1. **(Global Game)** We consider the same game considered in the last problem set. A large, that is a continuum, population with unit mass (so you can index player \( i \in [0,1] \)), must choose an action, "invest" or "not invest". There is a cost \( y \) to investing and assume \( 0 < y < 1 \). There is a benefit \( 1 \) of investing (investment "succeeds") if at least proportion \( 1 - \theta \) of the population invests (i.e., at most \( \theta \) do not invest), and as before state \( \theta \in \mathbb{R} \) represents "fundamentals" of the economy. We suppose that the players only have private and noisy information, that is we assume that

\[
\theta \sim \mathcal{U} [-M, M]
\]

for large \( M > 0 \), but each player only observe a signal \( x_i \)

\[
x_i = \theta + \frac{1}{z} \epsilon_i, \quad \epsilon_i \sim \mathcal{U} \left[ -\frac{1}{2}, \frac{1}{2} \right]
\]

and so \( z \) represent the accuracy of the signal \( x_i \).

We will show the following proposition. For all \( \tilde{\theta} \in [-M, M] \setminus \{y\} \), there exists a \( z \) small enough, such that in every Bayes Nash equilibrium of the incomplete information game, players invest when the realized state is \( \theta = \tilde{\theta} \) if and only if \( \tilde{\theta} \geq y \).

(a) Define formally a strategy \( s \) for a player in this game.

(b) Suppose an investment succeeds if \( \theta > \tilde{\theta} \), find the critical type \( \bar{x} \) that finds it a dominant strategy to invest. Similarly, suppose an investment does not succeed if \( \theta < \tilde{\theta} \), find the critical type \( \underline{x} \) that finds it a dominant strategy not to invest.

(c) Suppose an agent invests if he gets a signal \( x > \bar{x} \), find the critical state \( \tilde{\theta} \) such that the investment surely succeeds for \( \theta > \tilde{\theta} \). Similarly, suppose an agent does not invest if he gets a signal \( x < \underline{x} \), find the critical state \( \underline{\theta} \) such that the investment surely does not succeed for \( \theta < \underline{\theta} \).
(d) Define the following sets,

\[ S \triangleq \{ \text{the set of all available strategies for any given player} \} \]

\[ S_n \triangleq \left\{ \text{the set of strategies that are not strictly dominated for player } i, \text{ when all other players choose } s \in S_{n-1} \right\}. \]

\[ \bar{\theta}_n = \min\{ \theta : \text{investment succeeds, given that agents choose } s \in S_{n-1} \} \]

\[ \bar{\theta}_n = \max\{ \theta : \text{investment does not succeed, given that agents choose } s \in S_{n-1} \} \]

Using parts 1b and 1c, provide a characterization of \( \bar{\theta}_n \) and \( \bar{\theta}_n \) in terms of \( S_{n-1} \) and a characterization of \( S_n \) in terms of \( \bar{\theta}_n \) and \( \bar{\theta}_n \).

(e) Characterize \( S_\infty \).

(f) Conclude by arguing that for all \( z \), the investment succeeds if and only if \( \theta \geq y \). Moreover, as \( z \to \infty \) in any BNE agents invest if and only if the state is greater than \( y \).

2. **Optimal Taxation.** Exercise 3.1 in Salanie. This question explores the famous model of optimal taxation of Mirrlees (1971).

3. **Revelation Principle.** In class we stated the revelation principle for a single agent. Now, state and proof the revelation principle for many agents with:

(a) for pure strategy equilibrium in dominant strategies;

(b) for pure strategy Bayesian Nash equilibrium;

(c) what, if any, are differences in the proof of the revelation principle for dominant and Bayesian Nash equilibrium.

4. **First Price Auction with Private Values.** Consider a first-price sealed-bid auction of an object between two risk-neutral bidders. Each bidder \( i \) (for \( i = 1, 2 \)) simultaneously submits a bid \( b_i \geq 0 \). The bidder who submits the highest bid receives the object and pays his bid; both bidders win with equal probability in case they submit the same bid. Before the auction takes place, each bidder \( i \) privately observes the realization of a random variable \( t_i \) that is drawn independently from a uniform distribution over the interval \([0, 1]\). The actual valuation of the object to bidder \( i \) is equal to \( t_i + 0.5 \). Therefore, the payoff of bidder \( i \) is given by

\[ u_i = \begin{cases} 
  t_i + 0.5 - b_i & \text{if } b_i > b_j \\
  \frac{1}{2} (t_i + 0.5 - b_i) & \text{if } b_i = b_j \\
  0 & \text{if } b_i < b_j
\end{cases} \]
(a) Derive the symmetric linear Bayesian Nash equilibrium for this game (i.e., each bidder uses an equilibrium strategy of the form $b_i = \alpha t_i + \beta$).

(b) What is the conditionally expected payoff of bidder $i$ with type $t_i$ in this equilibrium?

5. **First Price Auction with Common Values.** Consider the first-price auction in problem (4), except that the actual valuation of the object to bidder $i$ is now equal to $t_i + t_j$ ($j \neq i$) and therefore the payoff of bidder $i$ now becomes

$$u_i = \begin{cases} 
    t_i + t_j - b_i & \text{if } b_i > b_j \\
    \frac{1}{2} (t_i + t_j - b_i) & \text{if } b_i = b_j \\
    0 & \text{if } b_i < b_j 
\end{cases}$$

Notice that, given his own private type $t_i$, the expected value of the object is $t_i + 0.5$ (i.e., the same as that in problem (4)).

(a) Derive the symmetric linear Bayesian Nash equilibrium for this game.

(b) Compare the equilibrium bid for any given type $t_i$ in this problem to that in problem (4). Interpret your result.

6. **First Price Auction.** Consider the first price auction in a symmetric environment with binary valuations, i.e. the value of bidder $i$ is given by $v_i \in \{v_l, v_h\}$ with $0 \leq v_l < v_h < \infty$. It is sufficient to consider the case of $i = 1, 2$.

(a) The prior probability is given by $Pr(v_i = v_h) = \alpha$ for all $i$. Characterize the equilibrium in the first price auction. (Hint: Can you find a pure strategy Bayesian Nash equilibrium?)

(b) The prior probability is now given by $Pr(v_i = v_h) = \alpha_i$ with $0 < \alpha_1 < \alpha_2 < 1$. Characterize the equilibrium in the first price auction. (Hint: Can you find a pure strategy Bayesian Nash equilibrium?)

(c) Does the revenue equivalence result between the first and the second price auction still hold with the binary payoff types.

**Reading MWG:** 23, S (=Salanie) 2 and 3