Comprehensive Exam

1. Consider the following model of price competition with differentiated products. At price vector \((p_1, p_2)\), the demands of the two firms are given by:

\[
\begin{align*}
q_1 (p_1, p_2) &= 1 - p_1 + \gamma p_2; \\
q_2 (p_1, p_2) &= 1 - p_2 + \gamma p_1.
\end{align*}
\]

Assume that the firms production costs are zero and hence the firms’ profits equal their revenues \(p_i q_i\). Assume also that \(\gamma < 1\).

(a) Find the Nash equilibrium of the game where the firms set their prices simultaneously.

(b) Compare the equilibrium profits to industry profit maximizing levels (i.e. find prices that maximize the sum of the firms’ payoffs).

(c) Suppose next that prior to engaging in price competition, the firms may invest in their own product. The investment shifts their own demand out by \(\alpha_i\) and the cost of the investment is \(\alpha_i^2\). After investing \(\alpha_i\), the demand for firm \(i\) is

\[
q_i (p_i, p_j; \alpha_i, \alpha_j) = 1 + \alpha_i - p_i + \gamma p_j.
\]

The firms’ investment levels are observed by both firms prior to the second stage where the firms compete in prices. Solve for the equilibrium investment levels and equilibrium prices.

2. Consider a common-value auction in which bidder \(i\)’s type is an element \(t_i \in [0, 1]\), and the common value for the good is \(v(t) = t_i + t_j\). The format is a second-price auction so that the winner has utility \(v(t) - p\), where \(p\) is the second-highest submitted bid, and losers have utility zero.

(a) Assume that the types are independently and uniformly distributed. Define the notion of a bidding strategy and the notion of a symmetric Bayesian Nash equilibrium for the second price auction

(b) Derive the symmetric Bayesian Nash equilibrium with two bidders.

(c) Suppose now that there are \(I > 2\) bidders and that the common value is given by \(v(t) = \sum_{i=1}^{t} t_i\). Generalize the equilibrium bidding rule from (b.) to the case of many bidders, maintaining independence and uniformity.
3. Consider a duopoly model with uncertain demand. The inverse demand function is given by:
\[ p(q_1, q_2) = \theta - q_1 + q_2 \]
where \( \theta \) is uniformly distributed between \([1, 2]\). Each firm has a quadratic cost function given by
\[ c(q_i) = \frac{1}{2}q_i^2. \]
The firms receive no additional information about the true level of demand before they have to make their strategic choice.

(a) Derive the pure strategy equilibrium in the Cournot duopoly under uncertainty. What is the quantity supplied into the market by each firm and what is the equilibrium price. How does the equilibrium quantity and price depend on the realization of \( \theta \)?

(b) Suppose now that the firms compete in supply functions. In other word each firm submits to the market a supply function \( q_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) which specifies the supply that firm \( i \) is willing to make if the equilibrium price is \( p \), i.e. \( q_i(p) \). An equilibrium \((q_1(p), q_2(p))\) in the game with supply functions competition has the property that for every realization of \( \theta \in [1, 2] \):
\[ p(q_1(p), q_2(p)) = \theta - q_1(p) + q_2(p) = p. \]
You may restrict to your attention to linear supply function strategies \( q_i(p) = a + bp \) for some \( a, b \in \mathbb{R}_+ \).

i. Derive the best response of firm \( i \) to a linear supply function offered by firm \( j \).

ii. Derive an equilibrium in linear supply functions. How does the equilibrium quantity and price depend on the realization of \( \theta \)?

4. Consider an agent and a principal who consider entering a contractual relationship. Both agent and principal are risk neutral, but the agent is also subject to a limited liability constraint. The agent can either work or shirk, \( e \in \{0, 1\} \) and the cost of working/shirking is \( e \). The outcome \( x \in \{0, 1\} \) is a random variable with a conditional probability \( \Pr(x | e) \) given by:
\[ \Pr(1 | 1) = \Pr(0 | 0) = p > \frac{1}{2}. \]
The principal can offer a wage contract \( w(x) \) conditional on \( x \), but not on \( e \). The utility function of the agent is given by
\[ u(w, e) = w - e \]
and his reservation value (outside option) is zero. The utility function of the principal is
\[ v(x, w) = ax - w \]
where $\alpha > 0$ is the marginal value of the outcome $x$ for the principal. The limited liability constraint for the agent requires that for every pair $(w, e)$ which may arise in equilibrium:

$$\forall w, e : w - e \geq 0.$$ 

(a) Carefully describe the optimization problem for the principal.

(b) Suppose initially that the limited liability constraint is absent. What is the nature of the optimal contract? When will the agent choose $e = 0$ or $e = 1$ in the optimal contract as function of the values of $\alpha$ and $p$?

(c) Suppose now that the limited liability constraint has to hold. Describe the optimal contract. Find the values of $\alpha$ and $p$ for which it is optimal for the principal to induce the agent to select $e = 0$ or $e = 1$. How does your answer differ from (4b)? Briefly interpret and explain the source of the difference.