Suggested Answers for Problem Set 2

1. Public Goods

(a) The pivot mechanism is given by the efficient decision rule:

\[ q^*(\theta) = 1 \text{ if } \sum_i \theta_i \geq c, \]

and

\[ q^*(\theta) = 1 \text{ if } \sum_i \theta_i < c, \]

and transfer rule

\[ t_i(\theta) = \theta q^*(\theta, \theta_{-i}) + (q^*(\theta) - q^*(\theta, \theta_{-i}))(c - \sum_{j \neq i} \theta_j) \]

for all \( i \) and all \( \theta \).

(b) It is obvious from the definition of the transfers in the pivot mechanism that the lowest type individual receives the outside utility 0. Since the payments across any two mechanisms with the same decision rule \( q \) differ by a constant (by the envelope calculation behind revenue equivalence), we know that the pivot mechanism extracts the largest transfer in the class of mechanisms that satisfy IC and IR and that implement the efficient rule.

Consider next the total payment in the mechanism. If \( q^*(\theta) = 0 \), then no agent pays any transfers and since there are no costs, the deficit is zero. If \( q^*(\theta) = 1 \), and \( q^*(\theta, \theta_{-i}) = 1 \) for all \( i \), then each agent pays \( \theta \) and since by assumption \( N\theta < c \), there is a deficit.

Consider next \( \theta \) such that If \( q^*(\theta) = 1 \), and \( q^*(\theta, \theta_{-i}) = 0 \) for some \( i \). Call such players pivotal and other players nonpivotal. We denote these cases by \( i \in P \) and \( i \in NP \) respectively. We compute the transfers as follows:
\[
\sum_{i \in P} (c - \sum_{j \neq i} \theta_j + \sum_{i \in NP} \theta) = Pc - (P - 1) \sum_{j \in NP} \theta_j - (P - 1) \sum_{i \in P} \theta_i \sum_{i \in NP} (\theta_i - \theta)
\]

\[
= Pc - (P - 1) \sum_{j \in NP} \theta_j - \sum_{i \in NP} (\theta_i - \theta).
\]

To see that this sum is no more that \( c \), note that

\[
= Pc - (P - 1) \sum_{j} \theta_j - \sum_{i \in NP} (\theta_i - \theta) \leq c
\]

\[
\Leftrightarrow (P - 1)c \leq (P - 1) \sum_{j} \theta_j + \sum_{i \in NP} (\theta_i - \theta).
\]

But this follows immediately from our assumption that \( q^*(\theta) = 1 \) and therefore

\[
c \leq \sum_{j} \theta_j.
\]

Hence the pivot mechanism results in a deficit in this case as well and therefore, there must be an expected deficit in the pivot mechanism.

(c) Suppose \( \lim_{N \to \infty} \Pr\{q(\theta) > \epsilon\} > \delta \) for some \( \delta, \epsilon > 0 \) in an IC, IR and BB mechanism. Then the average ayment must be strictly higher than \( \underline{\theta} \) by our assumption \( N\underline{\theta} < c \). The claim is proved if you can show that for any mechanism that provides the public good with a strictly positive probability, the probability with
which each individual player is pivotal goes to zero. This is intuitively clear from law of large numbers type arguments. For formal details, see

Asymmetric Information Bargaining Problems with Many Agents, with Andrew Postlewaite, Review of Economic Studies, 57 (July 1990), 351367.

2. Double Auctions

(a) If there are no IR constraints, just use an $n + 1^{st}$ price auction. If there are no IR constraints, then the goods can be confiscated and then reallocated in an auction.

(b) For this question, dominant strategy mechanisms (that respect IR) without budget balance are given by the usual VCG mechanisms. For example, the pivot mechanism can be given by the allocation functions $q_i(\theta)$ and transfer rules $t_S(\theta)$ and $t_B(\theta)$ for the sellers and buyers respectively (we denote sellers and buyers by $i \in S$ and $i \in B$):

$$q - i(\theta) = 1 \iff \#\{j \mid \theta_j < \theta_i\} \geq N.$$  

The transfer to a seller $i$ such that $q_i = 0$ is given by:

$$t_S(\theta_i, \theta_{-i}) = \min_{j \in B}\{\theta_j \mid q_j(\theta) = 1\}.$$  

For buyer $i$ such that $q_i = 1$, we have

$$t_B(\theta_i, \theta_{-i}) = \max_{j \in S}\{\theta_j \mid q_j(\theta) = 0\}.$$  

Unfortunately, it can be shown (along the lines of the previous problem) that the pivot mechanism runs a budget deficit.
Consider next the class of BB mechanisms that are ex post individually rational and that satisfy dominant strategy incentive compatibility. (In the case of double auctions, the BB constraint seems very natural).

For the case $N = 1$, Hagert and Rogerson (1987) 'Robust Trading Mechanisms' in Journal of Economic Theory 42, 94-107 (1987) shows that the class of dominant strategy mechanisms with ex post IR and BB are fixed price mechanisms, i.e. ones where trade occurs at a pre-set price $t^*$ if and only if both the seller and the buyer agree to the trade.

The generalization to markets with $N$ buyers and sellers are direct mechanisms where all players report their true valuations. Let $n_S = \#\{j \in S \mid \theta_j \leq t^*\}$ and $n_B = \#\{j \in S \mid \theta_j \geq t^*\}$. Furthermore let $n = \min\{n_S, n_B\}$. No seller with $\theta_j > t^*$ trades and no buyer with $\theta_i < t^*$ trades. There are $n$ trades in total with all the traders on the short side trading. The rationing on the long side is uniform.

Notice that by construction, we have a BB mechanism that satisfies IR and dominant strategy IC. This mechanism is otherwise very nice, but the rationing is not based on efficiency (i.e. the probability of being rationed is not dependent on reported type). This leads to efficiency losses. Then again, DIC would be lost if the probability of rationing did depend on reported types.

(c) Consider the mechanism where $t^* = F(\frac{1}{2})$. By the (weak) law of large numbers,

$$\frac{n_S - n_B}{N} \rightarrow 0$$

in probability as $N \rightarrow \infty$. 

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Therefore almost all profitable trades are executed in the limit as the market gets large. The difference to the previous problem is that here individual announcements have an effect on the (individual) allocation decisions but not on the transfer. In Problem 1, the probability of being pivotal is negligible, but the announcement has an effect on own transfer if the good is provided with positive probability.

3. Efficient Committees The hints were already provided (not meant to be included on the web site originally...) The last point is that the dynamic pivot mechanism results in a surplus. Therefore it is possible to have a BB mechanism that is efficient and incentive compatible (redistribute the surplus from the pivot mechanism to the players).