1. Question 1

1.1. Recall that
\[ CV = E(P_x, p_y, U) - E(p_x, p_y, U) = \int_{p_x}^{P_x} \frac{\partial E}{\partial p} dp. \]

By the envelope theorem, we know that
\[ \frac{\partial E(p, p_y, U)}{\partial p} = \frac{\partial L(h_x, h_y, p, p_y, U)}{\partial p} = \frac{\partial}{\partial p} (ph_x + p_y h_y + \lambda(U - u(h_x, h_y))) = h_x^* . \]

Therefore,
\[ \int_{p_x}^{P_x} \frac{\partial E}{\partial p} dp = \int_{p_x}^{P_x} h_x(p, p_y, U) dp. \]

We know that the revenue captured by the government will be equal to the increase in price multiplied by the number of goods sold under the new price:
\[ R = (P_x - p_x)h_x(P_x, p_y, U) = ((1 + t_x)p_x - p_x)h_x(P_x, p_y, U) = t_x p_x h_x(P_x, p_y, U). \]

The ratio of tax revenue to additional expenditure is
\[ \frac{t_x p_x h_x(P_x, p_y, U)}{\int_{p_x}^{P_x} h_x(p, p_y, U) dp} = \frac{\int_{p_x}^{P_x} h_x(P_x, p_y, U) dp}{\int_{p_x}^{P_x} h_x(p, p_y, U) dp}. \]

Note that in the numerator integral, \( h_x \) is a fixed quantity, while in the denominator integral, \( h_x \) is a function of \( p \) and is not fixed during integration. Since \( h_x(P_x, p_y, U) \leq h_x(p, p_y, U) \) for any \( p \in (p_x, P_x) \), we know that this ratio will be less than one and, therefore, that commodity tax revenue will always be less than the compensating variation.

See the end of the solutions for the graphs.
1.2. At $P_x$, the consumer needs $E(P_x, p_y, U)$ to buy the bundle $h_x(P_x, p_y, U)$ and get utility $U$. If the price were reduced to $p_x$, they would only need $E(p_x, p_y, U)$ to buy $h_x(p_x, p_y, U)$ which also gives them $U$. Therefore,

$$CV = E(p_x, p_y, U) - E(P_x, p_y, U) = -(E(P_x, p_y, U) - E(p_x, p_y, U)).$$

Since the loss of the consumer is the gain of the government under this lump-sum taxation,

$$T = -CV = E(P_x, p_y, U) - E(p_x, p_y, U).$$

See the end of the solutions for the graphs.

1.3. We begin with prices set at $p_x$ and the consumer with expenditures $E(P_x, p_y, U)$.

Under commodity taxation, the government raises prices to $P_x$, the consumer buys $h_x(P_x, p_y, U)$ (which they can afford) and $R = t_xP_xh_x(P_x, p_y, U)$. Under lump-sum taxation, the government takes $T$ directly from the consumer, the consumer now has expenditures $E(P_x, p_y, U) - T = E(p_x, p_y, U)$, and therefore the consumer buys $h_x(p_x, p_y, U)$. In both scenarios, the consumer gets utility $U$. However, as explained in the solution to 1.1, commodity tax revenue is always less than the compensating variation, which is equal to lump-sum tax revenue in this case. Therefore, $T - R > 0$.

See the end of the solutions for the graphs.

2. Question 2

2.1. First, we calculate the Hicksian demand and the expenditure function. Set up the Lagrangian and derive the first-order conditions:

$$L(p_x, p_y, U) = p_xx + p_yy + \lambda(U - \sqrt{xy})$$

$$\frac{\partial L}{\partial x} = p_x - \lambda \frac{\sqrt{y}}{2\sqrt{x}} = 0$$

$$\frac{\partial L}{\partial y} = p_y - \lambda \frac{\sqrt{x}}{2\sqrt{y}} = 0$$

$$\frac{\partial L}{\partial \lambda} = U - \sqrt{xy} = 0$$
Divide the first FOC by the second and substitute into the third to solve:

\[ \Rightarrow \frac{p_x}{p_y} = \frac{y}{x} \]
\[ \Rightarrow y = \frac{p_x x}{p_y} \]
\[ \Rightarrow U - x \sqrt{\frac{p_x}{p_y}} = 0 \]
\[ \Rightarrow h_x = \sqrt{\frac{p_y}{p_x}} U \]
\[ \Rightarrow h_y = \sqrt{\frac{p_x}{p_y}} U \]
\[ \Rightarrow E(p_x, p_y, U) = 2U \sqrt{p_x p_y}. \]

Next, we calculate the Marshallian demand and the indirect utility function. Again, set up the Lagrangian and derive the first-order conditions:

\[ L(p_x, p_y, I) = \sqrt{xy} + \lambda(I - p_x x - p_y y) \]
\[ \frac{\partial L}{\partial x} = \frac{\sqrt{y}}{2\sqrt{x}} - \lambda p_x = 0 \]
\[ \frac{\partial L}{\partial y} = \frac{\sqrt{x}}{2\sqrt{y}} - \lambda p_y = 0 \]
\[ \frac{\partial L}{\partial \lambda} = I - p_x x - p_y y = 0 \]
Divide the first FOC by the second and substitute into the third to solve:

$$ \frac{p_x}{p_y} = \frac{y}{x} \Rightarrow y = \frac{p_x x}{p_y} $$

$$ \Rightarrow I - p_x x - p_y \left( \frac{p_x}{p_y} x \right) = 0 $$

$$ \Rightarrow x^* = \frac{I}{2p_x} $$

$$ \Rightarrow y^* = \frac{I}{2p_y} $$

$$ \Rightarrow V(p_x, p_y, I) = \frac{I}{2\sqrt{p_x p_y}}. $$

2.2. If $I = 100$ and $p_x = p_y = 1$, then $x^* = y^* = 50$ and $V = 50$.

2.3. If $p_x$ rose to 1.21, then $x^* = \frac{50}{1.21}$, $y^* = 50$, and $V = \frac{50}{\sqrt{1.21}}$.

2.4. $R = (1.21 - 1)\left( \frac{50}{1.21} \right) = \frac{10.5}{1.21}$.

2.5. The compensating variation required to raise utility back up to $V = 50$ is $E(1.21, 1, 50) - E(1, 1, 50) = 100\sqrt{1.21} - 100 = 100(\sqrt{1.21} - 1)$.

Government revenue from the tax is not enough to provide the compensating variation. This has been explained in question 1. Intuitively, it is due to the deadweight loss in society caused by the tax.

2.6. The consumer originally purchased 50 units each of $x$ and $y$. To purchase those quantities again, the consumer would require an income of $(1.21)50 + 50 = 110.50$. Therefore, the Slutsky compensation is $110.50 - 100 = 10.50$.

Given this income, the consumer would not purchase the bundle that he did under the old price. Slutsky compensation counteracts the income effect of a price change, but the substitution effect remains and will impact the consumer’s optimal bundle. Indeed, we find that under the new prices with the Slutsky compensation as additional income, $x^* = \frac{55.25}{1.21}$, $y^* = 55.25$, and $V = \frac{55.25}{\sqrt{1.21}}$. 
2.7. The reduction in consumer surplus will be the following:

\[
\int_{1}^{1.21} \frac{50}{p} dp = 50 \ln(1.21).
\]

See the end of the solutions for the graphs.

Graphically, it is easy to see that the reduction in consumer surplus from the price change is less than the compensating variation required to keep the consumer at his former level of utility. Intuitively, it’s a little trickier. Compensating variation is the additional income required to stay at the same utility level under the new prices, while the reduction in consumer surplus is what the consumer lost (expressed in units of income) due to the new prices. Because prices are now higher, the marginal value of income is lower, so simply reimbursing the consumer what they lost in consumer surplus will not be enough to get them back to their former utility. Thus, the reduction in consumer surplus is less than the compensating variation.

3. Question 3

3.1. Note that unlike the case for Marshallian demand, we minimize with respect to \( x \) and maximize with respect to \( \lambda \). Therefore, a violation of the utility constraint should result in a positive value being added to the objective function, as this would be a penalty in this case. Therefore, the problem to solve is the minimization with respect to \( x \) and the maximization with respect to \( \lambda \) of

\[
L(p, U) = p \cdot x + \lambda(U - u(x))
\]

3.2. The envelope theorem tells us that if we have a function \( f^*(r) \), which is the function \( f(x, r) \) maximized or minimized with respect to \( x \) at the point \( x^* \), and we wish to find \( \frac{\partial f^*}{\partial r} \), then we can simply plug \( x^* \) into \( f(x, r) \) and calculate \( \frac{\partial f(x^*, r)}{\partial r} \) instead.

Therefore,

\[
\frac{\partial E}{\partial p_i} = \frac{\partial L(h^*, p, U)}{\partial p_i} = h_i(p, U).
\]
3.3. See the end of the solutions for the graphs.

Recall that the derivative gives us the instantaneous rate of change at a point. Here, it tells us the proportional change in expenditure to an change in price for infinitesimally small changes in price. We can think of these changes as so miniscule that they don’t actually change the optimal bundle; hence, there is no substitution effect to the price change. Thus, by drawing the tangent line at this point, we can find out how changes in price would impact expenditure if there were no substitution.

3.4. Starting from the hint,

\[ h'' \cdot p'' \geq \lambda E(p, U) + (1 - \lambda)E(p', U) \]

\[ \Rightarrow \quad h'' \cdot (\lambda p + (1 - \lambda)p') \geq \lambda E(p, U) + (1 - \lambda)E(p', U) \]

\[ \Rightarrow \quad \lambda(h'' \cdot p) + (1 - \lambda)(h'' \cdot p') - \lambda E(p, U) - (1 - \lambda)E(p', U) \geq 0 \]

\[ \Rightarrow \quad \lambda(h'' \cdot p - E(p, U)) + (1 - \lambda)(h'' \cdot p' - E(p', U)) \geq 0. \]

Recall that \( h'' \) gives the consumer utility \( U \), since we have fixed utility at \( U \) for this problem. Note that expenditure on a bundle \( h'' \) giving the consumer utility \( U \) cannot be lower than the minimum expenditure required to get a bundle giving the consumer utility \( U \). Therefore, by definition of the expenditure function, \( h'' \cdot p \geq E(p, U) \) and \( h'' \cdot p' \geq E(p', U) \). Since \( \lambda \in [0, 1] \), we have proven the inequality to be true.

3.5. Recall that in 3.2, we showed that \( \frac{\partial E}{\partial p_i} = h_i^* \). Taking the derivative of both sides with respect to \( p_i \) again, we find that \( \frac{\partial^2 E}{\partial p_i^2} = \frac{\partial h_i^*}{\partial p_i} \). Therefore, if \( \frac{\partial^2 E}{\partial p_i^2} \leq 0 \), then \( \frac{\partial h_i^*}{\partial p_i} \leq 0 \).
Note: $p_y$ and $U$ are fixed in this graph.