1. (a) The complementary slackness principle is related to constrained optimization: We are looking for a solution of the following problem

\[
\max_{x \in \mathbb{R}} f(x)
\]

subject to

\[\pi - x \geq 0.\]

We approach this problem by associating a Lagrange multiplier to the constraint, \(\lambda \in \mathbb{R}_+\), and define a function of \(x\) and \(\lambda\), called the Lagrangian function, \(\mathcal{L}\):

\[
\mathcal{L}(x, \lambda) = f(x) + \lambda(\pi - x).
\]

We impose the following constraint on the shape of the Lagrangian

\[
\lambda(\pi - x) = 0
\]

for all \(x\) and all \(\lambda\). The new constraint, (1), is often called the complementary slackness constraint. We now maximize the Lagrangian with respect to \(x\) (and minimize it with respect to \(\lambda\)):

\[
\max_x \mathcal{L}(x, \lambda).
\]

We now look at the first order conditions of the unconstrained problem

\[f'(x^*) - \lambda x^* = 0\]

and the auxiliary complementary slackness constraint:

\[\lambda(\pi - x^*) = 0.\]

If \(x^*\) is a corner solution with \(x^* = \pi\), then \(\lambda \geq 0\) and we can interpret \(\lambda\) as the price (of violating the constraint).

(b) The deadweight loss from monopoly is the total surplus (consumer plus producer surplus) that is lost in monopoly compared to the efficient allocation. We find the efficient allocation by equating marginal willingness to pay to marginal cost of production. Hence, in the efficient allocation, the good is produced as long as the marginal consumer is willing to pay more than producing an additional unit costs.
Since the monopolist can control the price of the good, he equates marginal cost to marginal revenue. This leads to a lower quantity and a higher price, giving a higher rent to the monopolist (by the amount A in the diagram below) and a lower rent to the consumer. The sum of consumer and producer surplus decreases by the amount represented by the triangles B and C in the diagram below.

2. (a)  
   i. Let \( p_h = 2/3 \) be the probability that the high risk type faces the loss, and let \( p_l = 1/3 \) be the probability that the low risk type faces the loss. For some risk of loss \( p \), if the insurance company sells $1 worth of insurance then the expected loss to the insurance company is \( p \cdot 1 + (1 - p) \cdot 0 = p \). The insurance company makes zero expected profit if the premium \( \pi \) is equal to the expected loss, that is, if \( \pi = p \). So the premium for the high risk consumer is \( \pi_h = 2/3 \) and the premium for the low risk consumer is \( \pi_l = 1/3 \).

   ii. Let \( x_b \) be the number of dollars that the consumer gets if the loss occurs (the “bad” state), and let \( x_g \) be the number of dollars that the consumer gets if the loss does not occur (the “good” state). By purchasing a dollar of insurance at premium \( \pi \), the consumer loses \( \pi \) dollars in both states, but gains $1 dollar in the bad state,
for a net gain in the bad state of $1 - \pi$ dollars. With dollars in the good state on the x-axis and dollars in the bad state on the y-axis, the slope of the consumer’s budget constraint is thus $(1 - \pi)/\pi$, since the consumer can trade a reduction of $\pi$ dollars in the good state for a gain of $1 - \pi$ dollars in the bad state. The indifference curve of the high risk consumer is tangent to the budget constraint at a higher point than the indifference curve of the low risk consumer, since the bad state is more likely to occur for the high risk consumer and so the high risk consumer values dollars in the bad state more highly. The diagram below shows these results graphically.

iii. A consumer of type $j$ solves
\begin{align*}
\max_y p_j \ln(10,000 - \pi_j y + y) + (1 - p_j) \ln(20,000 - \pi_j y).
\end{align*}

The first order condition is
\begin{align*}
(1 - \pi_j) \frac{p_j}{10,000 + (1 - \pi_j)y} = \pi_j \frac{1 - p_j}{20,000 - \pi_j y}.
\end{align*}

Simplifying yields
\begin{align*}
y = \frac{p_j}{\pi_j} \cdot 20,000 - \frac{1 - p_j}{1 - \pi_j} \cdot 10,000
\end{align*}

Since \( p_j = \pi_j \) for when firms make zero profits from both consumer segments, under the zero profit condition \( y = 10,000 \) for both consumer types. That is, both consumer types purchase full insurance.

(b) Suppose that the expected probability of loss in the whole population is \( p \), and suppose that insurance companies tried to offer the actuarially fair insurance premium \( \pi = p \). Then high risk consumers would purchase more insurance than low risk consumers, so the expected loss per customer for the insurance company would in fact be \( p' > p \), and the insurance company would make negative profits. In order to make zero profits the insurance companies would have to offer a premium \( \pi' > p \), and as a result the low risk consumers would not purchase full insurance, leading to a Pareto sub-optimal situation.

3. (a) **Edgeworth Box**

   See figure 2

   (b) **Pareto efficiency**

   Remember that the set of Pareto efficient allocations and the set of welfare maximizing allocation across all possible vectors of weights are identical. For every weight \( w \in [0,1] \), we need to find the allocation which would maximize the social surplus given the weights \( (w, 1-w) \); in other words, we are interested in finding the allocation \( (x_{A1}, x_{A2}, x_{B1}, x_{B2}) \) which maximizes the sum

\begin{align*}
wx_A(x_A) + (1-w)u_B(x_B)
\end{align*}

subject to the resource constraints of the economy.

When \( w = 0 \), we should allocate all recourse to individual \( B \) to maximize the weighted sum of utilities. Similarly, all the goods should be consumed by \( A \) when \( w = 1 \). Both allocations are pareto optimal.
Next, we look at the cases when $w \in (0, 1)$. The maximization problem is
\[
\begin{align*}
\max_{x_{A1}, x_{A2}, x_{B1}, x_{B2}} & \quad w(x_{A1} + \alpha x_{A2}) + (1 - w)(\ln x_{B1} + \ln x_{B2}) \\
\text{s.t.} & \quad x_{A1} + x_{B1} = 150 \\
& \quad x_{A2} + x_{B2} = 200.
\end{align*}
\]

The corresponding lagrange $L(x_{A1}, x_{A2}, x_{B1}, x_{B2}, \lambda_1, \lambda_2)$ is
\[
\begin{align*}
w(x_{A1} + \alpha x_{A2}) + (1 - w)(\ln x_{B1} + \ln x_{B2}) \\
- \lambda_1(x_{A1} + x_{B1} - 150) - \lambda_2(x_{A2} + x_{B2} - 200).
\end{align*}
\]

To solve this maximization problem, we should be careful about the corner solutions. Given positive weight $1 - w$ on individual $B$, it is never optimal to set $x_{B1}$ or $x_{B2}$ to zero since the marginal utility of good 1 or good 2 to individual $B$ approaches infinity at the consumption level goes to zero. This is not the case for individual $A$. Take the allocation of good 1 as an example. When the weight put on $A$ is sufficiently small, it might be that even though we allocate all the goods to $B$, the marginal utility to $B$ which is $(1 - w)/x_{B1}$ is still higher than, $w$, the marginal utility to $A$.

Hence, we start with solving for all the interior solutions and then check the conditions on weights $w$ that lead to corner solutions.

The first order conditions are
\[
\begin{align*}
w &= \lambda_1 \\
w\alpha &= \lambda_2 \\
(1 - w)/x_{B1} &= \lambda_1 \\
(1 - w)/x_{B2} &= \lambda_2.
\end{align*}
\]

Combining the FOCs and the resource constraints of the economy, we have the optimal allocations
\[
(x_{A1}, x_{A2}, x_{B1}, x_{B2}) = (150 - \frac{1 - w}{w}, 200 - \frac{(1 - w)}{\alpha w}, \frac{1 - w}{w}, \frac{(1 - w)}{\alpha w}).
\]

Notice that when $w$ is sufficiently small, the optimal allocation $x_{B2}$ or $x_{B1}$ would exceed what the economy can afford. Depending on the value of parameter $\alpha$, we have two cases, leading to two different shapes of pareto efficient allocation.

Case 1: $\alpha \in (0, \frac{3}{4}]$

i. $w < (0, \frac{1}{15}]$, the optimal allocation is
\[
(x_{A1}, x_{A2}, x_{B1}, x_{B2}) = (0, 0, 150, 200).
\]
ii. $w \in \left(\frac{1}{150}, \frac{1}{200+1}\right)$, the optimal allocation is
\[
(x_{A1}, x_{A2}, x_{B1}, x_{B2}) = (150 - \frac{1-w}{w}, 0, 1 - \frac{w}{w}, 200).
\]

iii. $w \in \left[\frac{1}{200+1}, 1\right)$, the optimal allocation is
\[
(x_{A1}, x_{A2}, x_{B1}, x_{B2}) = (150 - \frac{1-w}{w}, 200 - \frac{(1-w)}{w}, 1 - \frac{w}{w}, (1-w)\frac{1}{\alpha w}).
\]

Case 2: $\alpha \in \left(\frac{3}{4}, 1\right]$

i. $w < \left(0, \frac{1}{200+1}\right)$, the optimal allocation is
\[
(x_{A1}, x_{A2}, x_{B1}, x_{B2}) = (0, 0, 150, 200).
\]

ii. $w \in \left[\frac{1}{200+1}, \frac{1}{150}\right)$, the optimal allocation is
\[
(x_{A1}, x_{A2}, x_{B1}, x_{B2}) = (0, 200 - \frac{(1-w)}{\alpha w}, 150, \frac{1-w}{\alpha w}).
\]

iii. $w \in \left[\frac{1}{150}, 1\right)$, the optimal allocation is
\[
(x_{A1}, x_{A2}, x_{B1}, x_{B2}) = (150 - \frac{1-w}{w}, 200 - \frac{(1-w)}{w}, 1 - \frac{w}{w}, (1-w)\frac{1}{\alpha w}).
\]

See figure 2 for graphical illustration.

(c) Competitive equilibrium

We normalize the price level of this year by setting $p_1 = 1$.
Utility maximization problem for $A$ is
\[
\max_{x_{A1}, x_{A2}} x_{A1} + \alpha x_{A2}
\quad\text{s.t.}\quad x_{A1} + p_2 x_{A2} = 100.
\]

The lagrange for $A$ is
\[
L(x_{A1}, x_{A2}, \lambda_A) = x_{A1} + \alpha x_{A2} - \lambda_A(x_{A1} + p_2 x_{A2} - 100).
\]

In this case, if $p_2 > \alpha$, the solution is
\[
(x_{A1}, x_{A2}) = (100, 0).
\]

If $p_2 < \alpha$, the solution is
\[
(x_{A1}, x_{A2}) = \left(0, \frac{100}{p_2}\right).
\]

If $p_2 = \alpha$, the solution is
\[
(x_{A1}, x_{A2}) = \left(x, \frac{100 - x}{p_2}\right).
\]
where \( x \in [0, 100] \).

Utility maximization problem for \( B \) is

\[
\max_{x_B1, x_B2} \quad \ln x_{B1} + \ln x_{B2} \\
\text{s.t.} \quad x_{B1} + p_2 x_{B2} = 50 + 200p_2.
\]

The lagrange for \( B \) is

\[
L(x_{B1}, x_{B2}, \lambda_B) = \ln x_{B1} + \ln x_{B2} - \lambda_B(x_{B1} + p_2 x_{B2} - 50 - 200p_2).
\]

The first order conditions are

\[
\begin{align*}
\frac{1}{x_{B1}} - \lambda_B &= 0 \\
\frac{1}{x_{B2}} - \lambda_B p_2 &= 0 \\
x_{B1} + p_2 x_{B2} - 50 - 200p_2 &= 0.
\end{align*}
\]

The optimal consumption of \( B \) is

\[
\begin{align*}
x_{B1} &= \frac{50 + 200p_2}{2} \\
x_{B2} &= \frac{50 + 200p_2}{2p_2}.
\end{align*}
\]

The market clearing conditions are

\[
\begin{align*}
x_{A1} + x_{B1} &= 150 \\
x_{A2} + x_{B2} &= 200.
\end{align*}
\]

i. \( \alpha < p_2 \).

In this case, \( x_{A1} = 100 \). From the market clearing condition for good 1, we have

\[
x_{A1} + x_{B1} = 100 + \frac{50 + 200p_2}{2} = 150,
\]

which gives that \( p_2 = 1/4 \). The equilibrium is

\[
(x_{A1}, x_{A2}; x_{B1}, x_{B2}; p_1, p_2) = (100, 0; 50, 200; 1, \frac{1}{4}).
\]

Notice that this is the equilibrium when \( \alpha \in (0, \frac{3}{4}) \). The interest rate is

\[
r = -\frac{3}{4}.
\]
ii. $\alpha > p_2$
In this case, $x_{A1} = 0$. From the market clearing condition for good 1, we have
\[
x_{A1} + x_{B1} = 0 + \frac{50 + 200p_2}{2} = 150,\]
which gives that $p_2 = 5/4$. Contradiction to the assumption that $\alpha > p_2$.

iii. $\alpha = p_2$
In this case $x_{B1} = \frac{50+200\alpha}{2}$ and $x_{A1} = x \in [0, 100]$. From the market clearing condition for good 1, we have
\[
x_{A1} + x_{B1} = x + \frac{50 + 200\alpha}{2} = 150,\]
which gives that $x_{A1} = 125 - 100\alpha$. Since $x_{A1} \in [0, 100]$, we have the constraint that $\alpha$ should be weakly greater than $\frac{1}{4}$.

To sum, when $\frac{1}{4} \leq \alpha < 1$, the equilibrium is
\[
(x_{A1}, x_{A2}; x_{B1}, x_{B2}; p_1, p_2) = (125 - 100\alpha, 100 - \frac{25}{\alpha}; 25 + 100\alpha, \frac{25}{\alpha} + 100; 1, \alpha).
\]

The interest rate is $r = \alpha - 1$.

See figure 3 for graphical illustration.

(d) **Contract Curve**
See figure 3 for graphical illustration.
Figure 2: Edgeworth Box and Pareto Efficient allocation
Figure 3: Competitive Equilibrium and Contract Curve