \( k, l \) of commodities:

\[
\frac{\alpha_i \frac{\partial u^i(x^i)}{\partial x^i_k}}{\alpha_i \frac{\partial u^i(x^i)}{\partial x^i_l}} = \frac{\lambda_k}{\lambda_l}
\]

This means that the MRS between two goods \( k \) and \( l \) is same across individuals which is the condition for Pareto Optimality. Hence a specific profile of weights \( \alpha = (\alpha^1, \alpha^2, \ldots, \alpha^I) \) will give us a specific allocation among the set of Pareto efficient allocations. Therefore we have the following powerful theorem:
Theorem 1. The set of Pareto efficient allocations and the set of welfare maximizing allocations across all possible vectors of weights are identical.

Below we solve a particular example with Cobb-Douglas preferences.

Example 1. Let Ann and Bob have the following preferences:

\[ u^A(x^A_1, x^A_2) = \alpha \ln x^A_1 + (1 - \alpha) \ln x^A_2 \]
\[ u^B(x^B_1, x^B_2) = \beta \ln x^B_1 + (1 - \beta) \ln x^B_2 \]

Let the weight on Ann’s utility function be \( \gamma \) and therefore the weight on Bob’s utility function is \( (1 - \gamma) \). The Lagrange expression is then given by,

\[ L(x, \lambda) = \gamma u^A + (1 - \gamma) u^B + \lambda_1[e_1 - x^A_1 - x^B_1] + \lambda_2[e_2 - x^A_2 - x^B_2] \]

The F.O.C.s are then given by,

\[
\frac{\partial L(x, \lambda)}{\partial x^A_1} = \frac{\gamma \alpha}{x^A_1} - \lambda_1 = 0 \\
\frac{\partial L(x, \lambda)}{\partial x^A_2} = \frac{\gamma (1 - \alpha)}{x^A_2} - \lambda_2 = 0 \\
\frac{\partial L(x, \lambda)}{\partial x^B_1} = \frac{(1 - \gamma) \beta}{x^B_1} - \lambda_1 = 0 \\
\frac{\partial L(x, \lambda)}{\partial x^B_2} = \frac{(1 - \gamma) \beta}{x^B_2} - \lambda_2 = 0
\]

Hence we get,

\[
(1 - \gamma) \beta x^A_1 = \gamma \alpha x^B_1
\]

and

\[
(1 - \gamma) \beta x^A_2 = \gamma (1 - \alpha) x^B_2
\]

Thus from the feasibility conditions we get that the allocations would be,

\[
x^A_1 = \frac{\gamma \alpha}{\gamma \alpha + (1 - \gamma) \beta} e_1, \quad x^A_2 = \frac{\gamma (1 - \alpha)}{(1 - \gamma) \beta + \gamma (1 - \alpha)} e_2
\]

\[
x^B_1 = \frac{(1 - \gamma) \beta}{\gamma \alpha + (1 - \gamma) \beta} e_1, \quad x^B_2 = \frac{(1 - \gamma) \beta}{(1 - \gamma) \beta + \gamma (1 - \alpha)} e_2
\]

Here by varying the value of \( \gamma \) in its range \([0,1]\) we can generate the whole set of Pareto efficient allocations.
2 Lecture 13: Competitive Equilibrium Continued

Example 2. We now consider a simple example, where Friday is endowed with the only (perfectly divisible) banana and Robinson is endowed with the only coconut. That is \( e^F = (1, 0) \) and \( e^R = (0, 1) \). To keep things simple suppose that both agents have the same utility function

\[
u(x_B, x_C) = \alpha \sqrt{x_B} + \sqrt{x_C}\]

and we consider the case where \( \alpha > 1 \), so there is a preference for bananas over coconuts that both agents share. We can determine the indifference curves for both Robinson and Friday that correspond to the same utility level that the initial endowments provide. The indifference curves are given by

\[
u^F(e^F_B, e^F_C) = \alpha \sqrt{e^F_B} + \sqrt{e^F_C} = \alpha = u^F(1, 0)
\]

\[
u^R(e^R_B, e^R_C) = \alpha \sqrt{e^R_B} + \sqrt{e^R_C} = 1 = u^R(0, 1)
\]

All the allocations between these two indifference curves are Pareto superior to the initial endowment. We can define the net trade for Friday (and similarly for Robinson) by

\[
z^F_B = x^F_B - e^F_B
\]

\[
z^F_C = x^F_C - e^F_C
\]

Notice that since initially Friday had all the bananas and none of the coconuts

\[
z^F_B \leq 0
\]

\[
z^F_C \geq 0
\]

There could be many Pareto efficient allocations (e.g., Friday gets everything, Robinson gets everything, etc.), but we can calculate which allocations are Pareto optimal. If the indifference curves at an allocation are tangent then the marginal rates of substitution must be equated. In this case, the resulting condition is

\[
\frac{\partial u^F}{\partial x_B} = \frac{\alpha}{2\sqrt{x_C}} = \frac{\alpha}{2\sqrt{x_C}} = \frac{\partial u^R}{\partial x_B} = \frac{\partial u^R}{\partial x_B} = \frac{\partial u^R}{\partial x_C} = \frac{\partial u^R}{\partial x_C}
\]

which simplifies to

\[
\frac{\sqrt{x_C}}{\sqrt{x_B}} = \frac{\sqrt{x_C}}{\sqrt{x_B}} = \frac{\sqrt{x_R}}{\sqrt{x_B}}
\]
and, of course, since there is a total of one unit of each commodity, for market clearing we must have

\[ x^R_C = 1 - x^F_C \]
\[ x^R_B = 1 - x^F_B \]

so

\[ \sqrt{x^F_C} = \sqrt{1 - x^F_C} \]
\[ \sqrt{x^F_B} = \sqrt{1 - x^F_B} \]

and squaring both sides

\[ \frac{x^F_C}{x^F_B} = \frac{1 - x^F_C}{1 - x^F_B} \]

which implies that

\[ x^F_C - x^F_C x^F_B = x^F_B - x^F_C x^F_B \]

and so

\[ x^F_C = x^F_B \]
\[ x^R_C = x^R_B. \]

What are the conditions necessary for an equilibrium? First we need the conditions for Friday to be optimizing. We can write Robinson’s and Friday’s optimization problems as the corresponding Lagrangian, where we generalize the endowments to any \( e^R = (e^R_B, e^R_C) \) and \( e^F = (e^F_B, e^F_C) \):

\[ \mathcal{L}(x^F_B, x^F_C, \lambda^F) = \alpha \sqrt{x^F_B} + \sqrt{x^F_C} + \lambda \left( p_B e^F_B + e^F_C - p_B x^F_B - x^F_C \right), \tag{1} \]

where we normalize \( p_C = 1 \) without loss of generality. A similar Lagrangian can be set up for Robinson’s optimization problem. The first-order conditions for (1) are

\[ \frac{\partial \mathcal{L}}{\partial x^F_B} = \frac{\alpha}{2 \sqrt{x^F_B}} - \lambda^F p_B = 0 \tag{2} \]
\[ \frac{\partial \mathcal{L}}{\partial x^F_C} = \frac{1}{2 \sqrt{x^F_C}} - \lambda^F = 0 \tag{3} \]
\[ \frac{\partial \mathcal{L}}{\partial \lambda^F} = p_B e^F_B + e^F_C - p_B x^F_B - x^F_C = 0. \tag{4} \]

Solving as usual by taking the ratio of equations (2) and (3) we get the following expression for the relative (to coconuts) price of bananas

\[ p_B = \alpha \frac{\sqrt{x^F_C}}{\sqrt{x^F_B}} \]
so that we can solve for $x_F^C$ as a function of $x_B^F$ 

$$x_F^C = \left(\frac{p_B}{\alpha}\right)^2 x_B^F.$$ 

Plugging this into the budget constraint from equation (4) we get

$$p_B x_B^F + \left(\frac{p_B}{\alpha}\right)^2 x_B^F = p_B e_B^F + e_C^F.$$ 

Then we can solve for Friday’s demand for bananas

$$x_B^F = \frac{p_B e_B^F + e_C^F}{p_B + \left(\frac{p_B}{\alpha}\right)^2}$$

and for coconuts

$$x_F^C = \left(\frac{p_B}{\alpha}\right)^2 \frac{p_B e_B^F + e_C^F}{p_B + \left(\frac{p_B}{\alpha}\right)^2}.$$ 

The same applies to Robinson’s demand functions, of course.

Now we have to solve for the equilibrium price $p_B$. To do that we use the market clearing condition for bananas, which says that demand has to equal supply (endowment): 

$$x_B^F + x_R^B = e_B^F + e_B^R.$$ 

Inserting the demand functions yields

$$\frac{p_B e_B^F + e_C^F}{p_B + \left(\frac{p_B}{\alpha}\right)^2} + \frac{p_B e_B^R + e_C^R}{p_B + \left(\frac{p_B}{\alpha}\right)^2} = e_B^F + e_B^R = e_B,$$

where $e_B$ is the social endowment of bananas and we define $e_C = e_C^F + e_C^R$. We solve this equation to get the equilibrium price of bananas in the economy:

$$p_B^* = \alpha \sqrt{\frac{e_C}{e_B}}.$$ 

So we have solved for the equilibrium price in terms of the primitives of the economy. This price makes sense intuitively. It reflects relative scarcity in the economy (when there are relatively more bananas than coconuts, bananas are cheaper) and preferences (when consumers value bananas more, i.e., when $\alpha$ is larger, they cost more). We can then plug this price back into the previously found equations both for agents’ consumption and have an expression for consumption in terms of the primitives.
Now we mention the two fundamental welfare theorems which lay the foundation for taking competitive markets as the benchmark for any study of markets and prices. The first one states that competitive equilibrium allocations are always Pareto efficient and the second one states that any Pareto efficient allocation can be achieved as an outcome of competitive equilibrium.

**Theorem 2.** (First Welfare Theorem) Every Competitive Equilibrium allocation \( x^* \) is Pareto Efficient.

**Proof.** Suppose not. Then there exists another allocation \( y \), which is feasible, such that

- for all \( i \): \( u^i(y) \geq u^i(x^*) \)
- for some \( i' \): \( u^{i'}(y) > u^{i'}(x^*) \).

If \( u^i(y) \geq u^i(x^*) \), then the budget constraint (and monotone preferences) implies that

\[
\sum_{k=1}^{K} p_k y^i_k \geq \sum_{k=1}^{K} p_k x^*_k \tag{5}
\]

and for some \( i' \)

\[
\sum_{k=1}^{K} p_k y^{i'}_k > \sum_{k=1}^{K} p_k x^{i'}_k \tag{6}
\]

Equations (5) and (6) imply that

\[
\sum_{i=1}^{I} \sum_{k=1}^{K} p_k y^i_k > \sum_{i=1}^{I} \sum_{k=1}^{K} p_k x^i_k = \sum_{k=1}^{K} p_k e_k,
\]

where the left-most term is the aggregate expenditure and the right-most term the social endowment. This is a contradiction because feasibility of \( y \) means that

\[
\sum_{i=1}^{I} y^i_k \leq \sum_{i=1}^{I} e^i_k = e_k
\]

for any \( i \) and hence

\[
\sum_{i=1}^{I} \sum_{k=1}^{K} p_k y^i_k \leq \sum_{k=1}^{K} p_k e_k.
\]

\( \square \)

**Theorem 3.** (Second Welfare Theorem) Every Pareto efficient allocation can be decentralized as a competitive equilibrium. That is, every Pareto efficient allocation is the equilibrium for some endowments.