This problem set is due on Monday, 2/6/12, in class. To receive full credit, provide a complete defense of your answer.

1. In class we defined the notion of a concave function, in particular a concave utility function. We said that a function $f : \mathbb{R}^n \to \mathbb{R}$ is concave if for all $x, x' \in \mathbb{R}^n$, and all $\lambda \in (0, 1)$:

$$f(\lambda x + (1 - \lambda)x') \geq \lambda f(x) + (1 - \lambda) f(x').$$  \hfill (1)

1. See Figures 1. If $f$ is a concave function, for every $x$ between $z$ and $y$, the point $(x, f(x))$ on the graph of $f$ is above the straight line joining the points $(z, f(z))$ and $(y, f(y))$

2. If $f''(x) \leq 0$, $f'(x)$ is a decreasing function. Suppose $f(x)$ is a utility function representing a decision maker’s preference, decreasing $f'(x)$ corresponds to diminishing marginal utility.

3. See Figures 2. If $f$ is a quasiconcave function, for every $x$ between $z$ and $y$, the point $(x, f(x))$ on the graph of $f$ is above the minimum of $\{f(z), f(y)\}$.
Let’s start with the definition of concavity:

\[ f(\lambda x + (1 - \lambda) y) \geq \lambda f(x) + (1 - \lambda) f(y). \]

Since,

\[
\begin{align*}
  f(x) & \geq \min \{f(x), f(y)\} \\
  f(y) & \geq \min \{f(x), f(y)\},
\end{align*}
\]

we have

\[
\begin{align*}
  f(\lambda x + (1 - \lambda) y) & \geq \lambda \min \{f(x), f(y)\} + (1 - \lambda) \min \{f(x), f(y)\} \\
  & \geq \min \{f(x), f(y)\},
\end{align*}
\]

which implies

\[ f(\lambda x + (1 - \lambda) y) \geq \min \{f(x), f(y)\}, \]

the definition of quasi-concavity. Figure 2 also demonstrates that a quasiconcave function is not necessarily a concave function. We already show that it is a quasiconcave function. For \( z, y, \lambda \) as shown in the figure, we have

\[
\begin{align*}
  f(\lambda x + (1 - \lambda) y) & < \lambda f(x) + (1 - \lambda) f(y) \quad (2) \\
  f(\lambda x + (1 - \lambda) y) & > \min \{f(x), f(y)\}. \quad (3)
\end{align*}
\]

This counterexample shows that \( f(x) \) as depicted in figure 2 is not a concave function.

2. An equivalent description of a quasiconcave function is the following definition: \( f : \mathbb{R}^n \to \mathbb{R} \) is quasiconcave if for all \( y \), the upper contour set \( U_y \), defined as follows

\[ U_y = \{ x \in X | f(x) \geq y \} \]

is a convex set.
1. See Figures 3. The circle on the left hand side is a convex set. The shape on the left is not since the convex combinations of two end points on the right are not in that set.

![Figure 3: A concave function](image)

2. See Figure 4. For any $z \in \mathbb{R}$, the upper contour set $UC(z)$ is convex, which is the case in figure 4. Consider Figure 5 where the upper contour set is not convex as a counterexample. Obviously this function is not quasi-concave.

3. Suppose the consumer’s preferences are represented by function $u$. For any two bundles $x$ and $x'$, the following equivalence holds,

$$x \succeq x' \iff u(x) \geq u(x').$$

We need to show that $u$ is a quasiconcave function if the consumer’s preferences are convex. We first prove this statement for $x, y$ such that $x \sim y$ and $x \neq y$. Since for all $\lambda \in (0, 1)$,

$$\lambda x + (1 - \lambda) y \succeq x$$

we have

$$u(\lambda x + (1 - \lambda) y) \geq \min\{u(x), u(y)\}.$$ 

Next we prove the statement for bundles $x, y$ such that $x \succ y$. Without loss of generality, we assume that $x \succcurlyeq y$. In this case, for all $\lambda \in (0, 1)$, we have

$$\lambda x + (1 - \lambda) y \succcurlyeq y.$$ 

Therefore,

$$u(\lambda x + (1 - \lambda) y) \geq u(y) = \min\{u(x), u(y)\}.$$ 

To sum, we prove that convex preference implies that the utility function is quasiconcave.

3. A consumer’s preferences are represented by the following utility function:

$$u(x, y) = \ln x + 2\ln y.$$
Figure 4: Equivalence of the two definitions of strict quasi-concavity

Figure 5: A non-convex upper contour set
1. Recall that for any two bundles \( Z \) and \( Z' \) the following equivalence holds

\[ Z \succeq Z' \iff u(Z) \geq u(Z') \]

Calculate utility from each bundle.

\[
\begin{align*}
  u(A) &= u(1, 4) = \ln 1 + 2 \ln 4 = 0 + 2 \times 1.4 = 2.8 \\
  u(B) &= u(4, 1) = \ln 4 + 2 \ln 1 = 1.4 + 2 \times 0 = 1.4
\end{align*}
\]

Thus, since \( u(A) > u(B) \) the consumer prefers bundle \( A \) to bundle \( B \).

2. Convex preferences imply that for any three bundles \( X, Y, \) and \( Z \), if \( Y \succeq X \) and \( Z \succeq X \) then \( \alpha Y + (1 - \alpha) Z \succeq X \).

From the example above take \( Y \) to be bundle \( A \), and \( X \) to be bundle \( B \). Since preferences are reflexive \( B \succeq B \), thus, we can take \( Z \) to be bundle \( B \). Thus, convexity of preferences implies

\[ \alpha A + (1 - \alpha) B \succeq B \]

Notice that \( C = 1/2 A + 1/2 B \). Thus \( C \succeq B \). To verify the answer, calculate utility from \( C \)

\[
  u(C) = u(2.5, 2.5) = \ln(2.5) + 2 \ln(2.5) = 2.7 > 1.4 = u(B)
\]

3. A bundle \( Z \) with consumptions given by \((x_Z, y_Z)\) is indifferent to the given bundle \( B \) when it yields the same utility as bundle \( B \).

Utility that bundle \( Z \) yields

\[ u(Z) = u(x_Z, y_Z) = \ln x_Z + 2 \ln y_Z \]

should be equal to \( u(B) = 1.4 \). Thus, an equation for the indifference through the bundle \((x_B, y_B) = (4, 1)\) is

\[
\begin{align*}
  \ln x_Z + 2 \ln y_Z &= \ln 4 \\
  \ln x_Z y_Z^2 &= \ln 4 \\
  x_Z y_Z^2 &= \exp \ln 4 \\
  y_Z^2 &= \frac{4}{x_Z} \\
  y_Z &= \frac{2}{\sqrt[1/2]{x_Z}}
\end{align*}
\]
4. 

\[ MRS(x, y) = -\frac{MU_x}{MU_y} = \frac{1}{x} = \frac{1}{2y} = -\frac{y}{2x} \]

\[ MRS(2.5, 2.5) = \frac{-2.5}{2.25} = -\frac{1}{2} \]

MRS is the rate at which two goods should be exchanged to keep the overall utility constant. For example, at a point (2.5; 2.5) 1 unit of good \( x \) should be exchanged for 1/2 units of good \( y \).

4. Graphically, the function \( f(x) = a - (x - b)^2 \) look like this:

![Graph of f(x) = a - (x - b)^2](image)

We want to maximize \( f(x) = a - (x - b)^2 \), or more formally we want to solve:

\[
\max_x a - (x - b)^2. \tag{5}
\]

Note that there is no constraint. To find the local maxima/minima we solve \( \frac{d}{dx} f(x^*) = 0 \). For our problem we want to solve

\[
\frac{d}{dx} f(x^*) = -2(x^* - b) = 0 \tag{6}
\]

and so the solution to our first-order condition (FOC) gives us \( x^* = b \). To make sure that we in fact have a maximum and not a minimum we must check our second-order condition (SOC), or that \( \frac{d^2}{dx^2} f(x^*) < 0 \).

In this case \( \frac{d^2}{dx^2} f(x) = -2 < 0 \) so our SOC is satisfied everywhere and our solution gives us a maximum.

5. Consider the same function \( f(x) = a - (x - b)^2 \), and suppose now that the choice of the optimal \( x \) is constrained by \( x \leq \bar{x} \), where \( \bar{x} \) is an arbitrary number, satisfying

\[
0 < \bar{x} < b. \tag{7}
\]

1. Adding the constraint \( x \leq \bar{x} \) to our graph looks like this:
where everything to the left of the new vertical line at \( \bar{x} \) are feasible choices of \( x \) for the constrained optimization problem.

2. Our new constrained problem is:

\[
\max_x a - (x - b)^2 \text{ s.t. } x \leq \bar{x},
\]

where \( 0 < \bar{x} < b \). Let us define our Lagrangian as \( L(x, \lambda) = a - (x - b)^2 + \lambda(\bar{x} - x) \).

The optimal conditions include one FOC and one complementary condition

\[
\begin{align*}
\frac{\partial}{\partial x} L(x^*, \lambda^*) &= -2(x^* - b) - \lambda^* = 0 \\
\lambda^*(\bar{x} - x^*) &= 0.
\end{align*}
\]

There are two possibilities given our complementary slackness condition, \( \lambda^*(\bar{x} - x^*) = 0 \). Either \( \lambda^* > 0 \) and \( \bar{x} - x^* = 0 \) or \( \bar{x} - x^* > 0 \) and \( \lambda^* = 0 \) should be the case. If \( \lambda^* = 0 \) is the relevant case here, equation (9) implies that \( x^* = b \), but this contradicts our assumption that \( x^* < \bar{x} \) since \( \bar{x} < b \). Therefore, we have \( \lambda^* = 0 \) and \( x^* = \bar{x} \), which gives us our optimal choice of \( x \). Equation (9) implies that \( \lambda^* = -2(x^* - b) \), which can be combined with our first equation to give us \( \lambda^* = 2(b - \bar{x}) \) our optimal choice of \( \lambda \).

3. **Lagrange multiplier.** The value of the Lagrangian evaluated at our solution is \( L(x^*, \lambda^*) = L(\bar{x}, 2(b - \bar{x})) = a - (\bar{x} - b)^2 \). We can find the marginal effect of increasing our bound, \( \bar{x} \), by taking the partial derivative of our value function with respect to \( \bar{x} \). Namely, we are interested in

\[
\frac{\partial}{\partial \bar{x}} L(x^*, \lambda^*) = -2(\bar{x} - b) = 2(b - \bar{x}),
\]

and we find that \( \frac{\partial}{\partial \bar{x}} L(x^*, \lambda^*) = \lambda^* \).

4. **Interpretation of Lagrangian multiplier.** The price we would have to charge in order to get the decision maker to choose exactly \( x^* = \bar{x} \), even in the absence of a hard constraint, would be given by \( \lambda^* = 2(b - \bar{x}) \).
6. **Comparative Statics.** Substituting \( x^* = \bar{x} \) into function \( f \), we get

\[
y^* = f(x^*) = a - (\bar{x} - b)^2.
\]

To measure how \( y^* \) varies with \( b \), we simply take the derivative of \( y^* \) with respect to \( b \),

\[
\frac{dy^*(b)}{db} = -2(b - \bar{x}).
\]

We can get the same result using envelope theorem.

We start off with

\[
y^*(b) = f(x^*(b), b),
\]

\[
\frac{dy^*}{db} = \frac{\partial f}{\partial x} \frac{dx}{db} + \frac{\partial f}{\partial b} = \frac{\partial f(x, b)}{\partial b}.
\]

This is the envelope theorem which states that at the optimal solution, the impact of a marginal change in an exogenous parameter on the value of the objective function is equal to the direct marginal change that the exogenous variable has on the objective function. The indirect effects, i.e.

\[
\frac{\partial f}{\partial x} \frac{dx}{db} |_{x=\bar{x}} = 0
\]

all cancel out at the optimal solution. In our case \( \frac{dy}{db} = \frac{\partial f(x, b)}{\partial b} |_{x=\bar{x}} = 2(\bar{x} - b) \).

7. Consider the following utility function (often referred to as quasi-linear utility function as it is linear in the second element):

\[ u(x, y) = \ln(x) + y; \]

with prices and income given by: \( p_x = 1, p_y \in \mathbb{R}_+ \) and \( I \in \mathbb{R}_+ \).

1. For special case of \( p_y = 2 \) and \( I = 1 \), the tangency condition implies that

\[
\frac{\partial u(x, y)}{\partial x} = \frac{p_x}{p_y}, \quad \frac{\partial u(x, y)}{\partial y} = 0
\]

\[
\frac{1}{x} = 1, \quad \frac{1}{y} = 2.
\]

Since \( p_x x + p_y y = x + 2y \leq 1 \), we conclude that there is no positive consumption choice \( (x^*, y^*) \geq 0 \) which satisfies the tangency condition.

Figure 6 shows the green indifference curve is tangent to the budget constraint at point \( (x, y) = (2, -\frac{1}{2}) \). At the price level \( (p_x, p_y) = (1, 2) \) the tangency condition can be satisfied only if \( x = 2 \) which can not be sustained given that \( (p_x, p_y, I) = (1, 2, 1) \).
2. From figure 6 we can see that if we are restricted to bundles weakly positive (graphically, we require that we choose bundles above the x-axis), the highest indifference curve which intersects the budget constraint area is the red indifference curve. Therefore, the optimal consumption choice is \((x^*, y^*) = (1, 0)\). Note that at the optimal consumption bundle, the indifference curve is not tangent to the budget constraint line.

Analytically, we only need to check two corner solutions \((1, 0)\) and \((0, \frac{1}{2})\), whichever gives the higher utility is the optimal bundle.

3. Now solve by the method of substitution (using \(p_x x + p_y y = I\)) and do not insist on positive values of consumption of either \(x\) or \(y\).

1. To solve by substitution we start by rearranging the budget constraint into \(y = \frac{I - p_x x}{p_y}\) and then substitute this into the quasilinear utility function to obtain \(u(x) = \ln x + \left(\frac{I - p_x x}{p_y}\right)\). We could have just as well solved for \(x\) to obtain \(u(y)\). To find demand, we solve the following unconstrained problem:

\[
\max_x \ln x + \left(\frac{I - p_x x}{p_y}\right).
\]

The FOC is \(\frac{1}{x} - \frac{p_x}{p_y} = 0\), which yields the demand function \(x^* = \frac{p_y}{p_x}\). We can use the budget constraint to find the other demand function. This yields \(y^* = \frac{I - p_x x^*}{p_y}\). But note that for the demand functions that we have found, there is nothing to guarantee that \(p_x x^* \leq I\) or that \(y^* \geq 0\). We can plug in our demand functions into these conditions and with a bit of algebra we find that both conditions imply \(\frac{p_y}{I} \leq 1\). So if the parameters are such that \(\frac{p_y}{I} > 1\), our demand functions do not make sense (namely, they tell us to purchase negative quantities of \(y\) and spend more on \(x\) than our budget would allow). In this case, it must be that our budget does not allow the purchase of \(x\) to the point that our FOC is satisfied and so the best consumption bundle would be to spend all our money on \(x\) and none on \(y\). So the correct and complete demand functions are:
\[ x^* = \begin{cases} \frac{p_y}{p_x} & \text{if } \frac{p_x}{I} \leq 1 \\ \frac{L}{p_x} & \text{else} \end{cases} \]

\[ y^* = \begin{cases} \frac{I-p_x}{p_y} & \text{if } \frac{p_x}{I} \leq 1 \\ 0 & \text{else} \end{cases} \]

2. To solve using the Lagrange method we must first set up our Lagrange equation by incorporating our budget constraint and our nonnegativity constraints, \( x \geq 0 \) and \( y \geq 0 \). The Lagrangean then formally becomes

\[ L(x, y, \lambda_I, \lambda_x, \lambda_y) = \ln(x) + y + \lambda_I (I - p_x x - p_y y) + \lambda_x x + \lambda_y y. \]

This yields the FOCs with respect to \( x \) and \( y \) as well as complementary slackness conditions,

\[ \frac{\partial}{\partial x} L(x^*, y^*, \lambda_I^*, \lambda_x^*, \lambda_y^*) = \frac{1}{x^*} = \frac{\lambda_I^*}{x^*} + \lambda^* = 0 \quad (13) \]

\[ \frac{\partial}{\partial y} L(x^*, y^*, \lambda_I^*, \lambda_x^*, \lambda_y^*) = 1 - \lambda_I^* p_y + \lambda_y^* = 0 \quad (14) \]

\[ \lambda_I^* (I - p_x x^* - p_y y^*) = 0 \quad (15) \]

\[ \lambda_x^* \cdot x^* = 0 \quad (16) \]

\[ \lambda_y^* \cdot y^* = 0. \quad (17) \]

We start with equation (13). For \( 1/x^* \) to be well defined, it must be the case that \( x^* > 0 \) given that \( x^* \) is weakly positive. This result is confirmed by checking the marginal utility of \( x \). We can tell by the utility function that the nonnegativity constraint on \( x \) will never be binding at the optimum since \( \frac{\partial}{\partial x} u(0, y) = \infty \) for all \( y \). Therefore, \( \lambda_x = 0 \) and \( x^* > 0 \).

Substituting \( \lambda_x = 0 \) and \( x^* > 0 \) into equation (13) we conclude that \( \lambda_I p_x > 0 \), implying that \( \lambda_I > 0 \). Combining this condition with equation (15), We know that the budget constraint will be binding, which should be the case since the utility function is strictly monotonic. So our two possible cases/solutions involve \( y^* > 0 \) (and thus \( \lambda_y^* = 0 \)) or \( y^* = 0 \) (and thus \( \lambda_y^* > 0 \)).

3. We rewrite equations (13) and (14) into \( p_x x^* = \frac{1}{\lambda_I} \) and \( \lambda_I = \frac{1+\lambda_y}{p_y} \), respectively. Substituting these two conditions into budget constraint

\[ (I - p_x x^* - p_y y^*) = 0 \]

we have

\[ I = p_y \left( \frac{1}{1+\lambda_y} + y^* \right). \]

We know that either \( \lambda_y = 0 \) and \( y^* > 0 \) or \( \lambda_y > 0 \) and \( y^* = 0 \). In the former case, \( I = p_y (1 + y) > p_y \). In the latter case \( I = \frac{p_y}{1+\lambda_y} < p_y \). This shows that \( y^* = 0 \) if and only if the method of substitution would have given \( y^* < 0 \) as an answer. In summary, we have the following solution.

\[ x^* = \begin{cases} \frac{p_y}{p_x} & \text{if } \frac{p_x}{I} \leq 1 \\ \frac{L}{p_x} & \text{else} \end{cases} \]
4. The parameter values are $p_x = 2$, and $p_y = 1$. In order to find the income expansion path for the quasilinear utility function, we have to be mindful of when $I - p_y = 0$. So our income expansion path is given by:

$$
\text{IEP}(I) = \begin{cases} 
\left( \frac{1}{2}, I - 1 \right) & \text{if } I \geq 1 \\
\left( \frac{1}{2}, 0 \right) & \text{else}
\end{cases}
$$

4. With parameter values $p_x = 1$, $p_y = 2$, $I = 10$ we are in the case where $I > p_y$ so demand is given by $x^* = \frac{p_y}{p_x} = \frac{2}{1}$ and $y^* = \frac{I - p_y}{p_y} = 4$. The elasticity of the demand for good $i$ with respect to price $j$ is defined as $\varepsilon_{i,j} = \frac{\partial x_i}{\partial p_j} x_i$. So the own price elasticity for $x$ is given by

$$
\varepsilon_{x,x} = \frac{\partial x^*}{\partial p_x} \frac{p_x}{x^*} \\
= -\frac{p_y p_x}{p_x^2 x^*} \\
= -\frac{p_y p_x^2}{p_x^2 p_y} \\
= -1,
$$

and its cross price elasticity is

$$
\varepsilon_{x,y} = \frac{\partial x^*}{\partial p_y} \frac{p_y}{x^*} \\
= \frac{1}{p_x} \frac{p_y}{x^*} \\
= \frac{1}{p_x} \frac{p_x p_y}{p_y} \\
= 1.
$$
The own price elasticity for $y$ is given by

$$
\varepsilon_{y,y} = \frac{\partial y^* p_y}{\partial p_y y^*} = \frac{I y^* p_y}{p_y y^*} = \frac{10}{14} = \frac{5}{2},
$$

and its cross price elasticity is

$$
\varepsilon_{y,x} = \frac{\partial y^* p_x}{\partial p_x y^*} = 0\frac{p_x}{y^*} = 0.
$$