This is a closed-book exam. The exam lasts for 180 minutes. Please write clearly and legibly. Be especially careful in the definition of the game, the payoff function and the equilibrium notions. The allocated points are also a good indicator for your time budget. Please record the answers (1, 2), (3, 4), and 5 in separate bluebooks.
1. (30) Recall the definition of a signalling game. Nature draws player 1’s type \( \theta \in \Theta \), a finite set, according to some distribution \( p \) on \( \Theta \), where \( p(\theta) > 0 \) for all \( \theta \in \Theta \). Player 1 knows \( \theta \), player 2 does not. Player 1 takes an action \( \alpha_1 \in \Delta A_1 \), where \( A_1 \) is a finite set of actions. Player 2 observes the realization \( a_1 \) of \( \alpha_1 \) and chooses then \( \alpha_2 \in \Delta A_2 \), where \( A_2 \) is a finite set of actions. This is the end of the game. Payoffs are given by \( u_i(a, \theta) \), \( i = 1, 2 \).

(a) For this signalling game, define the notion of a weak perfect Bayesian equilibrium.

(b) For this signalling game, define the notion of a sequential equilibrium.

(c) Prove, or provide a counterexample for, the following claim: in a signalling game, weak perfect Bayesian equilibria and sequential equilibria coincide.

2. (40) Consider the following game between two sellers and a continuum of identical buyers. The sellers produce a homogenous good at zero (marginal) cost that every buyer values at \( v \). The sellers contact the buyers by making simultaneously a price offer of \( p_i \) for each firm \( i \in \{1, 2\} \). The buyers observe the offers and buys from the seller with the lower price if that price does not exceed \( v \). If the two prices are equal, then the market is equally split between the buyers.

(a) Find the pure strategy Nash equilibria of this pricing game.

(b) Suppose next that it costs \( c < v \) to make the price offer. The buyers can purchase only from a firm that has made an offer. Again, the buyers buy from the firm with the lowest offer and if the prices are equal, then the market is split between the firms. Are there pure strategy Nash equilibria in the game?

(c) Suppose now that there is no cost of making the price offer, but firm \( i \) incurs a privately observed transportation cost of \( d_i \) that is uniformly distributed on \([0, v]\). Hence firm \( i \) makes a profit of \( p_i - d_i \) on the units that it sells. Consider again the game where the firms announce prices simultaneously (and each firm has to announce a price).

i. What is the relevant solution concept for such a game? Define the strategy and the equilibrium for this game.

ii. Find an equilibrium of this game.
3. (30) Consider a finite normal form game with complete information.

(a) Define the notion of a strictly dominated strategy, first a strictly dominated pure strategy, then second a strictly dominated mixed strategy.

(b) Then prove or disprove: a mixed strategy \( \alpha_i \) is strictly dominated if and only if it assigns positive probability to a pure action \( a_i \) that is itself strictly dominated.

4. (40) Consider the following Bayesian game. Player 1 is interested in selling his watch to player 2. The cost of selling the watch is zero and his utility is \( p \) if he sells the watch at price \( p \). Player 2 has value \( v \) for the watch and would obtain utility \( v - p \) if he buys the watch for price \( p \). Only player 2 knows the value of \( v \). It is common-knowledge that player 1 believes that \( v \) is uniformly distributed on the interval \([0, 1]\). They will bargain over the price through a mediator. Each player \( i \) will submit a number \( b_i \) in a sealed envelope to the mediator. The mediator unseals the envelopes and implements a trade according to the following rules. If \( b_1 > b_2 \) then there will be no sale. If \( b_1 \leq b_2 \) then the watch will be sold to bidder 1 at price \( p = b_1 \).

(a) What are the sets of pure strategies in this Bayesian game for players 1 and 2?

(b) For what set of values \( b_1 \) does there exists a pure-strategy Bayesian Nash equilibrium in which player 1 submits \( b_1 \)? Explain carefully how you reach your conclusion.

(c) Give an example of a strategy for player 2 which is weakly dominated. (It makes no difference here or in the next part whether you use the \textit{ex ante} or \textit{interim} versions of weak dominance.)

(d) What is the set of Bayesian Nash equilibria in which player 2 does not use a weakly dominated strategy? Explain carefully how you reach your conclusion.
5. (40) Consumers buy insurance from a competitive insurance industry consisting of many individual firms. Each consumer is an expected utility maximizer with utility function $u(x) = \sqrt{x}$ for final consumption $x$. Each consumer has an initial wealth $\$20$. There are two types of consumers, high risk and low risk. High risk consumers have a probability $\frac{2}{3}$ of losing $\$10$, while low risk consumers have a probability $\frac{1}{3}$ of losing $\$10$. The insurance industry is perfectly competitive and risk-neutral, that is each insurance company is maximizing expected profit but because of perfect competition each firm earns zero expected profit from the insurance policy that it sells. The net utility of insuree is given by:

$$u = \begin{cases} \sqrt{w - lp}, & \text{if no loss occurs}, \\ \sqrt{w - l + i - lp}, & \text{if loss } l \text{ occurs}; \end{cases}$$

where $w > 0$ is the wealth, $l > 0$ is the loss, $i$ is the insurance compensation, the indemnity, and $p$ is the premium for $\$1$ of indemnity. The total premium payment, $i \cdot p$, has to be paid whether a loss occurs or not.

(a) First, suppose that insurance companies can distinguish high and low risk consumers.

i. In the competitive equilibrium, what premiums do they charge to the two types of consumers? Give an argument supporting your claim.

ii. In the competitive equilibrium, how much insurance, i.e. indemnity $i$, do the two types of consumers buy? Give an argument supporting your claim.

(b) Now suppose that insurance companies cannot distinguish high and low risk consumers.

i. What problem would arise if the insurance companies tried to sell insurance to consumers as described in part (a)?

ii. Suppose there were a "separating equilibrium," where different types of consumers bought different contracts. Describe what the equilibrium must look like graphically and algebraically. Each consumer can at most by one insurance contract characterized by the indemnity $i$ and the corresponding premium $p$. (Hint: You do not have solve for the equilibrium solution analytically. It suffices to describe the properties of the equilibrium allocation from the conditions of incentive compatibility and zero profit/free entry condition of the insurance firms.)

(c) Suppose that a proportion $\pi$ of consumers were high risk. For which values of $\pi$ would another firm be able to enter and earn positive profits selling to both high and low risk consumers relative to the "separating equilibrium" constructed in (b)?