Approximation in Algorithm Game Theory: The Price of Anarchy

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**Pigou's Example**

**Example:** one unit of traffic wants to go from \( s \) to \( t \)

\[
c(x) = x
\]

**cost depends on congestion**

\[
c(x) = 1
\]

**no congestion effects**

**Question:** what will selfish network users do?

- assume everyone wants smallest-possible cost
- [Pigou 1920]
Motivating Example

**Claim:** all traffic will take the top link.

**Reason:**
- \( \epsilon > 0 \implies \) traffic on bottom is envious
- \( \epsilon = 0 \implies \) equilibrium
  - all traffic incurs one unit of cost
Can We Do Better?

Consider instead: traffic split equally

\[ c(x) = x \quad \text{Flow} = \frac{1}{2} \]

\[ c(x) = 1 \quad \text{Flow} = \frac{1}{2} \]

Improvement:
- half of traffic has cost 1 (same as before)
- half of traffic has cost \( \frac{1}{2} \) (much improved!)
Braess’s Paradox

Initial Network:

Cost = 1.5
Braess's Paradox

Initial Network:          Augmented Network:

Cost = 1.5

Now what?
Braess’s Paradox

Initial Network:

Augmented Network:

Cost = 1.5

Cost = 2
Braess’s Paradox

Initial Network:          Augmented Network:

Cost = 1.5

Cost = 2

All traffic incurs more cost! [Braess 68]

• also has physical analogs [Cohen/Horowitz 91]
High-Level Overview

**Motivation:** equilibria of noncooperative network games typically *inefficient*

- e.g., Pigou's example + Braess's Paradox
- don't optimize natural objective functions

**Price of anarchy:** quantify inefficiency w.r.t some objective function

**Our goal:** when is the price of anarchy small?
- when does competition approximate cooperation?
- benefit of centralized control is small
Nonatomic Selfish Routing

- directed graph \( G = (V,E) \)
- source-destination pairs \((s_1,t_1), \ldots, (s_k,t_k)\)
- \( r_i = \) amount of traffic going from \( s_i \) to \( t_i \)
- for each edge \( e \), a cost function \( c_e(\cdot) \)
  - assumed continuous and nondecreasing

**Defn:** a multicommodity flow is an *equilibrium* if all traffic routed on shortest paths.
Our Objective Function

Definition of social cost: total cost $C(f)$ incurred by the traffic in a flow $f$.

Formally: if $c_p(f) =$ sum of costs of edges of $P$ (w.r.t. flow $f$), then:

$$C(f) = \sum_p f_p \cdot c_p(f)$$
Our Objective Function

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Formally: if $c_P(f) = \text{sum of costs of edges of } P \text{ (w.r.t. flow } f\text{)},$ then:

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Example:

Cost $= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{3}{4}$
The Price of Anarchy

**Defn:** price of anarchy of a game = \[
\frac{\text{obj fn value of worst equilibrium}}{\text{optimal obj fn value}}
\]

- definition from [Koutsoupias/Papadimitriou 99]

**Example:** POA = 4/3 in Pigou's example

\[
\begin{align*}
\text{Cost} &= \frac{3}{4} \\
\text{Cost} &= 1
\end{align*}
\]
A Nonlinear Pigou Network

Bad Example:

equilibrium has cost 1, min cost \( \approx 0 \)
A Nonlinear Pigou Network

Bad Example:

\[ x^d \]

(d large)

\[
\begin{array}{cc}
1 & 1 - \epsilon \\
1 - \epsilon & \epsilon
\end{array}
\]

\[ s \rightarrow t \]

equilibrium has cost 1, min cost \( \approx 0 \)

\[ \Rightarrow \text{price of anarchy unbounded as } d \rightarrow \infty \]

Goal: weakest-possible conditions under which P.O.A. is small.
When Is the Price of Anarchy Bounded?

Examples so far:

\[
\begin{align*}
\text{s} & \xrightarrow{x} 0 \xrightarrow{1} \text{t} \\
\text{s} & \xrightarrow{1} \xrightarrow{x} \text{t}
\end{align*}
\]

Hope: imposing additional structure on the cost functions helps
- worry: bad things happen in larger networks
Polynomial Cost Functions

Def: linear cost fn is of form $c_e(x) = a_e x + b_e$

Theorem: [Roughgarden/Tardos 00] for every network with linear cost functions:

$$\text{cost of Nash flow} \leq \frac{4}{3} \times \text{cost of opt flow}$$
Polynomial Cost Functions

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\[
\text{cost of Nash flow} \leq 4/3 \times \text{cost of opt flow}
\]

**Bounded-deg polys:** (w/nonneg coeffs) replace 4/3 by \( \approx \frac{d}{\ln d} \)
A General Theorem

Thm: [Roughgarden 02], [Correa/Schulz/Stier Moses 03] fix any set of cost fns. Then, a Pigou-like example 2 nodes, 2 links, 1 link w/constant cost fn) achieves worst POA

![Diagram showing a cycle with nodes s, x, d, t and edge (s, 1, t), indicating a tight example.](image-url)
Pigou Bound

Recall goal: want to show Pigou-like examples are always worst cases.

Pigou bound: given set of cost functions (e.g., degree-d polys), largest POA in a network:

• two nodes, two links
• one function in given set
• one constant function
  - constant = cost of fully congested top edge
Pigou Bound

**Defn:** the Pigou bound $\alpha(S)$ for $S$ is:

$$\max \frac{r \cdot c(r)}{y \cdot c(y) + (r-y) \cdot c(r)}$$

- $\max$ is over all choices of cost fns $c$ in $S$, traffic rate $r \geq 0$, flow $y \geq 0$
- choose $c(x) = x$; $r = 1$; $y = 1/2 \Rightarrow$ get $4/3$
- calculus: $\alpha(S) = 4/3$ when $S$ = affine functions
  - [d/ln d for deg-d polynomials]
Main Theorem (Formally)

Theorem: [Roughgarden 02, Correa/Schulz/Stier Moses 03]: For every set $S$, for every selfish routing network $G$ with cost functions in $C$, the POA in $G$ is at most $\alpha(S)$.
- POA always maximized by Pigou-like examples

That is, if $f$ and $f^*$ are Nash + optimal flows in $G$, then $C(f)/C(f^*) \leq \alpha(S)$.
- example: $\text{POA} \leq 4/3$ if $G$ has affine cost fns
Interpretation

**Bad news:** inefficiency of selfish routing grows as cost functions become "more nonlinear".
- think of "nonlinear" as "heavily congested"
- recall nonlinear Pigou's example

**Good news:** inefficiency does not grow with network size or # of source-destination pairs.
- in lightly loaded networks, no matter how large, selfish routing is nearly optimal

![Diagram](example)
Benefit of Overprovisioning

**Suppose:** network is overprovisioned by $\beta > 0$ ($\beta$ fraction of each edge unused).

**Then:** Price of anarchy is at most $\frac{1}{2}(1 + 1/\sqrt{\beta})$.

- arbitrary network size/topology, traffic matrix

**Moral:** Even modest (10%) over-provisioning sufficient for near-optimal routing.
Variational Inequality

Claim:

• if $f$ is a Nash flow and $f^*$ is feasible, then

$$\sum_{e} f_e \cdot c_e(f_e) \leq \sum_{e} f_e^* \cdot c_e(f_e)$$

• proof: use that Nash flow routes flow on shortest paths (w.r.t. costs $c_e(f_e)$)
Variational Inequality

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Thus:

$$\sum_e f_e \cdot c_e(f_e) \leq \sum_e f^*_e \cdot c_e(f^*_e) + \sum_e f^*_e \cdot [c_e(f_e) - c_e(f^*_e)]$$

relation to $C(f)$?
Geometry of Affine Case

Assume: $c_e(x) = a_e x + b_e$

Goal: compare

$f_e^* \cdot [c_e(f_e) - c_e(f_e^*)] \text{ vs. } f_e \cdot c_e(f_e)$

Interesting case: when $c_e(f_e) > c_e(f_e^*)$:
POA = 4/3 for Affine Costs

Assume: \( c_e(x) = a_e x + b_e \)

Thus: \( f_e^* \cdot [c_e(f_e) - c_e(f_e^*)] \leq [f_e \cdot c_e(f_e)]/4 \)

Thus: \( \sum_e f_e \cdot c_e(f_e) \leq \sum_e f_e^* \cdot c_e(f_e^*) + \sum_e f_e^* \cdot [c_e(f_e) - c_e(f_e^*)] \)

\[ \leq \sum_e [f_e \cdot c_e(f_e)]/4 \]

Thus: \( C(f) \leq 4/3 \cdot C(f^*) \)

- proof from [Correa/Schulz/Stier Moses 08]
Atomic Selfish Routing

Atomic networks: each of (finitely many) players picks a path on which to route one unit of traffic. (otherwise identical model)

AAE example: [refer to whiteboard for details] shows that the POA can be as high as 2.5 in this model, with affine cost functions.
Potential Functions

So: potential fn tracks deviations by players

Thus: equilibria of game = local optima of $\Phi$
• so finite potential games have pure-strategy Nash equilibria (proof: just do "best-response dynamics") [Monderer/Shapley 96]
  - precursors: [Rosenthal 73], [Beckmann et al 56]

Claim: every atomic selfish routing game has a potential function.
Proof of Potential Function

Define \( \Phi_e(k_e) = c_e(1) + c_e(2) + c_e(3) + \ldots + c_e(k_e) \)

where \( k_e \) is \# players using \( e \).

Let \( \Phi(S) = \sum_{e \in S} \Phi_e(S) \)

Consider some solution \( S \) (a path for each player).
Suppose player \( i \) is unhappy and decides to deviate.
What happens to \( \Phi(S) \)?
Proof of Potential Function

\( \Phi_e(k_e) = c_e(1) + c_e(2) + \ldots + c_e(k_e) \)

Suppose player i's new path includes \( e \).
   i pays \( c_e(k_e + 1) \) to use \( e \).
\( \Phi_e(k_e) \) increases by the same amount.

If player i leaves an edge \( e' \),
   \( \Phi_e(k_e) \) exactly reflects the change in i's cost.
Consequences for the Price of Anarchy?

Example: linear cost functions.

Compare cost + potential function:

\[ C(f) = \sum_e f_e \cdot c_e(f_e) = \sum_e [a_e f_e^2 + b_e f_e] \]
\[ \Phi(f) = \sum_e \Sigma_i c_e(i)dx \approx \sum_e [(a_e f_e)^2/2 + b_e f_e] \]

- cost, potential fn differ by factor of \( \leq 2 \)
- gives upper bound of 2 on price on anarchy?
  - \( C(f) \leq 2 \times \Phi(f) \leq 2 \times \Phi(f^*) \leq 2 \times C(f^*) \)
POA in Atomic Model

**Catch:** only bounds the cost of the *global* potential fn minimizer, not all Nash equilibria ($\approx$ *local* minimizers).

**Instead:** use variational inequality, modified for the atomic case:

$$\sum_e f_e \cdot c_e(f_e) \leq \sum_e f^*_e \cdot c_e(f_e+1)$$
A Technical Lemma

Claim:

- [Christodoulou/Koutsoupia 05]: for all integers $y, z$:
  $$y(z+1) \leq (5/3)y^2 + (1/3)z^2$$
A Technical Lemma

Claim:

• [Christodoulou/Koutsoupias 05]: for all integers $y, z$:
  $$y(z+1) \leq \frac{5}{3}y^2 + \frac{1}{3}z^2$$

• so: $ay(z+1) + by \leq \frac{5}{3}[ay^2 + by] + \frac{1}{3}[az^2 + bz]$
  
  - for all integers $y, z$ and $a, b \geq 0$
A Technical Lemma

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• so: $ay(z+1) + by \leq (5/3)[ay^2 + by] + (1/3)[az^2 + bz]$  
  - for all integers $y, z$ and $a, b \geq 0$

• so: $\sum_e [a_e(f_e+1) + b_e)f_e^*] \leq (5/3) \sum_e [(a_e f_e^* + b_e)f_e^*] + (1/3) \sum_e [(a_e f_e + b_e)f_e]$
A Technical Lemma

Claim:

- [Christodoulou/Koutsoupias 05]: for all integers $y,z$:
  \[ y(z+1) \leq \left( \frac{5}{3} \right) y^2 + \left( \frac{1}{3} \right) z^2 \]

- so: $ay(z+1) + by \leq \left( \frac{5}{3} \right) [ay^2 + by] + \left( \frac{1}{3} \right) [az^2 + bz]$ 
  - for all integers $y,z$ and $a,b \geq 0$

- so: $\sum_e [a_e(f_e+1) + b_e)f_e^*] \leq \left( \frac{5}{3} \right) \sum_e [(a_e f_e^* + b_e)f_e^*]$ 
  + $\left( \frac{1}{3} \right) \sum_e [(a_e f_e + b_e)f_e]$

- so: $C(f) \leq \sum_e f^* \cdot c_e(f_e+1) \leq \left( \frac{5}{3} \right) C(f^*) + \left( \frac{1}{3} \right) C(f)$

- so: $POA \leq 5/2$
Key Points of the Day

**Key Examples:** Pigou's example + nonlinear variant; Braess's Paradox; AAE example

- POA depends on cost fns, nonatomic vs. atomic
  - but proofs "structurally" the same (more on Friday)

**Key proof techniques:** (1) equilibrium satisfies a "variational inequality", which can be related to Nash & OPT costs; (2) parameterize POA bounds via "universal worst-case examples"

**Also:** equilibria (locally) minimize potential