Microeconomic Theory (501b)

Final Exam

This is a closed-book exam. The exam lasts for 180 minutes. Please write clearly and legibly. Be especially careful in the definition of the game, the payoff function and the equilibrium notions. The allocated points are also a good indicator for your time budget. Please record the answers (1, 2, 3) and 4 in two separate bluebooks.
1. (40) Consider simple case of an indivisible public project that has value \( S \) for the consumers. A single firm (monopolist) can realize the project. Its cost function is
\[
c = c(e, \beta) = \beta - e, \tag{1}
\]
where \( \beta \) is a known efficiency parameter and \( e \) is the managers’ effort. If the firm exerts effort level \( e \), it decreases the (monetary) cost of the project by \( e \) and incurs a disutility (in monetary units) of \( \psi(e) \). This disutility displays \( \psi' > 0 \), and satisfies \( \psi(0) = 0 \) and \( \lim_{e \to \beta} \psi(e) = +\infty \). The firm’s utility level is:
\[
U = t - (\beta - e) - \psi(e)
\]
The “reservation utility” of the firm is normalized to 0. Let \( \lambda > 0 \) denote the shadow cost of public funds. That is taxation inflicts a disutility \( (1 + \lambda) \) on taxpayers in order to levy \( $1 \) for the state. The net surplus of consumers/taxpayers if the project is realized is \( S - (1 + \lambda)t \). Hint: You may assume that it is socially optimal to realize the project.

(a) Assume first that cost and in particular effort is observable by the regulator. Describe the optimal solution \( \{e^*, t^*(e^*)\} \) for a utilitarian regulator, whose ex-post social welfare can be described by
\[
S + U - (1 + \lambda) t
\]
who has to respect the participation constraint by the firm. Briefly describe the intuition of your result.

(b) Show that the optimal solution can be implemented through fixed price contract such that \( t^*(e) = t^* \) for all \( e \) provided that the project is realized.

(c) Suppose now that the firm could either be efficient \( \beta_l \) or inefficient \( \beta_h \) with \( \beta_l < \beta_h \). The prior probability of each type is given by \( p_l \) and \( p_h \). The regulator only observes the realized cost \( c \) as defined in (1) and can make a transfer \( t \) to the firm. However, he does not observe \( \beta \) and \( e \) separately. A contract based on the observables \( t \) and \( c \) specifies a transfer-cost pair for each type of firm, namely \( \{t(\beta_l), c(\beta_l)\} \) for type \( \beta_l \) and \( \{t(\beta_h), c(\beta_h)\} \) for type \( \beta_h \). Define the optimization program for the regulator who wants to maximize social welfare and would like to make separate offers to low and high cost types of the firm. Hint: First write the utility of an agent of type \( \beta_i \) that chooses contract \( (t_j, c_j) \) so that (verify for yourself (!)):
\[
V(\beta_i, c_j, t_j) = t_j - c_j - \psi(\beta_i - c_j).
\]

(d) Derive the effort levels under the optimal regulation scheme. (Hint: First argue briefly which of the participation and incentive constraints are going to bind. After that it suffices to give a clean version of the first order conditions.)
(e) What can you say about efficiency and information rent for the firms.

[Solutions]

(a) From the statement of the problem, we know that:

\[ U = t - (\beta - e) - \psi(e) \]

Thus, the total surplus is given by:

\[ (S - \lambda t - \psi(e) - \beta + e) \cdot 1, \]

where 1 is equal to 1 if the project is realized and 0 otherwise. The participation constraint of the firm is given by:

\[ t - \beta + e - \psi(e) \geq 0. \]

Assuming the project must be realized, the utilitarian regulator must solve:

\[ (e^*, t^*) \in \arg \max_{e,t} S - \lambda t - \psi(e) + e - \beta \]

subject to: \( t - \beta + e - \psi(e) \geq 0. \)

Clearly the constraint will bind, thus we can just solve:

\[ e^* \in \arg \max_{e,t} S - \lambda(\beta - e + \psi(e)) - \psi(e) + e - \beta \]

Finding the F.O.C. we get:

\[ \psi'(e^*) = 1. \]

We can also easily recover the transfers needed to pay for the firm:

\[ t^* = \beta - e^* + \psi(e^*). \]

Thus, it is optimal for the project to be realized if and only if:

\[ (S - \lambda t^* - \psi(e^*) - \beta + e^*). \]

If it is optimal for the project to be realized, then the optimal effort is chosen to minimize the cost of construction.

(b) If it is optimal not to construct the project, then it is obviously optimal to just impose \( t^* = 0, \) and the firm will never construct the project. If it is optimal for the project to be realized, then the solution is still to impose transfers equal to the first best. That is, the regulator imposes transfers equal to:

\[ t^* = \beta - e^* + \psi(e^*), \]
if the project is realized. Note that in this case the regulator can impose any non-positive transfer if the project is not realized, in particular he imposes a transfer equal to 0 if the project is not realized. Note that, \( e - \psi(e) \leq e^* - \psi(e^*) \) for all \( e \). Thus, the firm will always exert the optimal effort if it decides to realize the project. Moreover, under the optimal contract the for the firm it is optimal to realize the project if and only if it is socially optimal, and always gets is reservation utility.

(c) The regulator must solve the following maximization problem:

\[
\max_{\{t_i, c_i, t_h, c_h\}} p_h(S - \lambda t_h - c_h - \psi(\beta_h - c_h)) + p_l(S - \lambda t_l - c_l - \psi(\beta_l - c_l))
\]

subject to:

\[
\begin{align*}
(t_h - \psi(\beta_l - c_h) - c_h) & \leq (t_l - c_l - \psi(\beta_l - c_l)) \\
(t_h - c_h - \psi(\beta_h - c_h)) & \geq (t_l - c_l - \psi(\beta_l - c_h)) \\
(t_h - \psi(\beta_l - c_h) - c_h) & \geq 0 \\
(t_h - c_h - \psi(\beta_h - c_h)) & \geq 0
\end{align*}
\]

(d) Note that the utility of an agent of type \( \beta_i \) that chooses contract \((t_j, c_j)\) is given by:

\[
V(\beta_i, c_j, t) = t_j - c_j - \psi(\beta_i - c_j).
\]

Moreover, it is easy to note that:

\[
\frac{\partial V(\beta_i, c, t)}{\partial \beta_i} = -\psi'(\beta_i - c) \leq 0,
\]

thus only the participation of \( \beta_h \) will bind. Second, if \((t_h^*, e_h^*)\) and \((t_l^*, e_l^*)\) as defined by the first best were to be offered, then type \( h \) would choose his own contract and get a utility of 0, while the \( l \) type would rather choose the contract of the \( l \) type as this would yield strictly positive utility. Thus, in the optimal contract we will have that the incentive constraint that the low type wants to pretend to be a high type is the one that binds (we check ex-post that this is true). Thus, the regulator can solve:

\[
\max_{\{t_i, c_i, t_h, c_h\}} p_h(S - \lambda t_h - c_h - \psi(\beta_h - c_h)) + p_l(S - \lambda t_l - c_l - \psi(\beta_l - c_l))
\]

subject to:

\[
\begin{align*}
(t_h - \psi(\beta_l - c_h) - c_h) & \leq (t_l - c_l - \psi(\beta_l - c_l)) \\
(t_h - c_h - \psi(\beta_h - c_h)) & \geq (t_l - c_l - \psi(\beta_l - c_h))
\end{align*}
\]

Since clearly both constraints will bind, we can just replace the transfers from the constraints into the object function and maximize over \((c_h, c_l)\). That is, we get the following objective function:

\[
\max_{\{t_i, c_i, t_h, c_h\}} p_h(S - (1+\lambda(1+\frac{p_h}{p_l}))(c_h + \psi(\beta_h - c_h)))+ p_l(S - (1+\lambda)(c_l + \psi(\beta_l - c_l))) + \lambda p_l(c_h + \psi(\beta_l - c_h))
\]
The first order conditions are given by (we denote by \( \tilde{c}_l \) and \( \tilde{c}_h \) the solutions to the second best):

\[
\psi'(\beta_l - c_l^*) = 1 \tag{2}
\]

\[
(1 + \lambda (1 + \frac{p_h}{p_l}))(1 - \psi'(\beta_h - \tilde{c}_h)) + \lambda p_l (1 - \psi'(\beta_l - \tilde{c}_h)) = 0 \tag{3}
\]

(e) From 2 we can see that the \( l \) type exerts his efficient effort, as this is the same as the problem solved by the social planner. Since \( \psi'' > 0 \) and \( \beta_l < \beta_h \), we have that:

\[
0 = (1 + \lambda (1 + \frac{p_h}{p_l}))(1 - \psi'(\beta_h - c_h^*)) + \lambda p_l (1 - \psi'(\beta_l - c_h^*))
\]

\[
< (1 + \lambda (1 + \frac{p_h}{p_l}))(1 - \psi'(\beta_h - c_h^*)) + \lambda p_l (1 - \psi'(\beta_l - c_h^*))
\]

where \( c_h^* \) is the cost that would implement the first best effort. Thus, we have that \( \tilde{c}_h > c_h^* \). Finally, we know that the \( l \) type gets the informational rents as

\[
\frac{\partial V(\beta_i, c, t)}{\partial \beta_i} = -\psi'(\beta_i - c) \leq 0
\]
2. (40) Consider a second price auction with two bidders and an entrance fee $f$. The valuation for each bidder is uniformly distributed in $[0, 1]$. The timing is as follows. First the seller offers a second price auction with an entrance fee $f > 0$ which is a fee that the bidders have to pay if they participate in the auction, independently of the outcome of the auction. Second, given the auction format, the bidders have to simultaneously decide whether or not to participate, and if so, how much to bid. If they don’t bid at all (or equivalently bid zero) then they don’t have to pay the fee but will also never get the object.

(a) Describe the payoff function of each bidder as a function of an arbitrary pair of bids.

(b) Define the notion of a strategy and the notion of a Bayesian Nash equilibrium for the auction game with the entry fee. Describe the expected payoff of the auctioneer as a function of the bidding strategies and the entry fee.

(c) Does the imposition of an entrance fee change the bidding strategy of the agents?

(d) Compute the symmetric Bayesian Nash equilibrium.

(e) Compute the seller’s expected revenue from the auction with an arbitrary entrance fee $f > 0$.

(f) Which entrance fee maximizes the expected revenue of the auctioneer? Is the resulting allocation ex-post efficient (assume the seller’s valuation of the good is zero)?

[Solution]

(a) The utility of an agent of type $t_i$ bidding $b_i$ when the other agent bids $b_j$ is given by:

$$u(t_i, b_i, b_j) = \begin{cases} 
0 & b_i = 0 \\
-f & 0 < b_i < b_j \\
t_i - b_j - f & 0 < b_j < b_i 
\end{cases}$$

(b) A strategy is given by a function, $b_i : [0, 1] \rightarrow \mathbb{R}$, which specifies the bid as a function of the type of an agent. The expected utility of the seller is given by:

$$f \cdot \mathbb{P} \{b_i(t_i) > 0\} + f \cdot \mathbb{P} \{b_j(t_j) > 0\} + \mathbb{E}[\min\{b_i(t_i), b_j(t_j)\}]$$

(c) Yes it does, it is easy to see that a type $t_i < f$ will always bid 0, independent of the bid of the other agent. This implies that the bidding strategy will be different than a regular second price auction, and it is easy to see that in general it will change with $f$. 


(d) It is easy to see that, conditional on paying the fee, the optimal strategy for an agent is to bid his true valuation. This is easy to see from the regular argument of second price auctions, as the entry fee conditional on being paid is just a sunk cost. Thus, there will be a critical type \( t^*_i \in [0,1] \), such that the bidding strategy will be:

\[
b_i(t_i) = \begin{cases} \ t_i & t_i \geq t^*_i \\ 0 & t_i < t^*_i \end{cases} \text{ for all } i \in \{1,2\}.
\]

The critical type will clearly have to be indifferent between entering and not entering. If the critical type enters, then he wins with probability \( \mathbb{P}\{t_j < t^*_i\} = t^*_i \) (that is, wins with the same probability that the other agent is below \( t^*_i \). If \( t^*_i \) wins the object, he will surely pay 0, as the other agent must have not entered. Thus, the indifference condition can be written as follows:

\[
\mathbb{P}\{t_j < t^*_i\} t^*_i - f = 0 \iff t^*_i = \sqrt{f}
\]

Thus, the bidding strategy of agents will be given by:

\[
b_i(t_i) = \begin{cases} \ t_i & t_i \geq \sqrt{f} \\ 0 & t_i < \sqrt{f} \end{cases} \text{ for all } i \in \{1,2\}.
\]

(e) The profits of the seller will be given by:

\[
2f \cdot \mathbb{P}\{b_i(t_i) > 0\} + \mathbb{E}[\min\{b_i(t_i), b_j(t_j)\}].
\]

Note that,

\[
\mathbb{P}\{b_i(t_i) > 0\} = (1 - \sqrt{f})
\]

and,

\[
\mathbb{E}[\min\{b_i(t_i), b_j(t_j)\}] = \mathbb{E}[\min\{t_i, t_j\}] \mathbb{P}\{t_i, t_j \geq \sqrt{f}\} = (1-\sqrt{f})^2 \mathbb{E}[\min\{t_i, t_j\}|t_i, t_j \geq \sqrt{f}]
\]

that is, the profits for the seller from the bids are equal to the bids of a normal second price auction, multiplied by the probability that both types are above \( \sqrt{f} \), as otherwise he gets 0 from the bids. Finally, note that the c.d.f. of the random variable \( q = \min\{t_i, t_j\} \) is equal to \( 1 - \frac{(1-q)^2}{(1-\sqrt{f})^2} \). Thus,

\[
(1-\sqrt{f})^2 \mathbb{E}[\min\{t_i, t_j\}|t_i, t_j \geq \sqrt{f}] = (1-\sqrt{f})^2 \int_{\sqrt{f}}^1 2q \frac{(1-q)}{(1-\sqrt{f})^2} dq = (1-f) - \frac{2(1-f^{3/2})}{3}.
\]

Thus, the profits of the seller are given by:

\[
\pi \triangleq 2f(1-\sqrt{f}) + (1-f) - \frac{2(1-f^{3/2})}{3}.
\]
(f) To calculate the optimal entry fee we can just take the first condition, in which case we get:

\[ 2(1 - \sqrt{f^*}) - \sqrt{f^*} - 1 + \sqrt{f^*} = 0 \]

\[ \Rightarrow f^* = \frac{1}{4}. \]

Note that this entry fee imposes the same allocation rule as a reservation price of 1/2. We know that the latter constitutes a optimal auction, thus by the revenue equivalence theorem we know the this auction must yield the same expected profits for the seller. It is not efficient as sometimes the object is not assigned even when this is efficient.
3. (40) Consider a firm that can invest an amount $I$ in a project generating a high observable cash flow $C > 0$ with probability $\theta$ and 0 otherwise. This probability depends on the firm’s efficiency (type) $0 < \theta_l < \theta_h < 1$. Let $\Pr(\theta = \theta_l) = \alpha$. The firm needs to raise $I$ from external investors who do not observe the value of $\theta$. Assume that
\[ \theta_h C - I > 0 > \theta_l C - I. \]
The external investors accept any contract that yields nonnegative profit in expectations. (The external investors are ”perfectly competitive”).

(a) Suppose that firms can only promise to repay an amount $R$ chosen by the firm (with $0 \leq R \leq C$) when cash flow is $C$ and 0 otherwise.

i. For this game carefully define the notion of a Perfect Bayesian Equilibrium.

ii. Can you give conditions under which there is a pure strategy separating PBE in which the good firm signal its type and receive funding?

iii. Can you give conditions under which there is a pure strategy pooling PBE in which both firms receive funding.

(b) Suppose now that the firm can promise to repay an amount $R$ chosen by the firm (with $0 \leq R \leq C$) when cash flow is $C$, but also has the possibility of pledging some other assets as a collateral for the loan. Should a “default” occur (the firm being unable to repay $R$), an asset of value $K$ to the firm is transferred to the creditor whose valuation is $xK$ with given and fixed $x : 0 < x < 1$. The size of the collateral $K$ is a choice variable of the entrepreneur.

i. For this extended game, define the notion of a Perfect Bayesian Equilibrium.

ii. Give a necessary and sufficient condition for the least cost separating equilibrium to exist. How does it depend on $\alpha$ and $x$?

[Solution]

(a) i. A PBE is given by:

I. A strategy for firms, which is given by:

\[ R : \{\theta_h, \theta_l\} \rightarrow [0, C] \]

and a strategy for the investors

\[ A : [0, C] \rightarrow \Delta\{0, 1\} \]

where $A(R) = 1$ means that the firm gets $I$ and $A(R) = 0$ means that the firm does not get funding.
II. A set of beliefs for the investors:

$$\beta : [0, C] \rightarrow \Delta \{\theta_h, \theta_l\},$$

where $\beta(R)$ is calculated according to Bayes rule whenever possible.

III. Firms maximize their expected utility given the strategy of the investors and the investors maximize given their beliefs.

(a)ii. It is clear that we CANNOT find a separating equilibrium in which any firms get funding. To see this just note that in a separating equilibrium, firms of type $\theta_l$ cannot get funding and thus get a payoff of 0. Nevertheless, firms of type $\theta_l$ would deviate to any strategy that allows them to get funding with positive probability.

(b)i. A PBE is given by:

I. A strategy for firms, which is given by:

$$R : \{\theta_h, \theta_l\} \rightarrow [0, C] \times \mathbb{R}$$

and a strategy for the investors

$$A : [0, C] \times \mathbb{R} \rightarrow \Delta \{0, 1\}$$

where $A(R, K) = 1$ means that the firm gets $I$ and $A(R) = 0$ means that the firm does not get funding.

II. A set of beliefs for the investors:

$$\beta : [0, C] \times \mathbb{R} \rightarrow \Delta \{\theta_h, \theta_l\},$$

where $\beta((R, K))$ is calculated according to Bayes rule whenever possible.

III. Firms maximize their expected utility given the strategy of the investors and the investors maximize given their beliefs.

(b)ii. In any separating equilibrium type $\theta_l$ will necessarily not get any funding. This just comes from the fact that this is inefficient, so it cannot happen that $\theta_l$ gets funding and both investor and firm $\theta_l$ are maximizing. Type $\theta_H$ on the other hand will offer contracts $(R, K)$ such that investors break even, low type don’t want to pretend to be high types and $\theta_h$ get weakly positive profits. Thus, $(R, K)$ must satisfy the following properties:

$$(C - R) \cdot \theta_h - (1 - \theta_h)K \geq 0$$

$$(C - R) \cdot \theta_l - (1 - \theta_l)K \leq 0$$

$$R \theta_h + xK (1 - \theta_h) = I$$
For the least costly separating equilibrium we can just impose the second inequality with equality and solve for \((R, K)\). Thus, we get:

\[
R = \frac{-Cx (1 - \theta_h) \theta_l + I (1 - \theta_l)}{\theta_h (1 - \theta_l) - x (1 - \theta_h) \theta_l}, \quad K = \frac{\theta_l (C \theta_h - I)}{\theta_h - \theta_h \theta_l + x \theta_h \theta_l - x \theta_l}.
\]

For \(x\) being too high it may fail to exist as \(R\) becomes negative.

If \((\alpha \theta_h + (1 - \alpha) \theta_l) C - I \geq 0\), then there also exists a pooling equilibrium in which firms get funding. In this case it is easy to see that there exists a pooling equilibrium in which firms get funding. In this case, the most efficient pooling equilibrium is equal to:

\[
K = 0; \quad R = \frac{I}{(\alpha \theta_h + (1 - \alpha) \theta_l)}.
\]

To compare the efficiency of both equilibria we can just compare the inefficiencies in both equilibria. That is, the least separating equilibrium is efficient if and only if:

\[
(1-x)K(1-\theta_h)\alpha = (1-x)(1-\theta_h)\alpha \cdot \frac{C \theta_h \theta_l}{\theta_h + \theta_l ((x-1) \theta_h - x)} \leq (1-\alpha) \cdot (\theta_l R - I)
\]
4. (60) Consider an economy with \( n \) consumers and two goods. Each consumer \( i \) is endowed with 1 unit of good 1 and no good 2, and has utility function \( u_i(x_{i1}, y) \), where \( x_{i1} \) is \( i \)'s consumption of good 1 and \( y \) is \( i \)'s consumption of good 2. Utility functions are differentiable, strictly increasing in both arguments, and quasi-concave. Good 2 is a public good, and every consumer consumes the same quantity \( y \) of good 2.

(a) Each consumer decides how much of her endowment of good 1 to consume and how much to contribute to the production of the public good. The amount of public good produced is given by \( y = f(\sum_{i=1}^{n}(1 - x_{i1})) \), and hence is a function of the total contribution to the public good. The function \( f \) is increasing, concave, and differentiable. Characterize the Pareto efficient allocations of this economy. If you find it helpful in simplifying the notation, it is fine to consider the case \( n = 2 \). Interpret your conditions.

(b) Find conditions characterizing the equilibrium allocation in this economy. Show that this equilibrium allocation is inefficient. What is the nature of the inefficiency?

(c) Now let us introduce prices. The price of good 1 is 1. We introduce \( n \) prices for good 2, consisting of a personalized price for each consumer, with \( p_i \) being the price at which consumer \( i \) can buy the public good. An equilibrium for this economy is a specification of the \( n \) prices \( p_1, \ldots, p_n \) and a quantity \( Y \) of the public good with the properties that (i) setting \( y = Y \) maximizes the profits \( \sum_{i=1}^{n} p_i y - f^{-1}(y) \) of single firm that produces the public good, (ii) each consumer \( i \)'s utility \( u_i(x_{i1}, y) \) is maximized by buying \( y = Y \) of the public good, given the price \( p_i \) faced by the consumer and given the budget constraint \( x_{i1} + p_i y \leq 1 + s_i \pi \), where \( s_i \) is the share of the firm owned by consumer \( i \) and \( \pi \) is the equilibrium profit level of the firm, and (iii) markets clear, meaning that \( \sum_{i=1}^{n} x_{i1} + f^{-1}(Y) \leq n \) (where \( n \) is the total endowment of good 1). Show that an equilibrium of this economy (called a Lindahl equilibrium) is Pareto efficient.

(d) Your answer to the previous question suggests that competitive equilibria would give efficient provision of public goods if only we had enough prices. How many prices do we need? In light of this, what are the weaknesses of this solution to the public goods problem?
(a)

Let $\mathcal{N} \equiv \{1, \ldots, n\}$ denote the index set of consumers. Let $x_{i1} \equiv x_i$ for all $i \in \mathcal{N}$ (since there are only two goods, omitting the index of the first one will cause no confusion). Let $((x^*_i)_{i \in \mathcal{N}}, y^*)$ denote any Pareto optimal allocation. For the sake of simplicity, assume that $x^*_i > 0$ for any $i \in \mathcal{N}$. We can characterize the conditions that are necessary for Pareto optimality as follows.

Since utilities are strictly increasing in both arguments, neither of the goods is wasted in any Pareto optimum:

$$y^* = f \left( \sum_{j \in \mathcal{N}} (1 - x^*_j) \right). \quad (4)$$

Suppose that we are considering a very slight reallocation between private- and public-good provision. Let, for each $i \in \mathcal{N}$,

$$u^x_i \equiv \frac{\partial u_i(x^*_i, y^*)}{\partial x_i},$$

$$u^y_i \equiv \frac{\partial u_i(x^*_i, y^*)}{\partial y}.$$  

Semi-formally, suppose that the amount of the public good is changed by $\Delta y > 0$ and the provision of the private good for any consumer $j \in \mathcal{N}$ is changed by $\Delta x_j < 0$. Moreover, suppose that this change is such that any consumer but perhaps consumer $i \in \mathcal{N}$ is indifferent between the status quo and this perturbed allocation:

$$u^x_j \Delta x_j + u^y_j \Delta y = 0 \quad \forall j \in \mathcal{N} \setminus \{i\}.$$  

Then, consumer $i$ must be indifferent, too:

$$u^x_i \Delta x_i + u^y_i \Delta y = 0,$$

because if this quantity were positive (negative), then we could slightly increase (decrease, respectively) public-good provision, leaving consumer $i$ strictly better off and all other people indifferent, contradicting Pareto optimality. Moreover, this change must be feasible, implying the following condition:

$$f' \left( \sum_{j \in \mathcal{N}} (1 - x^*_j) \right) \sum_{j \in \mathcal{N}} \Delta x_j + \Delta y = 0.$$
Since
\[ \Delta x_j = -\frac{u_j^y \Delta y}{u_j^x} \quad \forall j \in \mathcal{N}, \]

it follows that
\[
f' \left( \sum_{j \in \mathcal{N}} (1 - x_j^*) \right) \sum_{j \in \mathcal{N}} \left( -\frac{u_j^y \Delta y}{u_j^x} \right) + \Delta y = 0,
\]
or
\[
\sum_{j \in \mathcal{N}} \frac{\partial u_j(x_j^*, y^*)}{\partial y} \frac{\partial y}{\partial x_j} = \frac{1}{f' \left( \sum_{j \in \mathcal{N}} (1 - x_j^*) \right)}.
\]

(5)

It follows that conditions (4) and (5) necessarily hold in any (interior) Pareto optimum.

In fact, these two conditions are not only necessary but also sufficient for Pareto optimality. To see this, suppose that \((x_i^*, y^*)_{i \in \mathcal{N}}\) satisfies (4)–(5). Let, for each \(i \in \mathcal{N}\), \(u_i^* \equiv u_i(x_i^*, y^*)\) denote each person's level of utility according to this allocation. Fix any person \(i \in \mathcal{N}\) and consider the following problem:

\[
\begin{align*}
\max_{(x_i)_{i \in \mathcal{N}}, y} & \quad u_i(x_i, y) \\
\text{s.t.} & \quad y \leq f \left( \sum_{j \in \mathcal{N}} (1 - x_j^*) \right), \\
& \quad u_j(x_j, y) \geq u_j^* \quad \forall j \in \mathcal{N} \setminus \{i\}.
\end{align*}
\]

We will show that \((x_i^*, y^*)_{i \in \mathcal{N}}, y^*\) solves this problem for any \(i \in \mathcal{N}\), which will imply that this is the best feasible allocation for any \(i \in \mathcal{N}\) holding other people's welfare fixed; in other words, it is Pareto optimal. It is sufficient, in turn, to show that \((x_i^*, y^*)_{i \in \mathcal{N}}, y^*\) satisfies the necessary first-order conditions associated with this problem, since the objective is quasi-concave and the constraint set is convex.\(^1\)

\(^1\)It is easy to see that the set
\[
\left\{ (x_i)_{i \in \mathcal{N}}, y \in \mathbb{R}^{n+1}_+ \mid y \leq f \left( \sum_{j \in \mathcal{N}} (1 - x_j^*) \right) \right\}
\]
is convex, given our assumptions. The convexity of
\[
\left\{ (x_i)_{i \in \mathcal{N}}, y \in \mathbb{R}^{n+1}_+ \mid u_j(x_j, y) \geq u_j^* \right\}
\]
for each \(j \in \mathcal{N} \setminus \{i\}\) follows from the quasi-concavity of \(u_j(\cdot)\). The constraint set is the intersection of these \(n\) convex sets (one feasibility condition and \(n - 1\) be-no-worse-off conditions), and thus convex.
Letting $\lambda$ denote the multiplier on the feasibility constraint and $\mu_j$ the multiplier on the requirement that no person $j \in \mathcal{N} \setminus \{i\}$ be worse off, the first-order conditions associated with this problem are as follows:

$$\frac{\partial u_i(x_i, y)}{\partial x_i} - \lambda f' \left( \sum_{j \in \mathcal{N}} (1 - x_j) \right) = 0,$$

$$\frac{\partial u_i(x_i, y)}{\partial y} - \lambda + \sum_{j \in \mathcal{N} \setminus \{i\}} \mu_j \frac{\partial u_j(x_j, y)}{\partial y} = 0,$$

$$-\lambda f' \left( \sum_{j \in \mathcal{N}} (1 - x_j) \right) + \mu_j \frac{\partial u_j(x_j, y)}{\partial x_j} = 0 \quad \forall j \in \mathcal{N} \setminus \{i\}.$$

It is a routine exercise to check that if $((x^*_i)_{i \in \mathcal{N}}, y^*)$ satisfies (4)–(5), then it also satisfies these first order conditions with

$$\lambda = \frac{\frac{\partial u_i(x^*_i, y^*)}{\partial x_i} - 1}{f' \left( \sum_{j \in \mathcal{N}} (1 - x^*_j) \right)},$$

$$\mu_j = \frac{\frac{\partial u_i(x^*_i, y^*)}{\partial x_i}}{\frac{\partial u_j(x^*_j, y^*)}{\partial x_j}}.$$

and that it satisfies the inequality constraints is trivial. It follows that $((x^*_i)_{i \in \mathcal{N}}, y^*)$ is Pareto optimal. That is, aside from some special cases such as corners, (4)–(5) are both necessary and sufficient for Pareto optimality.\(^2\)

(b)

Let 1 be the price of the private good and $\hat{p} \geq 0$ be the price of the public good in equilibrium. Let $((\hat{x}_i, \hat{y}_i)_{i \in \mathcal{N}}, \hat{p})$ denote a competitive equilibrium. Note that with competitive provision of the public good, each person must decide for herself how much she wishes to purchase. Individual optimization requires that the marginal rate of substitution be equal to the price

\(^2\)Note also that the semi-formal argument establishing the necessity of (5) for Pareto optimality can be made rigorous. If $((x^*_i)_{i \in \mathcal{N}}, y^*)$ is Pareto optimal, then it must necessarily solve the constrained optimization problem presented in the sufficiency part of the proof for each $i \in \mathcal{N}$. Again, it is not difficult to check that the first-order conditions of this problem imply (5) after some rearrangement.
ratio for any person. That is, for any \( i \in \mathcal{N} \),

\[
\frac{\partial u_i}{\partial x_i} \left( \hat{x}_i, \hat{y}_i + \sum_{j \in \mathcal{N} \setminus \{i\}} \hat{y}_j \right) = \hat{p},
\]

assuming that \( \hat{p} > 0 \) and the solution is interior. Let \( \hat{y} \equiv \sum_{j \in \mathcal{N}} \hat{y}_j \) denote the aggregate amount of the public good.

Suppose that the public good is produced by a representative competitive firm. Profit maximization requires that

\[
\tilde{z} \in \arg \max_{\tilde{z} \geq 0} \{ \hat{p} f(\tilde{z}) - \tilde{z} \},
\]

where \( \tilde{z} \geq 0 \) denotes the amount of the private good used in the production of the public good. The first-order condition is

\[
\hat{p} f'(\tilde{z}) = 1.
\]

Market clearing requires that \( \hat{y} = f(\tilde{z}) \) and \( \tilde{z} = \sum_{j \in \mathcal{N}} (1 - \hat{x}_j) \). After some rearrangement, the following conditions must hold in this equilibrium:

\[
\sum_{j \in \mathcal{N}} \frac{\partial u_j(\hat{x}_j, \hat{y})}{\partial y} = n \hat{p} = \frac{n}{f' \left( \sum_{j \in \mathcal{N}} (1 - \hat{x}_j) \right)}.
\]

Comparing the equilibrium conditions (6)–(7) to the Pareto optimality conditions (4)–(5) immediately reveals that competitive equilibria are, in general, not Pareto optimal. In particular, the sum of the marginal rates of substitution across consumers is greater (by a factor of \( n \)) in equilibria than they are in optima. This suggests that the marginal utility of the public good is, at least “on average,” too large in competitive equilibria—that is, too little of it is provided.

(c)

Let \( ((\pi_i, \bar{p}_i))_{i \in \mathcal{N}}, \bar{y}) \) be an interior Lindahl equilibrium. Its characterization is similar to that of competitive equilibria, except we have to be
careful about personalized prices.\textsuperscript{3} The first-order conditions associated with consumer optimization are as follows:

$$\frac{\partial u_i(x_i, y)}{\partial y} = \frac{\partial u_i(x_i, y)}{\partial x_i} = \bar{p}_i \quad \forall i \in \mathcal{N}.$$  

The first-order condition for producer optimization is

$$\left( \sum_{j \in \mathcal{N}} \bar{p}_j \right) f'(\bar{x}) = 1,$$

where \( \bar{x} \) is such that \( \bar{y} = f(\bar{x}) \). Market clearing requires also that \( \bar{x} = \sum_{j \in \mathcal{N}} (1 - \bar{x}_j) \). After some rearrangement, the Lindahl equilibrium satisfies the following two conditions:

$$\bar{y} = f \left( \sum_{j \in \mathcal{N}} (1 - \bar{p}_j) \right),$$

$$\sum_{j \in \mathcal{N}} \frac{\partial u_j(x_j, y)}{\partial y} \frac{\partial u_j(x_j, y)}{\partial x_j} = \sum_{j \in \mathcal{N}} \bar{p}_j = \frac{1}{f' \left( \sum_{j \in \mathcal{N}} (1 - \bar{x}_j) \right)}.$$  

Hence, \(((\bar{x}_i, \bar{p}_i)_{i \in \mathcal{N}}, \bar{y})\) satisfies (4) and (5), so it must be Pareto optimal.

(d)

We see that efficiency in a decentralized equilibrium can be restored if different consumers can be charged different prices for the same public good. However, it is unlikely to be feasible in practice if those types of consumers that would be charged high prices can pretend to be one of those types to whom lower prices are offered. In this case, it might be reasonable for public goods to be produced by a government agency having sufficient legal authority and legitimacy to fund public-good provision via some sophisticated centralized mechanism or tax system.

\textsuperscript{3}Upon comparing competitive and Lindahl equilibria, it becomes clear that purchases of the public good differ across persons but the price of the public good is the same for everyone in the former, whereas it is the other way around in the latter.