Problem Set 3 - Solutions

March 2011

Question 1

Part a.

Show that every action in the support of a (mixed strategy) Nash equilibrium is independent rationalizable.

Let \( \sigma^* := \langle \sigma_1^* \ldots \sigma_N^* \rangle \) be a Nash Equilibrium. Take \( a_i \in supp(\sigma_i^*) \). We know:

\[ u_i(a_i, \sigma_{-i}^*) \geq u_i(a, \sigma_{-i}^*) \quad \forall a \in A_i. \]

(Using Sofia’s solution as a guide) Proof by contradiction: Suppose that there exists a \( a_i \in supp(\sigma_i^*) \) that is not rationalizable. Let \( \bar{k} \) denote the smallest natural such that there is a player \( i \) such that \( a_i \notin IR_{\bar{k}}^i \) (where \( a_i \in supp(\sigma_i^*) \)).

This implies that \( a_i \) is never an independent BR against \( \sigma_{-i} \) for any \( \sigma_{-i} \in \times_{j \neq i} IR_{j}^{k-1} \). Yet since no strategy in the support of the (mixed) NE has yet been eliminated by stage \( k-1 \) of iterative deletion, we know \( \sigma_{-i}^* \in \times_{j \neq i} IR_{j}^{k-1} \) and by definition \( a_i \) is a BR to \( \sigma_{-i}^* \).

Part b.

Show that every independent rationalizable action is (correlated) rationalizable.

(Using Dongkyu solution as a guide) Proof by induction. If at \( m = 0 \) we have \( IR_0 = R^0 \). Assume that \( IR^n \subseteq R^n \). If \( a_i \in IR_i^{n+1} \) we then have that \( \exists \sigma_{-i} \in \times_{j \neq i} IR_j^n \) such that:

\[ u_i(\sigma_i, \sigma_{-i}) \geq u_i(a_i', \sigma_{-i}) \quad \forall a_i'. \]

Since we assume \( IR^n \subseteq R^n \) we then have that \( \sigma_{-i} \in \times_{j \neq i} IR_j^n \). By the above condition, it follows that \( a_i \in IR_i^n \). Thus if \( a_i \in IR_i^n \implies a_i \in R_i^n \). This holds for all \( i \). Thus \( IR^n \subseteq R^n \) which implies that \( IR \subseteq R \).

Part c.

Show that every action in the support of a correlated equilibrium is (correlated) rationalizable.

(Using Sabyasachi’s solution as a guide) Proof by induction. Let \( \overline{\pi} \in \mu \) be in the support of the CE, \( \mu \). Since \( \overline{\pi} \in A \), where \( A := \times_{i=1}^N A_i \), \( \overline{\pi} \in R^0 = A \). Assume \( \overline{\pi} \in R^k \). Then \( \mu(a_{-i} | \overline{\pi}_i) \in \Delta(R_{-i}^k) \). By the definition of a CE we have:

\[ \overline{\pi}_i \in \arg \max_{a_i} \sum_{a_{-i} \in A_{-i}} \mu(a_{-i} | \overline{\pi}_i) u_i(a_i, a_{-i}). \]
Thus $\pi$ is a BR to $\mu(a_{-i}|\pi) \in \Delta(R_{-i}^k)$. This implies $\pi \in R_{i}^{k+1}$. Thus $\pi \in R_{i}^{n+1}$. This implies $\pi \in R$.

**Part d.**

Show (by example) that not every action in the support of a correlated equilibrium is independent rationalizable.

(Example by Sabyasachi)

Let player 1 be the row player, 2 the column player and 3 the matrix player.

\[
\begin{array}{c|cc}
 & L & R \\
\hline
U & 1,1,1 & 1,0,1 \\
D & 0,1,0 & 0,0,0 \\
\end{array}
\]

\[
\begin{array}{c|cc}
 & L & R \\
\hline
U & 2,2,.7 & 0,0,0 \\
D & 0,0,0 & 2,2,.7 \\
\end{array}
\]

\[
\begin{array}{c|cc}
 & L & R \\
\hline
U & 1,1,0 & 1,0,0 \\
D & 0,1,1 & 0,0,1 \\
\end{array}
\]

The following is a correlated equilibrium:

\[\mu(U, L, B) = \mu(D, R, B) = \frac{1}{2}.\]

Now assume players 1 and 2 randomize independently. Assume player 1 plays $U$ with probability $p$ and 2 plays $L$ with probability $q$. The payoffs of playing $A$, $B$ or $C$ are then:

\[A : p\]

\[B : .7pq + (1 - p)(1 - q)\]

\[C : 1 - p\]

Note that if $p \geq .5$

\[.7pq + (1 - p)(1 - q) \leq .7p < p\]

and if $p < .5$

\[.7pq + (1 - p)(1 - q) \leq .7(1 - p) < 1 - p\]

Thus we see that $B$ is never a BR to any belief of $p$ and $q$. Thus $B$ is not independent rationalizable.
Part e.

Show (by example) that there exist independent rationalizable actions that are not in the support of any correlated equilibrium.

(Example by Sabyasachi)

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>M</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>3,3</td>
<td>2,3</td>
<td>0,2</td>
</tr>
<tr>
<td>M</td>
<td>2,1</td>
<td>3,1</td>
<td>3,3</td>
</tr>
<tr>
<td>D</td>
<td>0,3</td>
<td>3,0</td>
<td>−5, −5</td>
</tr>
</tbody>
</table>

All actions are independent rationalizable. Yet it must be the case that $P(D) = 0$ in any CE. Suppose not. If $P(D, L) > 0$ or $P(D, R) > 0$, player 2 has a strictly profitable deviation to play $M$ if told to play $D$. Yet if we have $P(D, M) > 0$ and $P(D, L) = P(D, R) = 0$ then player 2 can profitably deviate to play $L$ when told to play $M$.

Question 2

Part a.

Is $\{R^n\}_{n=1}^\infty$ an iterative delete-non-best-response sequence? If yes, verify. If no, explain which property it fails to satisfy. (Answer given by Vitor)

Notice that $\{R^n\}_{n=1}^\infty$ is (IDBR). The first requirement is simple, given by definition. The second requirement is also true, with equality. Just notice that the left side of the requirement is just the definition of $R^{n+1}_i$. The third requirement is also easy to verify. If there is some player $i$ and some action $a_i \in R^n_i$ that is not a best response to $R^n_{-i}$ then (since this is the requirement that defines $R^{n+1}_i$) we know that $a_i \notin R^{n+1}_i$ and then $R^n = \times_i R^n_i \neq R^{n+1} = \times_i R^{n+1}_i$.

Part b.

Is $\{IR^n\}_{n=1}^\infty$ an iterative delete-non-best-response sequence? If yes, verify. If no, explain which property it fails to satisfy.

No. Refer to the example given in the solution to problem 1 d. $B$ is not independent rationalizable. In other words $B \notin IR^1_B$. Yet we see from the correlated equilibrium that $B$ is a BR to the correlated strategies $\mu_{-3}$ where $\mu_{-3}(U, L) = \mu_{-3}(D, R) = \frac{1}{2}$. Thus we see $B \in X^1_3$. Thus $\{IR^n\}_{n=1}^\infty$ fails to satisfy property 2.

Part c.

Do iterative delete-non-best-response sequences converge? If yes, what can you say about their limits?

(Answer given by Vitor)

By the definition of (IDBR), the sequence not necessarily converges. I will provide one example.

Consider the following game

$$
\begin{bmatrix}
1,1 & 1,0 \\
0,1 & 0,0
\end{bmatrix}.
$$
Name the strategies as \( A_1 = \{U, D\} \) and \( A_2 = \{L, R\} \) (usual meaning). Now define the following sequence of sets of action profiles: for the first 2 periods
\[
X_0^0 = A_i \text{ for all } i; \\
X_1^0 = \{U\} \text{ and } X_2^1 = \{L\},
\]
and for periods \( n \geq 2 \)
\[
X_1^n = \begin{cases} 
\{U\}, & \text{if } n \text{ even;} \\
\{U, D\}, & \text{if } n > 1 \text{ odd}
\end{cases}
\]
and
\[
X_2^n = \{L\}.
\]

In the first 2 periods it clearly satisfies the requirements since they are equal to the rationalizability steps.

After the second period I have to check the requirements. First notice that the set of best responses for player 1 is always \( \{U\} \) and for player 2 it is always \( \{L\} \). Then they are always contained in \( X^n \). Also it is only the case that in odd periods \( n \) there is a strategy (namely, \( D \)) that is not a best response to \( X_2^n \), and indeed we have that \( X_1^{n+1} \neq X_1^n \). (In fact, subsequent sets are always different by construction.) Then the 3 requirements are satisfied and \( \{X^n\} \) is (IDBR).

Notice now that \( \{X_1^n\}_{n=1}^\infty \) oscillates each period, and thus is not convergent. Hence the same is true for \( \{X^n\}_{n=1}^\infty \) even though this is a pretty simple case.

The correction necessary to rule out such a trivial example (that turns the definition of (IDBR) uninteresting) is to require that the sequence of sets \( \{X^n\} \) be a decreasing one. This is in better accordance with the word “delete” in the question, since we require that the set be reduced whenever possible (instead of just requiring that it is different).

If we include in the definition that the sequence \( \{X^n\}_{n=1}^\infty \) has to be a decreasing one, then it is always the case that it converges after a finite number of steps (we can only eliminate something finitely many times).

We can also say that, if \( \{X^n\}_{n=1}^\infty \) is (IDBR), then \( R_i^n \subseteq X_i^n \) for all \( n \in \mathbb{N} \). Notice that \( X_0^0 = R^0_i = A_i \). From property 2 of (IDBR) we have that
\[
\{a_i \in X_0^0 | a_i \text{ is a best response to } X_0^{0-i}\} = R^1_i \subseteq X_1^1.
\]

Now, for some arbitrary \( n \), assume that \( R^n_i \subseteq X_i^n \) for all \( i \). Then we have that, for each \( i \)
\[
R_i^{n+1} \equiv \{a_i \in R^n_i | a_i \text{ is a best response to } X_{i-1}^n\} \subseteq \{a_i \in X_i^n | a_i \text{ is a best response to } X_i^n\} \subseteq X_i^{n+1},
\]
the middle \( \subseteq \) is justified since in the definition of \( R_i^{n+1} \) we are considering player \( i \)’s action from a smaller set (since \( R^n_i \subseteq X^n_i \)) and the requirement we are imposing is stronger (it has to be a best response to something also in a smaller set, since \( R_{i-1}^n \subseteq X_{i-1}^n \)). We conclude that \( X^*_i = \cap_{n \geq 1} X^n_i \subseteq R^*_i \).

But notice that for any \( i \), if \( a_i \in X^*_i \), this means that \( a_i \) is a best response to some \( a_{i-1} \in X_{i-1}^* \). If this was not the case, then for \( n \) sufficiently large \( X_{i-1}^n = X^*_{i-1} \) and then \( a_i \) is not a best response.
received emails but the other agent might have received since player 1 knows when the state is bad in
happens. Player one’s information is given by the sets

Now we describe the information of each agent in this model. No one know exactly everything that
happens. Player one’s information is given by the sets

Plain text representation:

Question 3

(Answer given by Vitor)

In this exercise we consider an alternative form way to describe the email information system
discussed in class. Instead of using the type space structure, we can describe the situation as a
probability space, in which an element would describe totally the relevant information for this
model, and we can describe the differential information of each player through “coarse” partitions of
the sample space. This means that no agent knows everything that happens, but can distinguish
over a set of situations.

Consider the sample space \( \Omega = \{0, 1, 2, \ldots \} \), in which an element \( \omega \in \Omega \) represents how many
messages have been received in total (summing up player 1 and player 2). I should emphasize that
we consider as a message the fact that player 1 learns that the state is \( \theta_y \).

So \( \omega = 0 \) means that we are in state \( \theta_0 \) and so no one receives any messages. In the situation
\( \omega = 1 \), the state is \( \theta_y \), so player 1 receives the initial message and sends his first message to player
2, which does not reach its destiny. Notice that an element describes completely all randomness in
the model (from the possible states and the possible errors in the messages). So this description
does not throw away any relevant dimension of the problem.

Using the same probabilities described in class we have that

\[ \mathbb{P}(\omega) = (1 - \varepsilon)^\omega \varepsilon. \]

Now we describe the information of each agent in this model. No one know exactly everything that
happens. Player one’s information is given by the sets

\[ P_1 = \{ \{0\}, \{1, 2\}, \{3, 4\}, \ldots \}, \]

since player 1 knows when the state is bad, i.e. \( \omega = 0 \), and then he knows that he has received \( n \)
emails, but the other agent might have received \( n - 1 \) messages (and \( \omega = 2n - 1 \) or he might have
received \( n \) (and then \( \omega = 2n \)). So his information sets are of the form \( (2n - 1, 2n) \), for \( n \) integer.

Player 2’s information is given by

\[ P_2 = \{ \{0, 1\}, \{2, 3\}, \ldots \}. \]
If player 2 has received 0 messages, it might be the case that player 1 also did not receive any ($\omega = 0$) or it might be the case that he received one, then sent a message to player 2 and it did not reach ($\omega = 1$). Then whenever player 1 has received $n$ messages, he consider as possible that player 2 has also received $n$ messages or that player 1 might have received $n + 1$ messages. Then his information sets are of the form $(2n, 2n + 1)$ for $n$ integer.

For each player $i$ denote as $P_i(\omega)$ as the set in the partition $P_1$ that contains $\omega$.

The interpretation is that whenever $\omega$ happens, player $i$ can only know that something in $P_i(\omega)$ happened, but cannot differentiate between anything inside this set. Then the equivalent of a type profile is a double $(P_1, P_2) \in P_1 \times P_2$. Remember that, as in the previous case, most of these elements have zero probability.

Now an event is any set $E \subseteq \Omega$. So the event “state is good” can be written as $E_g = \{1, 2, \ldots\}$. Similarly to the case described before, we can say that player $i$ knows event $E$ at partition $P_i$ if $P_i \subseteq E$. And the knowledge operator can be defined as $K_i(E) = \{\omega \in \Omega | P_i(\omega) \subseteq E\}$. So we have that $K_1(E_g) = E_g$ and $K_2(E_g) = \{2, 3, \ldots\}$.

Similarly we can define $K_n = \cap_i K_i(E)$ and $C(E) = \cap_n K_n^i(E)$ where $K_n^i(E) = K_i(K_{n-1}^i(E))$. Notice that it is also the case that $C(E_g) = \emptyset$.

**Question 4**

Example (OR Ex. 56.3). Find the rationalizable actions of each player in the following game.

Since this is a 2 player game, we know that the set of actions surviving iterative strict dominance is equivalent to the set of actions which are rationalizable.

Note that $\frac{1}{2} b_1, \frac{1}{2} b_3$ strictly dominates $b_4$. Thus we may truncate the game and eliminate $b_4$. Then in the truncated game, $a_4$ is strictly dominated by $a_2$. Since we can check there are no more dominated actions, we see $R_1^* = \{a_1, a_2, a_3\}$ and $R_2^* = \{b_1, b_2, b_3\}$.

**Question 5**

Part a.

By solving the agents first order conditions, we find the unique NE is given by $a = < a_1 \ldots a_I >$ where:

$$a_i = \frac{1}{I+1} \forall i.$$ 

I=2

Note that $A_i \in [0, 1]$, and $g_i(a) = a_i (1 - a_1 - a_2)$. We may rewrite $g$ as:

$$g_i(a) = a_i (1 - a_i) - a_i a_j.$$ 

We then see that:

$$g_i(\frac{1}{2}, a_j) = \frac{1}{4} - \frac{1}{2} a_j > g(a_i, a_j) \forall a_i \in (.5, 1], a_j \in [0, 1]$$
since \( \frac{1}{4} > a_i(1 - a_i) \forall a_i \in (0.5, 1], a_j \in [0, 1] \) and \(-\frac{1}{2}a_j > -a_i a_j \forall a_i \in (0.5, 1], a_j \in [0, 1]. \) Thus all \( a_i > \frac{1}{2} \) are strictly dominated by \( a_i = \frac{1}{2}. \)

Thus \( R_1^1 = [0, \frac{1}{2}]. \)

Assume \( R_i^n = [low^n, high^n] \) (note by symmetry we need not index this by \( i \)). We already know it must be the case that \( 0 \leq low \leq high \leq \frac{1}{2}. \) We know want to find \( R_i^{n+1}. \) If agent \( i \) believes agents \( j \) plays the strategy \( f(a_j) \) with support over \( R_j^n \) we have that \( i \)'s best response is given by:

\[
a_i = \frac{1 - \mathbb{E}[a_j]}{2}.
\]

Since we know \( \mathbb{E}[a_j] \in [low^n, high^n] \) we get that \( R_i^{n+1} = [\frac{1 - high^n}{2}, \frac{1 - low^n}{2}] \cap R_i^n. \) Noting that \( low^1 = 0, \) \( high^1 = \frac{1}{2} \) we see:

\[
low^{n+1} = \frac{1 - high^n}{2}
\]
and

\[
high^{n+1} = \frac{1 - low^n}{2}.
\]

This implies:

\[
high^{n+2} = \frac{1 - \frac{1 - high^n}{2}}{2} = \frac{1}{4} + \frac{high^n}{2}
\]
and

\[
low^{n+2} = \frac{1}{4} + \frac{low^n}{2}.
\]

Thus using the starting points \( high^1 = \frac{1}{2} \) and \( low^1 = 0 \) we see:

\[
\lim_{n \to \infty} high^n = \lim_{n \to \infty} low^n = \frac{1}{3}.
\]

Thus we see the NE outcome is the only rationalizable, and thus independent rationalizable outcome.

**I \geq 3**

Using the same argument as above, we can show that \( R_1^1 = [0, \frac{1}{2}]. \) We can then see \( R_2^1 = [0, \frac{1}{2}]. \) The reason behind this is is that \( x \in [0, \frac{1}{2}] \) is a best reply to \( a_{-i} = < a_1 \ldots a_{i-1}, a_{i+1} \ldots a_I > \) where \( a_j = \frac{1 - 2x}{I - 1} \forall i \neq j. \) Since \( x \in [0, \frac{1}{2}] \) we know \( a_{-i} \in R_1^1. \) This gives us \( R_1^1 = R_2^1 = [0, \frac{1}{2}]. \) which implies \( R_i^* = [0, \frac{1}{2}] \forall i. \)

Precisely the same argument can be used to show that \( IR_i^* = [0, \frac{1}{2}] \forall i. \)

**Part b.**

(Answer provided by Vitor)
Nash Equilibria:

Now let’s characterize an arbitrary Nash equilibria $\vec{a}$. Suppose (WLOG) that firm 1 produces 0, this means that

$$1 - \gamma \sum_{j \geq 2} a_j \leq 0.$$ 

Then we have that $\sum_{j > 2} a_j \geq 1 > 0$ and so at least some firm is producing positively. Take a firm $i$ that has $\vec{a}_i > 0$, then

$$g_i(\vec{a}) = \vec{a}_i \left( 1 - \vec{a}_i - \gamma \sum_{j \neq i} \vec{a}_j \right)$$

$$= \vec{a}_i \left( 1 - \gamma \sum_{j \geq 2} \vec{a}_j - (1 - \gamma) \vec{a}_i \right) < 0,$$

but this is impossible since each firm has to have payoff at least zero! Then all firms produce positively, which means that

$$\vec{a}_i = \frac{1 - \gamma \sum_{j \neq i} \vec{a}_j}{2},$$

then summing up over $i$ (and using $\vec{Q} = \sum_i \vec{a}_i$),

$$\vec{Q} = \frac{1}{2} (I - \gamma (I - 1) \vec{Q})$$

$$\downarrow$$

$$\vec{Q} = \frac{I}{2 + \gamma (I - 1)}.$$

And we also have

$$\vec{a}_i = \frac{1 - \gamma \sum_{j \neq i} \vec{a}_j}{2} = \frac{1 - \gamma \vec{Q} + \gamma \vec{a}_i}{2}$$

$$\vec{a}_i = \frac{1 - \gamma \vec{Q}}{2 - \gamma}$$

$$\vec{a}_i = \frac{1 - \gamma \frac{I}{2 + \gamma (I - 1)}}{2 - \gamma} = \frac{1}{2 + \gamma (I - 1)}.$$

So this is the only Nash Equilibria of this game.

Rationalizable strategies:

we start with $R_0^0 = [0, 1]$. Once again we can look at best responses of only pure strategies by players $-i$ since everyone only cares about the expected joint production of the opponents (and the pure strategy set is convex). Since the best response function is continuous, the set of best responses to $R_k^i$ is going to also be an interval, with limits

$$q^i = \max \left\{ 0, \frac{1 - \gamma (I - 1)}{2} \right\};$$

$$q^i = \frac{1}{2}.$$
So we have $R_i^1 = [q^1, \bar{q}^1]$ for all $i$.

Notice that $q^1 > 0$ if
$$\frac{1}{\gamma} > I - 1.$$ 

And $q^1 = 0$ if $\frac{1}{\gamma} \leq I - 1$.

So now we have two possibilities:

1) If $q^1 = 0$, then we have that
$$\bar{q}^2 = \frac{1 - \gamma 0}{2} = \frac{1}{2},$$

and
$$q^2 = \max \left\{ 0, \frac{1 - \gamma (I - 1)}{2} \right\},$$

Once again there are 2 possibilities

$$q^2 = 0 \quad \Downarrow$$
$$\frac{1}{\gamma} \leq \frac{I - 1}{2}.$$ 

Or we have that $q^2 > 0$ if
$$\frac{1}{\gamma} > \frac{I - 1}{2}.$$ 

So in case $\frac{1}{\gamma} \leq \frac{I - 1}{2}$ we have that $R_i^2 = [q^2, \bar{q}^2] = R_i^1 = [q^1, \bar{q}^1]$ and so we have that $R_i^* = [0, \frac{1}{2}]$.

Now in the case where $\frac{1}{\gamma} > \frac{I - 1}{2}$, we have that $q^2 > 0$ and $\bar{q}^2 = \frac{1}{2}$, so from this period one we have that
$$q^{k+1} = \frac{1 - \gamma (I - 1)q^k}{2};$$
$$\bar{q}^{k+1} = \frac{1 - \gamma (I - 1)\bar{q}^k}{2}.$$ 

This means that best responses will always be strictly positive from then on.

Given the assumption on parameters, we have that this mapping is a contraction. We also have that, using as $q^* = \frac{1}{2 + \gamma (I - 1)} = \frac{1 - \gamma (I - 1)q^*}{2}$ (given the solution of Nash equilibria) we have that
$$q^* - q^{k+1} = \frac{\gamma (I - 1)}{2} (q^k - q^*);$$
$$q^{k+1} - q^* = \frac{\gamma (I - 1)}{2} (q^* - q^k).$$ 

And remember that the case we are focusing is equivalent to $\frac{\gamma (I - 1)}{2} < 1$, then both this distances converge to zero and then
$$R_i^* = \{q^*\} = \left\{ \frac{1}{2 + \gamma (I - 1)} \right\}.$$ 

And the only rationalizable strategy is the NE one.
Part b) So the requirement needed so that Rationalizable strategies and Nash Equilibrium concepts are the same is that
\[
\frac{\gamma (I - 1)}{2} < 1.
\]
which means that either the number of firms is not very large (so that for reasonable production levels every firm still considers producing positively) or it is the case that substitution between firms is very small (and each firm does not care so much about what others are producing).

In the case
\[
\frac{\gamma (I - 1)}{2} \geq 1,
\]
we have that \( R^*_i = [0, \frac{1}{2}] \) and rationalizability has considerably less predictive restrictions than Nash.

**Question 6**

(Answers provided by Vitor)

**Part a.**

Characterize the set of correlated equilibria.

First we solve for the set of correlated equilibria, I will name the probabilities as
\[
\begin{bmatrix}
  p_{11} & p_{12} \\
  p_{21} & p_{22}
\end{bmatrix}.
\]

So we have to make sure that, whenever an agent plays some action with positive probability, this action is optimal given the conditional expected play for the other agents. For player 1 we have 2 optimality conditions. The condition for him to play “Hawk” is
\[
p_{11}0 + p_{12}5 \geq p_{11}1 + p_{12}4 \iff p_{12} \geq p_{11},
\]
and the condition for him wanting to play “Dove” is
\[
p_{12}1 + p_{22}4 \geq p_{21}0 + p_{22}5 \iff p_{21} \geq p_{22}.
\]

For player 2. The condition under which he wants to play “hawk” is
\[
p_{11}0 + p_{21}5 \geq p_{11}1 + p_{21}4 \iff p_{21} \geq p_{11},
\]
and the condition for him to want to play “Dove” is
\[
p_{12} + p_{22}4 \geq p_{12}0 + p_{22}5 \iff p_{12} \geq p_{22}.
\]

Notice that each of this conditions has to hold whenever an agent plays with positive probability an action. And in the case where an agent does not play an action with positive probability, the conditions are also satisfied trivially (since both sides are zero). Therefore these have to be true in any Correlated Equilibria.
So we have a characterization of the set of correlated equilibria, CE:

$$CE = \{ (p_{11}, p_{12}, p_{21}, p_{22}) \in \Delta (A) \mid (1), (2), (3) \text{ and } (4) \text{ hold} \}.$$ 

Basically the requirement for a distribution to be a CE is that the diagonal entries are not played with large probabilities. This is so because whenever an agent believes that a diagonal play is occurring with large probability, he has an incentive to deviate (people do not want to “hawk” each other and to not want to “dove” each other).

**Part b.**

Characterize the set of expected payoff vectors that players can achieve in correlated equilibrium.

See graph in answer to f.

**Part c.**

Characterize the set of Nash equilibria.

In this game we have 3 possible NE (2 pure and one mixed)

$$NE = \left\{ (\text{hawk, dove}), (\text{dove, hawk}), \left( \left( \frac{1}{2}, \frac{1}{2} \right), \left( \frac{1}{2}, \frac{1}{2} \right) \right) \right\}.$$ 

**Part d.**

Characterize the set of expected payoff vectors that players can achieve in Nash equilibrium.

The payoffs vectors achieved with these 3 equilibria respectively are \((5,1), (1,5)\) and \((\frac{5}{2}, \frac{5}{2})\).

**Part e.**

Find the correlated equilibrium that maximizes the sum of the players payoffs and minimizes the sum of the players payoffs.

Since the set of correlated equilibria is defined by a set of linear inequalities, characterizing the symmetric optimal allocation is a simple linear programming problem. we want to find

$$p^{\text{max}} = \arg \max_{p \in CE} \, 8p_{22} + 6(p_{12} + p_{22}) \; \text{s.t. } p \in CE.$$ 

Clearly if we were not constrained by equilibrium conditions, the optimal would be to just play \((p_{22} = 1)\) and get payoffs \((4,4)\) (which have maximal sum =8). Clearly this is not in CE since each player would want to deviate do “Hawk”. Then binding constraints will be the ones for each player playing dove optimally. So the solution (given the linearity of the problem) is basically to reduce \(p_{22}\) and increase \(p_{12}\) and \(p_{21}\) so that the binding constraint holds with equality, i.e.,

$$p_{12} = p_{21} = p_{22}.$$ 

Using \(p_{11} = 0\) is clearly optimal since this is the worst possible realization in terms of payoff and \(p_{11} = 0\) makes the other incentive constraints hold always. So the optimal is

$$p_{12} = p_{21} = p_{11} = \frac{1}{3}; \quad p_{11} = 0.$$
And the payoff vector achieved is \((\frac{16}{3}, \frac{16}{3})\) which has joint payoff of \(\frac{20}{3}\) (notice that it is bigger than the joint payoff of any NE).

**Part f.**

Plot your answers to (b) and (d) on a graph.